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INVERSE SPECTRAL PROBLEMS FOR DIFFERENTIAL OPERATORS ON A GRAPH WITH A ROOTED CYCLE

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Abstract. An inverse spectral problem is studied for Sturm-Liouville differential operators on graphs with a cycle and with standard matching conditions in internal vertices. A uniqueness theorem is proved, and a constructive procedure for the solution is provided.

1. Introduction

1.1. We study inverse spectral problems for Sturm-Liouville differential operators on graphs with the so-called rooted cycle. Inverse spectral problems consist in recovering operators from their spectral characteristics. The main results on inverse spectral problems on an *interval* are presented in the monographs [1]−[8]. Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [9, 10] and the references therein). Most of the works in this direction are devoted to the so-called direct problems of studying properties of the spectrum and the root functions for operators on graphs. Inverse spectral problems, because of their nonlinearity, are more difficult to investigate, and nowaday there are only a number of papers in this area. In particular, inverse spectral problems of recovering coefficients of differential operators on trees (i.e on graphs without cycles) were studied in [11]−[17] and other papers. The inverse spectral problem for graphs with a cycle is solved in [18] but only for a very particular case. In this paper we study more general graphs than in [18]. We give a formulation and obtain the solution of the inverse spectral problem for Sturm-Liouville operators on graphs with a rooted cycle and with standard matching conditions in the internal vertex. We prove the corresponding uniqueness theorem and provide a constructive procedure for the solution of this class of inverse problems. For solving the inverse problem we develop ideas from [12] and [18].

1.2. Consider a compact graph G in $\mathbf{R^m}$ with the set of vertices $V = \{v_0, \ldots, v_r\}$ and the set of edges $\mathcal{E} = \{e_0, \ldots, e_r\}$, where e_0 is a cycle, $v_0 \in e_0$. The graph has the form $G = e_0 \cup T$, where T is a tree (i.e. graph without cycles) with the root v_0, the set of

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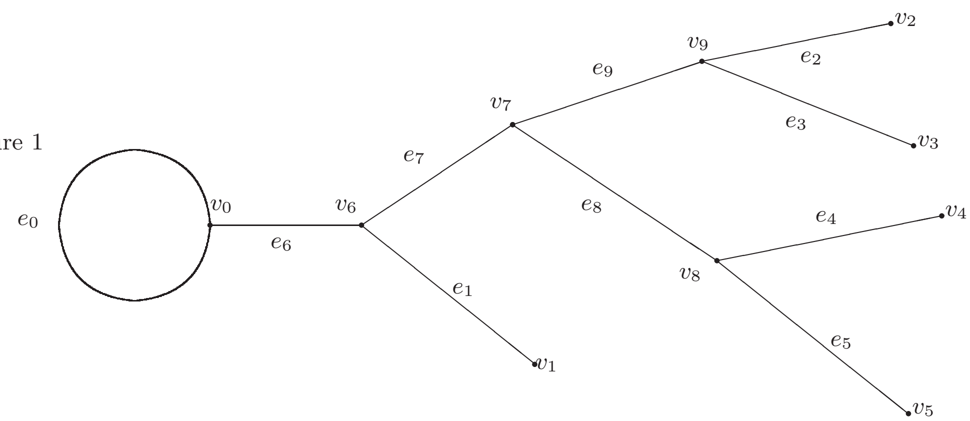
vertices $\{v_0, \dots, v_r\}$ and the set of edges $\{e_1, \dots, e_r\}$, $T \cap e_0 = v_0$, and v_0 is a boundary vertex for T, but v_0 is an internal vertex for the graph G.

For two points $a, b \in T$ we will write $a \le b$ if a lies on a unique simple path connecting the root v_0 with b. We will write $a < b$ if $a \le b$ and $a \ne b$. The relation $<$ defines a partial ordering on T. If $a < b$ we denote $[a, b] := \{z \in T : \ a \le z \le b\}$. In particular, if $e = [v, w]$ is an edge, we call v its initial point, w its end point and say that e emanates from v and terminates at w. For each internal vertex v we denote by $R(v) := \{e \in \mathcal{E} : \ e = [v, w], \ w \in V\}$ the set of edges emanated from v. For each $v \in V$ we denote by $|v|$ the number of edges between v_0 and v. For any $v \in V$ the number $|v|$ is a non-negative integer, which is called the order of v. For $e \in \mathcal{E}$ its order is defined as the order of its end point. The number $\sigma := \max_{j=\overline{1,r}} |v_j|$ is called the height of the tree T. Let $V^{(\mu)} := \{v \in V : \ |v| = \mu\}$, $\mu = \overline{0, \sigma}$ be the set of vertices of order μ, and let $\mathcal{E}^{(\mu)} := \{e \in \mathcal{E} : \ e = [v, w], \ v \in V^{(\mu-1)}, w \in V^{(\mu)}\}$, $\mu = \overline{1, \sigma}$ be the set of edges of order μ.

For definiteness we enumerate the vertices v_j as follows: $\Gamma := \{v_1, \dots, v_p\}$ are boundary vertices of G, $v_{p+1} \in V^{(1)}$, and v_j, $j > p+1$ are enumerated in order of increasing $|v_j|$. We enumerate the edges similarly, namely: $e_j = [v_{j_k}, v_j]$, $j = \overline{1, r}$, $j_k < j$. In particular, $E := \{e_1, \dots, e_p\}$ is the set of boundary edges, $e_{p+1} = [v_0, v_{p+1}]$. The edge e_{p+1}, emanated from the root v_0, is called the rooted edge of T. Clearly, $e_j \in \mathcal{E}^{(\mu)}$ iff $v_j \in V^{(\mu)}$. As an example see figure 1 where $r = 9$, $p = 5$, $\sigma = 4$.

Figure 1



Let d_j be the length of the edge e_j, $j = \overline{0, r}$. Each edge $e \in \mathcal{E}$ is viewed as a segment $[0, d_j]$ and is parameterized by the parameter $x_j \in [0, d_j]$. It is convenient for us to choose the following orientation: for $j = \overline{1, r}$ the end vertex v_j corresponds to $x_j = 0$, and the initial vertex v_{j_k} corresponds to $x_j = d_j$; for the cycle e_0 both ends $x_0 = +0$ and $x_0 = d_0 - 0$ correspond to v_0.

1.3. An integrable function Y on G may be represented as $Y = \{y_j\}_{j=\overline{0,r}}$, where the function $y_j(x_j)$ is defined on the edge e_j. Let $q = \{q_j\}_{j=\overline{0,r}}$ be an integrable real-valued function on G which is called the potential. Consider the Sturm-Liouville equation on G:

$$-y_j''(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j), \quad x_j \in [0, d_j], \tag{1}$$

where $j = \overline{0, r}$, λ is the spectral parameter, the functions $y_j(x_j)$, $y_j'(x_j)$ are absolutely continuous on $[0, d_j]$ and satisfy the following matching conditions in the internal vertices v_0 and v_k, $k = \overline{p+1, r}$: For $k = \overline{p+1, r}$,

$$y_j(d_j) = y_k(0) \text{ for all } e_j \in R(v_k), \qquad \sum_{e_j \in R(v_k)} y_j'(d_j) = y_k'(0), \tag{2}$$

and for v_0,

$$y_{p+1}(d_{p+1}) = y_0(d_0) = y_0(0), \quad y_{p+1}'(d_{p+1}) + y_0'(d_0) = y_0'(0). \tag{3}$$

Matching conditions (2)-(3) are called the standard conditions. In electrical circuits, (2) expresses Kirchhoff's law; in elastic string network, it expresses the balance of tension, and so on.

Let us consider the boundary value problem $L_0(G)$ for equation (1) with the matching conditions (2)-(3) and with the Dirichlet boundary conditions at the boundary vertices $v_1, \dots, v_p$:

$$y_j(0) = 0, \quad j = \overline{1, p}.$$

Moreover, we also consider the boundary value problems $L_k(G)$, $k = \overline{1, p}$ for equation (1) with the matching conditions (2)-(3) and with the boundary conditions

$$y_k'(0) = 0, \quad y_j(0) = 0, \ j = \overline{1, p} \setminus k.$$

We denote by $\Lambda_k = \{\lambda_{kn}\}_{n \geq 1}$ the eigenvalues (counting with multiplicities) of $L_k(G)$, $k = \overline{0, p}$. We recall that the multiplicity of an eigenvalue is a number of linear independent eigenfunctions related to this eigenvalue. In contrast to the case of trees (see [12]), here the specification of the spectra Λ_k, $k = \overline{0, p}$, does not uniquely determines the potential, and we need an additional information. Let $S_j(x_j, \lambda)$, $C_j(x_j, \lambda)$, $j = \overline{0, r}$ be the solutions of equation (1) on the edge e_j with the initial conditions

$$S_j(0, \lambda) = C_j'(0, \lambda) = 0, \quad S_j'(0, \lambda) = C_j(0, \lambda) = 1.$$

For each fixed $x_j \in [0, d_j]$, the functions $S_j^{(\nu)}(x_j, \lambda)$, $C_j^{(\nu)}(x_j, \lambda)$, $j = \overline{0, r}$, $\nu = 0, 1$, are entire in λ of order 1/2. Moreover,

$$\langle C_j(x_j, \lambda), S_j(x_j, \lambda) \rangle \equiv 1,$$

where $\langle y, z \rangle := yz' - y'z$ is the Wronskian of y and z. Denote

$$h(\lambda) := S_0(d_0, \lambda), \quad H(\lambda) := C_0(d_0, \lambda) - S_0'(d_0, \lambda).$$

Let $\{\nu_n\}_{n\geq 1}$ be zeros of the entire function $h(\lambda)$, and put $\omega_n := \operatorname{sign} H(\nu_n)$, $\Omega = \{\omega_n\}_{n\geq 1}$. The inverse problem is formulated as follows.

Inverse problem 1. Given $\Lambda_k,\ k = \overline{0,p}$ and Ω, construct the potential q on G.

Let us formulate the uniqueness theorem for the solution of Inverse Problem 1. For this purpose together with q we consider a potential $\tilde{q}$. Everywhere below if a symbol α denotes an object related to q, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{q}$.

Theorem 1. *If $\Lambda_k = \tilde{\Lambda}_k,\ k = \overline{0,p}$, and $\Omega = \tilde{\Omega}$, then $q = \tilde{q}$ on G. Thus, the specification of $\Lambda_k,\ k = \overline{0,p}$ and Ω uniquely determines the potential q on G.*

This theorem will be proved in section 3. Moreover, we give a constructive procedure for the solution of Inverse Problem 1. In section 2 we introduce the main notions and prove some auxiliary propositions.

2. Characteristic functions

2.1. Fix $k = \overline{p+1,r}$. Denote $Q_k := \{z \in T:\ v_k < z\},\ G_k := \overline{G \setminus Q_k}$. Then

$$Q_k = \bigcup_{e_i \in R(v_k)} T_{ki}\,,$$

where T_{ki} is the tree with the root v_k and with the rooted edge e_i. Clearly, $G_k = e_0 \cup T_k$, where $T_k = \overline{T \setminus Q_k}$.

Notation. If D is a graph, then we will denote by $L_0(D)$ the boundary value problem for equation (1) on D with the standard matching conditions in internal vertices and with the Dirichlet boundary conditions in boundary vertices. Let $\{Y\}_D := \{y_j\}_{e_j \in D}$. If v_j is a boundary vertex of D, then $L_j(D)$ will denote the boundary value problem for equation (1) on D with the standard matching conditions in internal vertices, with the Neuman boundary condition $Y'_{|v_j} = 0$ at v_j and with the Dirichlet boundary conditions in all other boundary vertices. For example, $L_0(G_k)$ is the boundary value problem on G_k with the boundary conditions $y_m(0) = 0$, $e_m \in E \cap G_k$, and $L_k(G_k)$ is the boundary value problem on G_k with the boundary conditions $y'_k(0) = 0$, $y_m(0) = 0$, $e_m \in (E \cap G_k) \setminus e_k$. We also consider the BVP $L^1(T)$ for equation (1) on T with the boundary conditions $Y'_{|v_0} = 0,\ Y_{|v_j} = 0,\ j = \overline{1,p}$.

Fix $k = \overline{1,p}$. Let $\Phi_k = \{\Phi_{kj}\}_{j=\overline{0,r}}$, be solutions of equation (1) satisfying the matching conditions (2)-(3) and the boundary conditions

$$\Phi_{kj}(0,\lambda) = \delta_{kj}, \quad j = \overline{1,p}, \tag{4}$$

where δ_{kj} is the Kronecker symbol. Denote

$$M_k(\lambda) := \Phi'_{kk}(0,\lambda), \quad k = \overline{1,p}.$$

The function $M_k(\lambda)$ is called the Weyl function with respect to the boundary vertex v_k.

Denote $M^0_{kj}(\lambda) = \Phi'_{kj}(0, \lambda)$, $M^1_{kj}(\lambda) = \Phi_{kj}(0, \lambda)$, $j = \overline{0, r}$. Then

$$\Phi_{kj}(x_j, \lambda) = M^1_{kj}(\lambda) C_j(x_j, \lambda) + M^0_{kj}(\lambda) S_j(x_j, \lambda), \quad j = \overline{0, r}. \tag{5}$$

In particular, $M^0_{kk}(\lambda) = M_k(\lambda)$, $M^1_{kk}(\lambda) = 1$, $M^1_{kj}(\lambda) = 0$ for $j = \overline{1, p} \setminus k$. Hence

$$\Phi_{kk}(x_k, \lambda) = C_k(x_k, \lambda) + M_k(\lambda) S_k(x_k, \lambda),$$

and consequently,

$$\langle \Phi_{kk}(x_k, \lambda), \, S_k(x_k, \lambda) \rangle \equiv 1.$$

Substituting (5) into (2)-(3) and (4) we obtain a linear algebraic system s_k with respect to $M^0_{kj}(\lambda), M^1_{kj}(\lambda)$, $j = \overline{0, r}$. The determinant $\Delta_0(\lambda, G)$ of this system does not depend on k and has the form

$$\Delta_0(\lambda, G) = \Delta_0(\lambda, T) d(\lambda) + \Delta^1(\lambda, T) h(\lambda), \tag{6}$$

where

$$d(\lambda) = C_0(d_0, \lambda) + S'_0(d_0, \lambda) - 2, \quad h(\lambda) = S_0(d_0, \lambda), \tag{7}$$

$\Delta_0(\lambda, T)$ and $\Delta^1(\lambda, T)$ are the characteristic functions of the boundary value problems $L_0(T)$ and $L^1(T)$ respectively, which were defined and studied in [17]. For convenience of the readers in the Appendix at the end of the paper we provide formulae for constructing the functions $\Delta_0(\lambda, T)$ and $\Delta^1(\lambda, T)$ from [17]. The function $\Delta_0(\lambda, G)$ is entire in λ of order 1/2, and its zeros (counting with multiplicities) coincide with the eigenvalues of the boundary value problem $L_0(G)$. Solving the algebraic system s_k we get by Cramer's rule: $M^\nu_{kj}(\lambda) = \Delta^\nu_{kj}(\lambda, G)/\Delta_0(\lambda, G)$, $\nu = 0, 1$, $j = \overline{0, r}$, where the determinant $\Delta^\nu_{kj}(\lambda, G)$ is obtained from $\Delta_0(\lambda, G)$ by the replacement of the column which corresponds to $M^\nu_{kj}(\lambda)$ with the column of free terms. In particular,

$$M_k(\lambda) = -\frac{\Delta_k(\lambda, G)}{\Delta_0(\lambda, G)}, \quad k = \overline{1, p}, \tag{8}$$

where $\Delta_k(\lambda, G)$, $k = \overline{1, p}$, is obtained from $\Delta_0(\lambda, G)$ by the replacement of $S^{(\nu)}_k(d_k, \lambda)$, $\nu = 0, 1$, with $C^{(\nu)}_k(d_k, \lambda)$. The zeros of $\Delta_k(\lambda, G)$ (counting with multiplicities) coincide with eigenvalues of the boundary value problem $L_k(G)$. The function $\Delta_k(\lambda, G)$, $k = \overline{0, p}$, is called the characteristic function for the boundary value problem $L_k(G)$.

Fix $k = \overline{p + 1, r}$. Let $\Delta_0(\lambda, G_k)$ and $\Delta_k(\lambda, G_k)$ be the characteristic functions for $L_0(G_k)$ and $L_k(G_k)$, respectively. Using (6), (7) and formulae for $\Delta_0(\lambda, T)$, $\Delta^1(\lambda, T)$ from [17] (see also the Appendix) one can get

$$\Delta_0(\lambda, G) = \Delta_0(\lambda, Q_k) \Delta_0(\lambda, G_k) + \Big(\prod_{e_i \in R(v_k)} \Delta_0(\lambda, T_{ki}) \Big) \Delta_k(\lambda, G_k), \tag{9}$$

where $\Delta_0(\lambda, Q_k)$ and $\Delta_0(\lambda, T_{ki})$ are the characteristic functions for $L_0(Q_k)$ and $L_0(T_{ki})$, respectively, which were defined and studied in [17] (see also the Appendix). Similarly, for $e_j \in E \cap T_{ks}$,

$$\Delta_j(\lambda, G) = \Delta_j(\lambda, Q_k)\Delta_0(\lambda, G_k) + \Big(\Delta_j(\lambda, T_{ks}) \prod_{e_i \in R(v_k),\ i \neq s} \Delta_0(\lambda, T_{ki})\Big)\Delta_k(\lambda, G_k), \quad (10)$$

where $\Delta_j(\lambda, Q_k)$ and $\Delta_j(\lambda, T_{ki})$ are constructed from $\Delta_0(\lambda, Q_k)$ and $\Delta_0(\lambda, T_{ki})$ by the replacement of $S_j^{(\nu)}(d_j, \lambda)$, $j = 0, 1$, with $C_j^{(\nu)}(d_j, \lambda)$.

Example 1. Let $\sigma = 1$. Then $r = 1$,

$$\Delta_0(\lambda, G) = S_1(d_1, \lambda)d(\lambda) + S_1'(d_1, \lambda)h(\lambda), \quad \Delta_0(\lambda, T) = S_1(d_1, \lambda), \quad \Delta^1(\lambda, T) = S_1'(d_1, \lambda).$$

Example 2. Let $\sigma = 2$. Then $r = p + 1$, $T_{p+1,i} = \{e_i\}$, $i = \overline{1, p}$, hence

$$\Delta_0(\lambda, T) = \prod_{m=1}^{p} S_m(d_m, \lambda)\Big(\sum_{i=1}^{p} \frac{S_i'(d_i, \lambda)}{S_i(d_i, \lambda)} S_{p+1}(d_{p+1}, \lambda) + C_{p+1}(d_{p+1}, \lambda)\Big),$$

$$\Delta^1(\lambda, T) = \prod_{m=1}^{p} S_m(d_m, \lambda)\Big(\sum_{i=1}^{p} \frac{S_i'(d_i, \lambda)}{S_i(d_i, \lambda)} S_{p+1}'(d_{p+1}, \lambda) + C_{p+1}'(d_{p+1}, \lambda)\Big).$$

In particular, for $p = 2$,

$$\Delta_0(\lambda, T) = (S_1(d_1, \lambda)S_2'(d_2, \lambda) + S_1'(d_1, \lambda)S_2(d_2, \lambda))S_3(d_3, \lambda) + S_1(d_1, \lambda)S_2(d_2, \lambda)C_3(d_3, \lambda),$$

$$\Delta^1(\lambda, T) = (S_1(d_1, \lambda)S_2'(d_2, \lambda) + S_1'(d_1, \lambda)S_2(d_2, \lambda))S_3'(d_3, \lambda) + S_1(d_1, \lambda)S_2(d_2, \lambda)C_3'(d_3, \lambda),$$

and (9) takes the form

$$\begin{aligned}\Delta_0(\lambda, G) &= (S_1(d_1, \lambda)S_2'(d_2, \lambda) + S_1'(d_1, \lambda)S_2(d_2, \lambda))(S_3(d_3, \lambda)d(\lambda) + S_3'(d_3, \lambda)h(\lambda)) \\ &\quad + S_1(d_1, \lambda)S_2(d_2, \lambda)(C_3(d_3, \lambda)d(\lambda) + C_3'(d_3, \lambda)h(\lambda)) \\ &= \Delta_0(\lambda, Q_3)\Delta_0(\lambda, G_3) + \Delta_0(\lambda, T_{31})\Delta_0(\lambda, T_{32})\Delta_3(\lambda, G_3),\end{aligned}$$

where

$$\Delta_0(\lambda, Q_3) = S_1(d_1, \lambda)S_2'(d_2, \lambda) + S_1'(d_1, \lambda)S_2(d_2, \lambda),$$
$$\Delta_0(\lambda, G_3) = S_3(d_3, \lambda)d(\lambda) + S_3'(d_3, \lambda)h(\lambda),$$
$$\Delta_0(\lambda, T_{31}) = S_1(d_1, \lambda), \quad \Delta_0(\lambda, T_{32}) = S_2(d_2, \lambda),$$
$$\Delta_3(\lambda, G_3) = C_3(d_3, \lambda)d(\lambda) + C_3'(d_3, \lambda)h(\lambda).$$

2.2. Let $\lambda = \rho^2$, $\operatorname{Im}\rho \geq 0$. Denote $\Lambda := \{\rho :\ \operatorname{Im}\rho \geq 0\}$, $\Lambda^\delta := \{\rho :\ \arg\rho \in [\delta, \pi - \delta]\}$. It is known (see [19]) that for each fixed $j = \overline{0, r}$ on the edge e_j, there exists a fundamental system of solutions of equation (1) $\{e_{j1}(x_j, \rho), e_{j2}(x_j, \rho)\}$, $x_j \in [0, d_j]$, $\rho \in \Lambda$, $|\rho| \geq \rho^*$ with the properties:

(1) the functions $e_{js}^{(\nu)}(x_j,\rho)$, $\nu=0,1$, are continuous for $x_j\in[0,d_j]$, $\rho\in\Lambda$, $|\rho|\ge\rho^*$;

(2) for each $x_j\in[0,d_j]$, the functions $e_{js}^{(\nu)}(x_j,\rho)$, $\nu=0,1$, are analytic for $\operatorname{Im}\rho>0$, $|\rho|>\rho^*$;

(3) uniformly in $x_j\in[0,d_j]$, the following asymptotical formulae hold

$$e_{j1}^{(\nu)}(x_j,\rho)=(i\rho)^\nu\exp(i\rho x_j)[1],\ e_{j2}^{(\nu)}(x_j,\rho)=(-i\rho)^\nu\exp(-i\rho x_j)[1],\ \rho\in\Lambda,\ |\rho|\to\infty, \tag{11}$$

where $[1]=1+O(\rho^{-1})$.

Fix $k=\overline{1,p}$. One has

$$\Phi_{kj}(x_j,\lambda)=A_{kj}^1(\rho)e_{j1}(x_j,\rho)+A_{kj}^0(\rho)e_{j2}(x_j,\rho),\quad x_j\in[0,d_j],\ j=\overline{0,r}. \tag{12}$$

Substituting (12) into (2)-(3) and (4) we obtain a linear algebraic system s_k^0 with respect to $A_{kj}^\nu(\lambda)$, $\nu=0,1,\ j=\overline{0,r}$. The determinant $\delta_0(\rho)$ of s_k^0 does not depend on k and has the form

$$\delta_0(\rho)=2(-2i\rho)^{r+1}\Delta_0(\lambda,G),\quad \rho\in\Lambda.$$

Moreover,

$$\delta_0(\rho)=\beta\exp\Big(-i\rho\sum_{j=0}^r d_j\Big)[1],\quad \rho\in\Lambda^\delta,\ |\rho|\to\infty,\ \beta\ne0.$$

Solving the algebraic system s_k^0 by Cramer's rule and using (11), we get

$$A_{kk}^1(\rho)=[1],\ A_{kk}^0(\rho)=\beta_k\exp(2i\rho d_k)[1],\quad \rho\in\Lambda^\delta,\ |\rho|\to\infty,$$

where β_k are constants. Together with (11) and (12) this yields for each fixed $x_k\in[0,d_k),\ \nu=0,1$:

$$\Phi_{kk}^{(\nu)}(x_k,\lambda)=(i\rho)^\nu\exp(i\rho x_k)[1],\quad \rho\in\Lambda^\delta,\ |\rho|\to\infty. \tag{13}$$

In particular, $M_k(\lambda)=(i\rho)[1],\ \rho\in\Lambda^\delta,\ |\rho|\to\infty$. Moreover, uniformly in $x_j\in[0,d_j]$,

$$S_j^{(\nu)}(x_j,\lambda)=\frac{1}{2i\rho}\Big((i\rho)^\nu\exp(i\rho x_j)[1]-(-i\rho)^\nu\exp(-i\rho x_j)[1]\Big),\ \rho\in\Lambda,\ |\rho|\to\infty, \tag{14}$$

$$C_j^{(\nu)}(x_j,\lambda)=\frac12\Big((i\rho)^\nu\exp(i\rho x_j)[1]+(-i\rho)^\nu\exp(-i\rho x_j)[1]\Big),\ \rho\in\Lambda,\ |\rho|\to\infty. \tag{15}$$

Let $\lambda_{kn}^0=(\rho_{kn}^0)^2,\ k=\overline{0,p}$, be the eigenvalues of the boundary value problem $L_k^0(G)$ with the zero potential $q=0$, and let $\Delta_k^0(\lambda,G)$ be the characteristic functions of $L_k^0(G)$. Clearly, $\Delta_k^0(\lambda,G)$ can be calculated by (9) and (10) but with $\cos\rho x_j$ and $\frac{\sin\rho x_j}{\rho}$ instead of $C_j(x_j,\lambda)$ and $S_j(x_j,\lambda)$, $j=\overline{0,r}$, respectively.

Using (6), (7), (9), (10), (14) and (15), by the well-known method (see, for example, [20]), one can obtain the following properties of the eigenvalues of $L_k(G)$, $k=\overline{0,p}$:

(1) There exists $h > 0$ such that the eigenvalues $\lambda_{kn} = \rho_{kn}^2$ lie in the domain $|\mathrm{Im}\,\rho| < h$.

(2) The number $N_{\xi k}$ of zeros of $\Delta_k(\lambda, G)$ in the rectangle $\Pi_\xi = \{\rho : \ |\mathrm{Im}\,\rho| \le h, \ \mathrm{Re}\,\rho \in [\xi, \xi+1]\}$ is bounded with respect to ξ.

(3) For $\rho \in \Lambda^\delta, \ |\rho| \to \infty$,

$$\Delta_k(\lambda, G) = \Delta_k^0(\lambda, G)(1 + O(\rho^{-1})).$$

(4) For $n \to \infty$,

$$\rho_{kn} = \rho_{kn}^0 + O\Big(\frac{1}{\rho_{kn}^0}\Big).$$

2.3. Now we study the reconstruction of the characteristic functions from their zeros. Denote

$$\lambda_{kn}^{01} = \begin{cases} \lambda_{kn}^0 & \text{if} \quad \lambda_{kn}^0 \neq 0, \\ 1 & \text{if} \quad \lambda_{kn}^0 = 0. \end{cases} \tag{16}$$

By Hadamard's factorization theorem [21],

$$\Delta_k^0(\lambda, G) = A_k^0 \prod_{n=0}^{\infty} \frac{\lambda_{kn}^0 - \lambda}{\lambda_{kn}^{01}}, \tag{17}$$

where

$$A_k^0 = \frac{(-1)^{s_k}}{s_k!} \Big(\frac{\partial^{s_k}}{\partial \lambda^{s_k}} \Delta_k^0(\lambda, G) \Big)_{|\lambda=0}, \tag{18}$$

and $s_k \ge 0$ is the multiplicity of the zero eigenvalue of $L_k^0(G)$.

Let us show that

$$\Delta_k(\lambda, G) = A_k^0 \prod_{n=1}^{\infty} \frac{\lambda_{kn} - \lambda}{\lambda_{kn}^{01}}. \tag{19}$$

Indeed, by Hadamard's factorization theorem,

$$\Delta_k(\lambda, G) = A_k \prod_{n=1}^{\infty} \frac{\lambda_{kn} - \lambda}{\lambda_{kn}^{1}}, \tag{20}$$

where $A_k \neq 0$ is a constant, and

$$\lambda_{kn}^{1} = \begin{cases} \lambda_{kn} & \text{if} \quad \lambda_{kn} \neq 0, \\ 1 & \text{if} \quad \lambda_{kn} = 0. \end{cases}$$

It follows from (17) and (20) that

$$\frac{\Delta_k(\lambda, G)}{\Delta_k^0(\lambda, G)} = \frac{A_k}{A_k^0} \prod_{n=1}^{\infty} \frac{\lambda_{kn}^{01}}{\lambda_{kn}^{1}} \prod_{n=1}^{\infty} \Big(1 + \frac{\lambda_{kn} - \lambda_{kn}^0}{\lambda_{kn}^0 - \lambda} \Big).$$

Using properties of the characteristic functions and the eigenvalues one gets for negative λ:

$$\lim_{\lambda\to-\infty}\frac{\Delta_k(\lambda,G)}{\Delta_k^0(\lambda,G)}=1,\qquad \lim_{\lambda\to-\infty}\prod_{n=1}^{\infty}\Big(1+\frac{\lambda_{kn}-\lambda_{kn}^0}{\lambda_{kn}^0-\lambda}\Big)=1,$$

and consequently,

$$A_k=A_k^0\prod_{n=1}^{\infty}\frac{\lambda_{kn}^1}{\lambda_{kn}^{01}}\,.$$

Substituting this relation into (20) we arrive at (19).

Thus, the specification of the spectrum $\Lambda_k=\{\lambda_{kn}\}_{n\ge1}$ uniquely determines the characteristic function $\Delta_k(\lambda,G)$ by (19) where $\{\lambda_{kn}^{01}\}$ and A_k^0 are defined by (16) and (18).

3. Solution of inverse problem 1

3.1. In this section we provide a constructive procedure for the solution of Inverse Problem 1 and prove its uniqueness. First we consider the following auxiliary inverse problem for G on the edge e_k, $k=\overline{1,p}$, which is called $IP(k)$.

IP(k). *Given $M_k(\lambda)$, construct $q_k(x_k)$, $x_k\in[0,d_k]$.*

In IP(k) we construct the potential only on the edge e_k, but the Weyl function $M_k(\lambda)$ brings a global information from the whole graph, i.e. IP(k) is not a local inverse problem related only to the edge e_k. Let us prove the uniqueness theorem for the solution of IP(k).

Theorem 2. *If $M_k(\lambda)=\tilde M_k(\lambda)$, then $q_k(x_k)=\tilde q_k(x_k)$ a.e. on $[0,d_k]$. Thus, the specification of the Weyl function M_k uniquely determines the potential q_k on the edge e_k.*

Proof. Let us define the matrix $P^k(x_k,\lambda)=[P_{js}^k(x_k,\lambda)]_{j,s=1,2}$ by the formula

$$P^k(x_k,\lambda)\begin{bmatrix}\tilde\Phi_{kk}(x_k,\lambda) & \tilde S_k(x_k,\lambda)\\ \tilde\Phi_{kk}'(x_k,\lambda) & \tilde S_k'(x_k,\lambda)\end{bmatrix}=\begin{bmatrix}\Phi_{kk}(x_k,\lambda) & S_k(x_k,\lambda)\\ \Phi_{kk}'(x_k,\lambda) & S_k'(x_k,\lambda)\end{bmatrix}.$$

Then

$$\left.\begin{aligned}\Phi_{kk}(x_k,\lambda)&=P_{11}^k(x_k,\lambda)\tilde\Phi_{kk}(x_k,\lambda)+P_{12}^k(x_k,\lambda)\tilde\Phi_{kk}'(x_k,\lambda),\\ S_k(x_k,\lambda)&=P_{11}^k(x_k,\lambda)\tilde S_k(x_k,\lambda)+P_{12}^k(x_k,\lambda)\tilde S_k'(x_k,\lambda).\end{aligned}\right\}\tag{21}$$

Since $\langle\Phi_{kk}(x_k,\lambda),\,S_k(x_k,\lambda)\rangle\equiv1$, one has

$$P_{1s}^k(x_k,\lambda)=(-1)^s\Big(\Phi_{kk}(x_k,\lambda)\tilde S_k^{(2-s)}(x_k,\lambda)-\tilde\Phi_{kk}^{(2-s)}(x_k,\lambda)S_k(x_k,\lambda)\Big).\tag{22}$$

It follows from (13), (14) and (22) that

$$P_{1s}^k(x_k,\lambda)=\delta_{1s}+O(\rho^{-1}),\quad \rho\in\Lambda^\delta,\ |\rho|\to\infty,\ x_k\in(0,d_k],\ s=0,1.\tag{23}$$

Using (22) and the relation $\Phi_{kk}(x_k,\lambda)=C_k(x_k,\lambda)+M_k(\lambda)S_k(x_k,\lambda)$, we calculate

$$P^k_{1s}(x_k,\lambda)=(-1)^s\Big(\Big(C_k(x_k,\lambda)\tilde S_k^{(2-s)}(x_k,\lambda)-\tilde C_k^{(2-s)}(x_k,\lambda)S_k(x_k,\lambda)\Big)$$

$$+(M_k(\lambda)-\tilde M_k(\lambda))S_k(x_k,\lambda)\tilde S_k^{(2-s)}(x_k,\lambda)\Big).$$

Since $M_k(\lambda)=\tilde M_k(\lambda)$, it follows that for each fixed x_k, the functions $P^k_{1s}(x_k,\lambda)$ are entire in λ of order $1/2$. Together with (23) this yields $P^k_{11}(x_k,\lambda)\equiv 1$, $P^k_{12}(x_k,\lambda)\equiv 0$. Substituting these relations into (21) we get $\Phi_{kk}(x_k,\lambda)\equiv\tilde\Phi_{kk}(x_k,\lambda)$ and $S_k(x_k,\lambda)\equiv\tilde S_k(x_k,\lambda)$ for all x_k and λ, and consequently, $q_k(x_k)=\tilde q_k(x_k)$ a.e. on $[0,d_k]$.

Using the method of spectral mappings [5] for the Sturm-Liouville operator on the edge e_k one can get a constructive procedure for the solution of the inverse problem IP(k). Here we only explain ideas briefly; for details and proofs see [5]. Take $\tilde q=0$. Then $\tilde S_k(x_k,\lambda)=\frac{\sin\rho x_k}{\rho}$. Fix $k=\overline{1,p}$. Denote $\lambda'=\min\limits_{l\ge 1}(\lambda_{0l},\tilde\lambda_{0l})$ and take a fixed $\delta>0$. In the λ- plane we consider the contour γ (with counterclockwise circuit) of the form $\gamma=\gamma^+\cup\gamma^-\cup\gamma'$, where $\gamma^\pm=\{\lambda:\ \pm\mathrm{Im}\,\lambda=\delta;\,\mathrm{Re}\,\lambda\ge\lambda'\}$, $\gamma'=\{\lambda:\ \lambda-\lambda'=\delta\exp(i\alpha),\ \alpha\in(\pi/2,3\pi/2)\}$. For each fixed $x_k\in[0,d_k]$, the function $S_k(x_k,\lambda)$ is the unique solution of the following linear integral equation

$$S_k(x_k,\lambda)=\tilde S_k(x_k,\lambda)+\frac{1}{2\pi i}\int_\gamma \tilde D_k(x_k,\lambda,\mu)S_k(x_k,\mu)\,d\mu, \tag{24}$$

where

$$\tilde D_k(x,\lambda,\mu)=\int_0^x \tilde S_k(t,\lambda)\tilde S_k(t,\mu)\hat M_k(\mu)\,dt,\quad \hat M_k(\mu):=M_k(\mu)-\tilde M_k(\mu).$$

The potential q_k on the edge e_k can be constructed from the solution of the integral equation (24) via the formula

$$q_k(x_k)=\frac{1}{2\pi i}\int_\gamma (S_k(x_k,\lambda)\tilde S_k(x_k,\lambda))'\hat M_k(\lambda)\,d\lambda$$

or by the formula $q_k(x_k)=\lambda+S''_k(x_k,\lambda)/S_k(x_k,\lambda)$. It is also possible to construct the potential from the discrete spectral data. For this purpose one can calculate the contour integral in (24) by the residue theorem and transform the integral equation (24) to the equation in a space of sequences; for details see [5].

3.2. Consider the following auxiliary inverse problem on the edge e_0, which is called IP(0).

IP(0). Given $d(\lambda),h(\lambda),\Omega$, construct $q_0(x_0)$, $x_0\in[0,d_0]$.

This inverse problem were studied in [22, 23] and other papers. For convenience of the readers we describe here the solution of IP(0). We recall that

$$d(\lambda) = C_0(d_0,\lambda) + S_0'(d_0,\lambda), \quad H(\lambda) = C_0(d_0,\lambda) - S_0'(d_0,\lambda), \quad \omega_n = \operatorname{sign} H(\nu_n),$$

where $\{\nu_n\}_{n\ge1}$ are zeros of $h(\lambda)$. Clearly,

$$S_0'(d_0,\nu_n) = (d(\nu_n) - H(\nu_n))/2. \tag{25}$$

Since $\langle C_0(d_0,\lambda), S_0(d_0,\lambda)\rangle \equiv 1$, it follows that

$$H^2(\lambda) - d^2(\lambda) = -4(1 + C_0'(d_0,\lambda)h(\lambda)),$$

and consequently,

$$H(\nu_n) = \omega_n\sqrt{d^2(\nu_n) - 4}. \tag{26}$$

Denote

$$\alpha_n := \int_0^{d_0} S_0^2(t,\nu_n)\,dt.$$

Then (see [1,4])

$$\alpha_n = \dot h(\nu_n)S_0'(d_0,\nu_n), \quad \dot h(\lambda) := \frac{dh(\lambda)}{d\lambda}. \tag{27}$$

The data $\{\nu_n,\alpha_n\}_{n\ge1}$ are called the spectral data for the potential q_0. It is known (see [1,4]) that the function q_0 can be uniquely constructed from the given spectral data $\{\nu_n,\alpha_n\}_{n\ge1}$. Thus, IP(0) has been solved, and the following theorem is valid.

Theorem 3. *The specification of $d(\lambda), h(\lambda), \Omega$ uniquely determines the potential $q_0(x_0)$ on $[0,d_0]$. The function q_0 can be constructed by the following algorithm.*

Algorithm 1. Given $d(\lambda), h(\lambda), \Omega$.

(1) Find $\{\nu_n\}_{n\ge1}$ as the zeros of $h(\lambda)$.

(2) Calculate $H(\nu_n)$ by (26).

(3) Find $S_0'(d_0,\nu_n)$ by (25).

(4) Calculate $\{\alpha_n\}_{n\ge1}$ using (27).

(5) Construct q_0 from the given spectral data $\{\nu_n,\alpha_n\}_{n\ge1}$ by solving the classical inverse Sturm-Liouville problem.

3.3. Now we are ready to provide a constructive procedure for the solution of Inverse problem 1 and prove its uniqueness. Let $\Lambda_k,\ k=\overline{0,p}$ and Ω be given. The procedure for the solution of Inverse Problem 1 consists in the realization of the so-called D_μ-procedures successively for $\mu = \sigma, \sigma-1, \ldots, 1, 0$ where σ is the height of T. Let us describe D_μ- procedures.

$\mathbf{D}_\sigma$- procedure.

(1) For each $k = \overline{0,p}$, we construct $\Delta_k(\lambda, G)$ by (19) where $\{\lambda_{kn}^{01}\}$ and A_k^0 are defined by (16) and (18).

(2) For each $k = \overline{1,p}$, we calculate the Weyl function $M_k(\lambda)$ via (8).

(3) For each edge $e_k \in \mathcal{E}^{(\sigma)}$, we solve the inverse problem IP(k) and find $q_k(x_k),\ x_k \in [0, d_k]$ on the edge e_k.

(4) For each $e_k \in \mathcal{E}^{(\sigma)}$, we calculate $C_k^{(\nu)}(d_k, \lambda),\ S_k^{(\nu)}(d_k, \lambda),\ \nu = 0, 1.$

(5) For each fixed $v_k \in V^{(\sigma-1)} \setminus \Gamma$ we choose and fix s and j such that $e_j \in E \cap T_{ks}$. Solving the linear algebraic system (9)-(10), we find $\Delta_0(\lambda, G_k)$ and $\Delta_k(\lambda, G_k)$.

(6) For each fixed $v_k \in V^{(\sigma-1)} \setminus \Gamma$ we construct the Weyl function $M_k(\lambda)$ for G_k by the formula
$$M_k(\lambda) = -\frac{\Delta_k(\lambda, G_k)}{\Delta_0(\lambda, G_k)}. \tag{28}$$

Now we carry out D_μ- procedures for $\mu = \overline{2, \sigma - 1}$ by induction. Fix $\mu = \overline{2, \sigma - 1}$, and suppose that $D_\sigma, \ldots, D_{\mu+1}$- procedures have been already carried out. Let us carry out D_μ- procedure.

$\mathbf{D}_\mu$- procedure.

(1) For each edge $e_k \in \mathcal{E}^{(\mu)}$, we solve the inverse problem IP(k) on G_k and find $q_k(x_k),\ x_k \in [0, d_k]$ on the edge e_k.

(2) For each $e_k \in \mathcal{E}^{(\mu)}$, we calculate $C_k^{(\nu)}(d_k, \lambda),\ S_k^{(\nu)}(d_k, \lambda),\ \nu = 0, 1$

(3) For each fixed $v_k \in V^{(\mu-1)} \setminus \Gamma$ we choose and fix s and j such that $e_j \in E \cap T_{ks}$. Solving the linear algebraic system (9)-(10), we find $\Delta_0(\lambda, G_k)$ and $\Delta_k(\lambda, G_k)$.

(4) For each fixed $v_k \in V^{(\mu-1)} \setminus \Gamma$ we calculate $M_k(\lambda)$ for G_k via (28).

$\mathbf{D}_1$- procedure.

(1) We solve the inverse problem IP(p+1) on G_{p+1} and find $q_{p+1}(x_{p+1}),\ x_{p+1} \in [0, d_{p+1}]$ on the rooted edge e_{p+1}.

(2) We calculate $C_{p+1}^{(\nu)}(d_{p+1}, \lambda),\ S_{p+1}^{(\nu)}(d_{p+1}, \lambda),\ \nu = 0, 1.$

(3) Solving the linear algebraic system
$$\Delta_0(\lambda, G_{p+1}) = S_{p+1}(d_{p+1}, \lambda)d(\lambda) + S'_{p+1}(d_{p+1}, \lambda)h(\lambda),$$
$$\Delta_{p+1}(\lambda, G_{p+1}) = C_{p+1}(d_{p+1}, \lambda)d(\lambda) + C'_{p+1}(d_{p+1}, \lambda)h(\lambda),$$
we find $d(\lambda)$ and $h(\lambda)$.

D_0- procedure.
Using $d(\lambda),\ h(\lambda)$ and Ω, we construct $q_0(x_0),\ x_0 \in [0, d_0]$ on e_0 by Algorithm 1.

Thus, executing successively $D_\sigma, D_{\sigma-1}, \dots, D_0$- procedures we obtain the solution of Inverse problem 1 and prove its uniqueness.

4. Appendix

Consider the tree T and fix $k = \overline{p+1, r}$. Denote $Q_k := \{z \in T : \ v_k < z\}$, $T_k := T \setminus Q_k$. Then $Q_k = \bigcup_{e_i \in R(v_k)} T_{ki}$, where T_{ki} is the tree with the root v_k and with the rooted edge e_i.

Let us define functions $\Delta_0(\lambda, T)$ and $\Delta^1(\lambda, T)$ recurrently with respect to σ, where σ is the height of T. For $\sigma = 1$ the tree T is the segment $e_1 = [0, d_1]$, and we put

$$\Delta_0(\lambda, T) = S_1(d_1, \lambda), \quad \Delta^1(\lambda, T) = S_1'(d_1, \lambda).$$

For each $\sigma \geq 2$, we put

$$\Delta_0(\lambda, T) = \prod_{e_k \in R(v_{p+1})} \Delta_0(\lambda, T_{p+1,k}) \Big(\sum_{e_i \in R(v_{p+1})} \frac{\Delta^1(\lambda, T_{p+1,i})}{\Delta_0(\lambda, T_{p+1,i})} S_{p+1}(d_{p+1}, \lambda) + C_{p+1}(d_{p+1}, \lambda) \Big), \tag{29}$$

$$\Delta^1(\lambda, T) = \prod_{e_k \in R(v_{p+1})} \Delta_0(\lambda, T_{p+1,k}) \Big(\sum_{e_i \in R(v_{p+1})} \frac{\Delta^1(\lambda, T_{p+1,i})}{\Delta_0(\lambda, T_{p+1,i})} S_{p+1}'(d_{p+1}, \lambda) + C_{p+1}'(d_{p+1}, \lambda) \Big). \tag{30}$$

Denote

$$\Delta_0(\lambda, Q_{p+1}) = \prod_{e_k \in R(v_{p+1})} \Delta_0(\lambda, T_{p+1,k}) \sum_{e_i \in R(v_{p+1})} \frac{\Delta^1(\lambda, T_{p+1,i})}{\Delta_0(\lambda, T_{p+1,i})}.$$

Then relations (29)-(30) take the form

$$\Delta_0(\lambda, T) = \Delta_0(\lambda, Q_{p+1}) S_{p+1}(d_{p+1}, \lambda) + \prod_{e_k \in R(v_{p+1})} \Delta_0(\lambda, T_{p+1,k}) C_{p+1}(d_{p+1}, \lambda), \tag{31}$$

$$\Delta^1(\lambda, T) = \Delta_0(\lambda, Q_{p+1}) S_{p+1}'(d_{p+1}, \lambda) + \prod_{e_k \in R(v_{p+1})} \Delta_0(\lambda, T_{p+1,k}) C_{p+1}'(d_{p+1}, \lambda).$$

The functions $\Delta_0(\lambda, T), \Delta^1(\lambda, T)$ and $\Delta_0(\lambda, Q_{p+1})$ are entire in λ of order 1/2. We note that $\Delta^1(\lambda, T)$ is obtained from $\Delta_0(\lambda, T)$ by the replacement of $S_{p+1}(d_{p+1}, \lambda)$ and $C_{p+1}(d_{p+1}, \lambda)$ with $S_{p+1}'(d_{p+1}, \lambda)$ and $C_{p+1}'(d_{p+1}, \lambda)$ respectively.

Example 3. Let $\sigma = 2$. Then $r = p + 1$, $T_{p+1,i} = \{e_i\}$, $i = \overline{1, p}$, hence

$$\Delta_0(\lambda, T) = \prod_{k=1}^{p} S_k(d_k, \lambda) \Big(\sum_{i=1}^{p} \frac{S_i'(d_i, \lambda)}{S_i(d_i, \lambda)} S_{p+1}(d_{p+1}, \lambda) + C_{p+1}(d_{p+1}, \lambda) \Big).$$

In particular, for $p = 2$,

$$\Delta_0(\lambda, T) = (S_1(d_1,\lambda)S_2'(d_2,\lambda)+S_1'(d_1,\lambda)S_2(d_2,\lambda))S_3(d_3,\lambda)+S_1(d_1,\lambda)S_2(d_2,\lambda)C_3(d_3,\lambda).$$

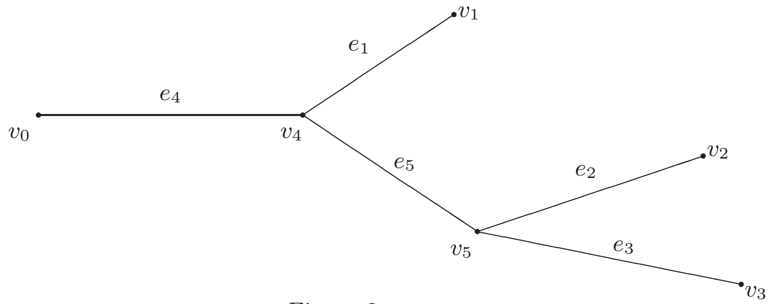


Figure 2

Example 4. Consider the tree on fig. 2. Then

$$\Delta_0(\lambda, T) = \Delta_0(\lambda, Q_4)S_4(d_4,\lambda) + S_1(d_1,\lambda)(S_2(d_2,\lambda)S_3'(d_3,\lambda)S_5(d_5,\lambda)$$
$$+S_2'(d_2,\lambda)S_3(d_3,\lambda)S_5(d_5,\lambda) + S_2(d_2,\lambda)S_3(d_3,\lambda)C_5(d_5,\lambda))C_4(d_4,\lambda),$$
$$\Delta_0(\lambda, Q_4) = S_1'(d_1,\lambda)(S_2(d_2,\lambda)S_3'(d_3,\lambda)S_5(d_5,\lambda) + S_2'(d_2,\lambda)S_3(d_3,\lambda)S_5(d_5,\lambda)$$
$$+S_2(d_2,\lambda)S_3(d_3,\lambda)C_5(d_5,\lambda)) + S_1(d_1,\lambda)(S_2(d_2,\lambda)S_3'(d_3,\lambda)S_5'(d_5,\lambda)$$
$$+S_2'(d_2,\lambda)S_3(d_3,\lambda)S_5'(d_5,\lambda) + S_2(d_2,\lambda)S_3(d_3,\lambda)C_5'(d_5,\lambda)).$$

Let $\Delta_j(\lambda, T)$, $\Delta_j(\lambda, T_{ks})$, $\Delta_j(\lambda, T_j)$ and $\Delta_j(\lambda, Q_k)$ be constructed from $\Delta_0(\lambda, T)$, $\Delta_0(\lambda, T_{ks})$, $\Delta_0(\lambda, T_j)$ and $\Delta_0(\lambda, Q_k)$ respectively by the replacement $S_j^{(\nu)}(d_j,\lambda)$ with $C_j^{(\nu)}(d_j,\lambda)$, $\nu = 0, 1$.

Fix $k = \overline{p+1, r}$. Then regrouping terms in (31) one can get

$$\Delta_0(\lambda, T) = \Delta_0(\lambda, Q_k)\Delta_0(\lambda, T_k) + \Big(\prod_{e_i \in R(v_k)} \Delta_0(\lambda, T_{ki})\Big)\Delta_k(\lambda, T_k).$$

Similarly, for $j = \overline{1, p}$, $e_j \in E \cap T_{ks}$,

$$\Delta_j(\lambda, T) = \Delta_j(\lambda, Q_k)\Delta_0(\lambda, T_k) + \Big(\Delta_j(\lambda, T_{ks}) \prod_{e_i \in R(v_k),\ i \neq s} \Delta_0(\lambda, T_{ki})\Big)\Delta_k(\lambda, T_k).$$

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