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Triharmonic curves along Riemannian submersions

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Abstract. The purpose of this paper is to study triharmonic curves along Riemannian submersions from Riemannian manifolds onto Riemannian manifolds. We obtain necessary and sufficient conditions for a triharmonic curve on the total manifold of Riemannian submersion from a space form to a Riemannian manifold to be triharmonic curve on the base manifold. The above research problem is also studied in the complex setting of the manifold on which the Riemannian submersion is defined. In this case, first the condition of a given curve to be a triharmonic curve in a complex space form is given, and then the character of the triharmonic curve during Riemann submersion is examined. In addition, we give several results involving curvature conditions for a triharmonic curves along Riemannian submersions.

Keywords. Riemannian manifold, complex space form, Riemannian submersion, biharmonic curves, triharmonic curves.

1 Introduction

Riemannian submersions between Riemannian manifolds were studied by O'Neill [10] and Gray [1]. This type of submersions has been used as an effective tool to obtain new manifolds with certain curvatures and to compare their geometry when given two manifolds. Riemannian submersions between Riemannian manifolds equipped with an additional structure of almost complex type, called Hermitian submersions, was first studied by Watson in [15]. See reference [2] for Hermitian submersions in complex geometry and their corresponding extensions in other manifolds.

Harmonic maps $F : (M, g) \to (N, g_N)$ between Riemannian manifolds are the critical points of the energy $E(F) = \frac{1}{2}\int_M |dF|^2 v_g$, and they are therefore the solutions of the corresponding Euler-Lagrange equation. This equation is giving by the vanishing of the tension field $\tau(F) = trace\nabla dF$. On the other hand, Jiang [3] studied first and second variation formulas of the bienergy functional $E_2(F)$ whose critical points are called as biharmonic maps. There have been a rich literature on biharmonic maps like as harmonic maps [11] and [9]. In 1989, S. B. Wang [14] studied the first variational formula of the tri-energy E_3. The critical points are called triharmonic maps. Notice that, every harmonic curve is a triharmonic curve. However, as proved by Maeta in [6], biharmonic curves are not necessary triharmonic curves and, vice versa, triharmonic curves do not need to be biharmonic.

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In this paper, we study curves along Riemannian submersions from Riemannian manifolds (and complex space forms) onto Riemannian manifolds. In 2, we present the basic information needed for this paper. In 3, we investigate necessary and sufficient conditions for the curves along Riemannian submersions from Riemannian manifold to be tri-harmonic. We obtain necessary and sufficient conditions for the Frenet curves along Riemannian submersions from Riemannian manifold to be tri-harmonic. In 4, we study triharmonic curves along Riemannian submersions from complex space forms. Then, we give necessary and sufficient conditions for the Frenet curves along Riemannian submersions from complex space forms to be triharmonic. Moreover, as a result of these theorems, many properties of triharmonic curves along Riemannian submersions were obtained depending on the curvatures.

2 Preliminaries

In this section, we recall some basic notions and results which will be needed throughout the paper from [2], [5], [7], [8], [10], [12], [11], [13], [16].

The geometry of Riemannian submersion is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{2.1}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.2}$$

where ∇ is the Levi-Civita connection of g_M [10]. It is easy to see that a Riemannian submersion $F : M \to N$ has totally geodesic fibres if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew -symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. That is, $\mathcal{A}_E$ and $\mathcal{T}_E$ are anti-symmetric with respect to g. It is easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor fields $\mathcal{T}$ is symmetric on the vertical distribution and $\mathcal{A}$ is anti-symmetric on the horizontal distribution. On the other hand, from (2.1) and (2.2) we have

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V \tag{2.3}$$
$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{2.4}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$ [10]. A Riemannian submersion is called a Riemannian submersion with totally umbilical fibers if

$$\mathcal{T}_U W = g(U, W)H \tag{2.5}$$

for $U, W \in \Gamma(kerF_*)$ [16]. Let $\alpha : I \to M$ be a curve parametrized by arc length in an n-dimensional Riemannian manifold (M, g). If there exists orthonormal vector fields $E_1, E_2, \cdots, E_r$ along α such that

$$\begin{aligned} E_1 &= \alpha' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\ &\vdots \\ \nabla_T E_r &= -\kappa_{r-1} E_{r-1}. \end{aligned} \tag{2.6}$$

then α is called a Frenet curve of osculating order r, where $\kappa_1, \cdots, \kappa_{r-1}$ are positive functions on I and $1 \leq r \leq n$ [2]. A Frenet curve of osculating order 1 is a geodesic; a Frenet curve of osculating order 2 is called a circle if κ_1 is a nonzero positive constant; a Frenet curve of osculating order $r \geq 3$ is called a helix of order r if $\kappa_1, \cdots\cdots \kappa_{r-1}$ are nonzero positive constants; a helix of order 3 is shortly called a helix. Following S. Maeda and Y. Ohnita [4], we define the complex torsions of the curve α by $\tau_{ij} = g(E_i, JE_j)$, $1 \leq i < j \leq r$. A helix of order r is called a holomorphic helix of order r if all the complex torsions are constant.
Let $F : (M, g) \to (N, h)$ be a Riemannian map between two Riemannian manifolds of dimensions m and n respectively. The second fundamental form of a map is defined by

$$(\nabla F_*)(X, Y) = \overset{N}{\nabla}{}^F{}_X F_* Y - F_*(\overset{M}{\nabla}_X Y) \tag{2.7}$$

for any vector fields X, Y on M, where $\overset{M}{\nabla}$ is the Levi-Civita connection of M and $\overset{N}{\nabla}{}^F$ is the pull-back of the connection $\overset{N}{\nabla}$ of N to the induced vector bundle $F^{-1}(TN)$. It is well known that ∇F_* is symmetric [13].
A map $F : (M, g) \to (N, h)$ between Riemannian manifolds is a tri-harmonic map if the tri-tension field $\tau_3(F)$ of F

$$\tau_3(F) = J(\bar{\Delta}(\tau(F))) - \sum_{i=1}^{m} R^N(\overset{N}{\nabla}\tau(F), \tau(F))F_*. \tag{2.8}$$

vanishes [8]. Let $\alpha : I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then the tritension field is given by [5]

$$\tau_3(\alpha) = \nabla_T^5 T + R^M(\nabla_T^3 T, T)T - R^M(\nabla_T^2 T, \nabla_T T)T. \tag{2.9}$$

Let (M, g) be a Riemannian manifold and $\alpha : I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then , using Frenet equations, the tritension field of α becomes [7]

$$\begin{aligned}
\tau_3(\alpha) &= (-10\kappa_1^{'}\kappa_1^{''} - 5\kappa_1\kappa_1^{'''} + 10\kappa_1^3\kappa_1^{'} + 5\kappa_1\kappa_1^{'}\kappa_2^2 + 5\kappa_1^2\kappa_2\kappa_2^{'})E_1 \\
&+(-15\kappa_1(\kappa_1)^2 - 10\kappa_1^2\kappa_1^{''} + \kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1^{''''} - 6\kappa_1^{''}\kappa_2^2 - 12\kappa_1^{'}\kappa_2\kappa_2^{'} \\
&-3\kappa_1(\kappa_2^{'})^2 - 4\kappa_1\kappa_2\kappa_2^{''} + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2)E_2 + (-9\kappa_1^2\kappa_1^{'}\kappa_2 + 4\kappa_1^{'''}\kappa_2 \\
&-4\kappa_1^{'}\kappa_2^3 - 6\kappa_1\kappa_2^2\kappa_2^{'} + 6\kappa_1^{''}\kappa_2^{'} - \kappa_1^3\kappa_2^{'} + 4\kappa_1^{'}\kappa_2^{''} + \kappa_1\kappa_2^{'''} - 4\kappa_1^{'}\kappa_2\kappa_3^2 \\
&-3\kappa_1\kappa_2^{'}\kappa_3^2 - 3\kappa_1\kappa_2\kappa_3\kappa_3^{'})F_*E_{3h} + (6\kappa_1^{''}\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 \\
&+8\kappa_1^{'}\kappa_2^{'}\kappa_3 + 3\kappa_1\kappa_2^{''}\kappa_3 - \kappa_1\kappa_2\kappa_3^3 + 4\kappa_1^{'}\kappa_2\kappa_3^{'} + 3\kappa_1\kappa_2^{'}\kappa_3^{'} + \kappa_1\kappa_2\kappa_3^{''} \\
&-\kappa_1\kappa_2\kappa_3\kappa_4^2)F_*E_{4h} + (4\kappa_1^{'}\kappa_2\kappa_3\kappa_4 + 3\kappa_1\kappa_2^{'}\kappa_3\kappa_4 + 2\kappa_1\kappa_2\kappa_3^{'}\kappa_4 \\
&+\kappa_1\kappa_2\kappa_3\kappa_4^{'})F_*E_{5h} + \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 F_*E_{6h} + R^M(\nabla_T^3 T, T)T \\
&-R^M(\nabla_T^2 T, \nabla_T T)T.
\end{aligned} \tag{2.10}$$

3 Triharmonic Curves along Riemannian Submersions from Riemannian Manifolds

In this section, we study triharmonic curves along Riemannian submersions from Riemannian manifolds. Then, we will investigate necessary and sufficient conditions for the curves along Riemannian submersions from Riemannian manifolds to be triharmonic.

Theorem 3.1. *Let $F : (M(c), g_M) \to (N, g_N)$ be a Riemannian submersion from a real space form $(M(c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(c), g_M)$ be a triharmonic horizontal curve with parallel horizontal tensor field $\mathcal{A}$. Then $F \circ \alpha : \gamma : I \to (N, g_N)$ is a triharmonic curve.*

Proof. Let $F : (M(c), g_M) \to (N, g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(c), g_M)$ be a triharmonic horizontal curve. Then, we have

$$\alpha^{'} = T = E_{1h}, \qquad \gamma^{'} = F_*T = \tilde{T}, \tag{3.1}$$

where E_{1h} is horizontal part of $T = E_1$. Note that $\gamma^{'} = \tilde{T}$ is the unit tangent vector field along the curve. Since F is Riemannian submersion, then $(\nabla F_*)(X, Y) = 0$, where $X, Y \in \Gamma(\mathcal{H})$, using (2.6) and (2.7) we get

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} = \kappa_1 F_* E_{2h}. \tag{3.2}$$

and

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = \kappa_1^{'} F_* E_{2h} + \kappa_1((\nabla F_*)(E_{1h}, E_{2h}) + F_*\overset{M}{\nabla}_{E_{1h}} E_{2h}). \tag{3.3}$$

From (2.3), (2.4) and Frenet formulas, we have

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}} E_{2h} = -\kappa_1 E_{1h} + \kappa_2 E_{3h} - \mathcal{A}_{E_{1h}} E_{2v}. \tag{3.4}$$

Using (3.4) in (3.3), we derive

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = \kappa_1^{'} F_* E_{2h} - \kappa_1^2 F_* E_{1h} + \kappa_1\kappa_2 F_* E_{3h} - \kappa_1 F_* \mathcal{A}_{E_{1h}} E_{2v}. \tag{3.5}$$

Taking the covariant dervivative of (3.5) and using the second fundamental form of the Riemannian submersion, we get

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} &= \kappa_1^{''} F_* E_{2h} + \kappa_1^{'} F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}} E_{2h} - 2\kappa_1\kappa_1^{'} F_* E_{1h} - \kappa_1^2 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}} E_{1h} \\ &+\kappa_1^{'}\kappa_2 F_* E_{3h} + \kappa_1\kappa_2^{'} F_* E_{3h} + \kappa_1\kappa_2 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}} E_{3h} - \kappa_1^{'} F_*\mathcal{A}_{E_{1h}} E_{2v} \\ &-\kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}} E_{2v}.\end{aligned} \tag{3.6}$$

Since

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}} E_{3h} = -\kappa_2 E_{2h} + \kappa_3 E_{4h} - \mathcal{A}_{E_{1h}} E_{3v} \tag{3.7}$$

due (3.2), (3.4) and Frenet formulas, using (3.7), we arrive at

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} &= -3\kappa_1\kappa_1^{'} F_* E_{1h} + (\tilde{\kappa}_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2) F_* E_{2h} + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'}) F_* E_{3h} \\ &+\kappa_1\kappa_2\kappa_3 F_* E_{4h} - 2\kappa_1^{'} F_*\mathcal{A}_{E_{1h}} E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}} E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}} E_{2v}.\end{aligned} \tag{3.8}$$

Taking the covariant dervivative of (3.8), using (2.3),(2.4) and Frenet formulas, we have the following equation.

$$\overset{N}{\nabla}{}^4_{\tilde{T}}\tilde{T} = (-3(\kappa_1^{'})^2 - 4\kappa_1\kappa_1^{''} + \kappa_1^4 + \kappa_1^2\kappa_2^2) F_* E_{1h} + (-6\kappa_1^2\kappa_1^{'} + \kappa_1^{'''}$$

$$
\begin{aligned}
&-3\kappa_1^{'}\kappa_2^2 - 3\kappa_1\kappa_2\kappa_2^{'})F_*E_{2h} + (3\kappa_1^{''}\kappa_2 - \kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + 3\kappa_1^{'}\kappa_2^{'} \\
&+\kappa_1\kappa_2^{''} - \kappa_1\kappa_2\kappa_3^2)F_*E_{3h} + (3\kappa_1^{'}\kappa_2\kappa_3 + 2\kappa_1\kappa_2^{'}\kappa_3 + \kappa_1\kappa_2\kappa_3^{'})F_*E_{4h} \\
&+\kappa_1\kappa_2\kappa_3\kappa_4F_*E_{5h} - (3\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)F_*\mathcal{A}_{E_{1h}}E_{2v} - (3\kappa_1^{'}\kappa_2 \\
&+2\kappa_1\kappa_2^{'})F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1\kappa_2\kappa_3F_*\mathcal{A}_{E_{1h}}E_{4v} - 2\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\
&-\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\
&-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.
\end{aligned} \tag{3.9}
$$

Using the formula of the second fundamental form of the map F and (3.2), we also have

$$
\begin{aligned}
&\overset{N}{\nabla}{}^5_{\tilde{T}}\tilde{T} = (-10\kappa_1^{'}\kappa_1^{''} - 5\kappa_1\kappa_1^{'''} + 10\kappa_1^3\kappa_1^{'} + 5\kappa_1\kappa_1^{'}\kappa_2^2 + 5\kappa_1^2\kappa_2\kappa_2^{'})F_*E_{1h} \\
&+(-15\kappa_1(\kappa_1^{'})^2 - 10\kappa_1^2\kappa_1^{''} + \kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1^{''''} - 6\kappa_1^{''}\kappa_2^2 - 12\kappa_1^{'}\kappa_2\kappa_2^{'} \\
&-3\kappa_1(\kappa_2^{'})^2 - 4\kappa_1\kappa_2\kappa_2^{''} + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2)F_*E_{2h} + (-9\kappa_1^2\kappa_1^{'}\kappa_2 + 4\kappa_1^{'''}\kappa_2 \\
&-4\kappa_1^{'}\kappa_2^3 - 6\kappa_1\kappa_2^2\kappa_2^{'} + 6\kappa_1^{''}\kappa_2^{'} - \kappa_1^3\kappa_2^{'} + 4\kappa_1^{'}\kappa_2^{''} + \kappa_1\kappa_2^{'''} - 4\kappa_1^{'}\kappa_2\kappa_3^2 \\
&-3\kappa_1\kappa_2^{'}\kappa_3^2 - 3\kappa_1\kappa_2\kappa_3\kappa_3^{'})F_*E_{3h} + (6\kappa_1^{''}\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 \\
&+8\kappa_1^{'}\kappa_2^{'}\kappa_3 + 3\kappa_1\kappa_2^{''}\kappa_3 - \kappa_1\kappa_2\kappa_3^3 + 4\kappa_1^{'}\kappa_2\kappa_3^{'} + 3\kappa_1\kappa_2^{'}\kappa_3^{'} + \kappa_1\kappa_2\kappa_3^{''} \\
&+\kappa_1\kappa_2\kappa_3\kappa_4^2)F_*E_{4h} + (4\kappa_1^{'}\kappa_2\kappa_3\kappa_4 + 3\kappa_1\kappa_2^{'}\kappa_3\kappa_4 + 2\kappa_1\kappa_2\kappa_3^{'}\kappa_4 \\
&+\kappa_1\kappa_2\kappa_3\kappa_4^{'})F_*E_{5h} + \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5F_*E_{6h} + (9\kappa_1^2\kappa_1^{'} - 4\kappa_1^{'''} + 4\kappa_1^{'}\kappa_2^2 \\
&+5\kappa_1\kappa_2\kappa_2^{'})F_*\mathcal{A}_{E_{1h}}E_{2v} + (-6\kappa_1^{''}\kappa_2 + \kappa_1^3\kappa_2 + \kappa_1\kappa_2^3 - 8\kappa_1^{'}\kappa_2^{'} - 3\kappa_1\kappa_2^{''} \\
&+\kappa_1\kappa_2\kappa_3^2)F_*\mathcal{A}_{E_{1h}}E_{3v} - (4\kappa_1^{'}\kappa_2\kappa_3 + 3\kappa_1\kappa_2^{'}\kappa_3 + 2\kappa_1\kappa_2\kappa_3^{'})F_*\mathcal{A}_{E_{1h}}E_{4v} \\
&-\kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{A}_{E_{1h}}E_{5v} + (-6\kappa_1^{''} + \kappa_1^3 + \kappa_1\kappa_2^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\
&-(4\kappa_1^{'}\kappa_2 + 3\kappa_1\kappa_2^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v} \\
&-4\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} \\
&-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.
\end{aligned} \tag{3.10}
$$

On the other hand, using (3.8), we derive

$$
\begin{aligned}
&R^N(\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T},\tilde{T})\tilde{T} = -3\kappa_1\kappa_1^{'}R^N(F_*E_{1h},F_*E_{1h})F_*E_{1h} + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2) \\
&R^N(F_*E_{2h},F_*E_{1h})F_*E_{1h} + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})R^N(F_*E_{3h},F_*E_{1h})F_*E_{1h} \\
&+\kappa_1\kappa_2\kappa_3R^N(F_*E_{4h},F_*E_{1h})F_*E_{1h} - 2\kappa_1^{'}R^N(F_*\mathcal{A}_{E_{1h}}E_{2v},F_*E_{1h})F_*E_{1h} \\
&-\kappa_1\kappa_2R^N(F_*\mathcal{A}_{E_{1h}}E_{3v},F_*E_{1h})F_*E_{1h} \\
&-\kappa_1R^N(F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v},F_*E_{1h})F_*E_{1h}.
\end{aligned} \tag{3.11}
$$

Using horizontal and vertical part of E_2, we can write

$$R^M(E_2,E_{1h})E_{1h} = R^M(E_{2v},E_{1h})E_{1h} + R^M(E_{2h},E_{1h})E_{1h}. \tag{3.12}$$

Hence, we get

$$F_*(R^M(E_2,E_{1h})E_{1h}) = F_*(R^M(E_{2v},E_{1h})E_{1h}) + F_*(R^M(E_{2h},E_{1h})E_{1h}). \tag{3.13}$$

Since F is a Riemannian submersion, we get

$$R^N(F_*E_{2h}, F_*E_{1h})F_*E_{1h} = F_*(R^M(E_2, E_{1h})E_{1h}) - F_*(R^M(E_{2v}, E_{1h})E_{1h}). \tag{3.14}$$

Using curtavure tensor formula, we have

$$R^N(F_*E_{2h}, F_*E_{1h})F_*E_{1h} = cF_*E_{2h}. \tag{3.15}$$

Similarly, we get

$$R^N(F_*E_{3h}, F_*E_{1h})F_*E_{1h} = cF_*E_{3h}. \tag{3.16}$$

$$R^N(F_*E_{4h}, F_*E_{1h})F_*E_{1h} = cF_*E_{4h}. \tag{3.17}$$

$$R^N(F_*\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{1h})F_*E_{1h} = cF_*\mathcal{A}_{E_{1h}}E_2. \tag{3.18}$$

$$R^N(F_*\mathcal{A}_{E_{1h}}E_{3v}, F_*E_{1h})F_*E_{1h} = cF_*\mathcal{A}_{E_{1h}}E_3. \tag{3.19}$$

$$\begin{aligned}&R^N(F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{1h})F_*E_{1h} = cF_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}\\&+c\kappa_1 g_N(F_*\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{2h})F_*E_{1h}.\end{aligned} \tag{3.20}$$

Thus putting (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) in (3.11), we have

$$\begin{aligned}&R^N(\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T}, \tilde{T})\tilde{T} = c(\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2)F_*E_{2h} + c(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h}\\&+c\kappa_1\kappa_2\kappa_3 F_*E_{4h} - 2c\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - c\kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v}\\&-c\kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} + c\kappa_1^2 g_N(F_*\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{2h})F_*E_{1h}.\end{aligned} \tag{3.21}$$

and

$$R^N(\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = c\kappa_1^3 F_*E_{2h}. \tag{3.22}$$

Thus putting (3.10), (3.21) and (3.22) in (2.9), we have

$$\begin{aligned}&\tau_3(\gamma) = (-10\kappa_1^{'}\kappa_1^{''} - 5\kappa_1\kappa_1^{'''} + 10\kappa_1^3\kappa_1^{'} + 5\kappa_1\kappa_1^{'}\kappa_2^2 + 5\kappa_1^2\kappa_2\kappa_2^{'}\\&+c\kappa_1^2 g_N(F_*\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{2h}))F_*E_{1h} + (-15\kappa_1(\kappa_1^{'})^2 - 10\kappa_1^2\kappa_1^{''} + \kappa_1^5\\&+2\kappa_1^3\kappa_2^2 + \kappa_1^{''''} - 6\kappa_1^{''}\kappa_2^2 - 12\kappa_1^{'}\kappa_2\kappa_2^{'} - 3\kappa_1(\kappa_2^{'})^2 - 4\kappa_1\kappa_2\kappa_2^{''} + \kappa_1\kappa_2^4\\&+\kappa_1\kappa_2^2\kappa_3^2 + c(\kappa_1^{''} - 2\kappa_1^3 - \kappa_1\kappa_2))F_*E_{2h} + (-9\kappa_1^2\kappa_1^{'}\kappa_2 + 4\kappa_1^{'''}\kappa_2 - 4\kappa_1^{'}\kappa_2^3\\&-6\kappa_1\kappa_2^2\kappa_2^{'} + 6\kappa_1^{''}\kappa_2^{'} - \kappa_1^3\kappa_2^{'} + 4\kappa_1^{'}\kappa_1^{''} + \kappa_1\kappa_2^{'''} - 4\kappa_1^{'}\kappa_2\kappa_3^2 - 3\kappa_1\kappa_2^{'}\kappa_3^2\\&-3\kappa_1\kappa_2\kappa_3\kappa_3^{'} + c(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'}))F_*E_{3h} + (6\kappa_1^{''}\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3\\&-\kappa_1\kappa_2^3\kappa_3 + 8\kappa_1^{'}\kappa_2^{'}\kappa_3 + 3\kappa_1\kappa_2^{''}\kappa_3 - \kappa_1\kappa_2\kappa_3^3 + 4\kappa_1^{'}\kappa_2\kappa_3^{'} + 3\kappa_1\kappa_2^{'}\kappa_3^{'}\\&+\kappa_1\kappa_2\kappa_3^{''} - \kappa_1\kappa_2\kappa_3\kappa_4^2 + c\kappa_1\kappa_2\kappa_3)F_*E_{4h} + (4\kappa_1^{'}\kappa_2\kappa_3\kappa_4 + 3\kappa_1\kappa_2^{'}\kappa_3\kappa_4\\&+2\kappa_1\kappa_2\kappa_3^{'}\kappa_4 + \kappa_1\kappa_2\kappa_3\kappa_4^{'})F_*E_{5h} + \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 F_*E_{6h} + (9\kappa_1^2\kappa_1^{'} - 4\kappa_1^{'''}\\&+4\kappa_1^{'}\kappa_2^2 + 5\kappa_1\kappa_2\kappa_2^{'} - 2c\kappa_1^{'})F_*\mathcal{A}_{E_{1h}}E_{2v} + (-6\kappa_1^{''}\kappa_2 + \kappa_1^3\kappa_2 + \kappa_1\kappa_2^3\end{aligned}$$

$$
\begin{aligned}
&-8\kappa_1'\kappa_2' - 3\kappa_1\kappa_2'' + \kappa_1\kappa_2\kappa_3^2 - c\kappa_1\kappa_2)F_*\mathcal{A}_{E_{1h}}E_{3v} - (4\kappa_1'\kappa_2\kappa_3 + 3\kappa_1\kappa_2'\kappa_3 \\
&+2\kappa_1\kappa_2\kappa_3')F_*\mathcal{A}_{E_{1h}}E_{4v} - \kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{A}_{E_{1h}}E_{5v} + (-6\kappa_1'' + \kappa_1^3 + \kappa_1\kappa_2^2 \\
&-c\kappa_1)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - (4\kappa_1'\kappa_2 + 3\kappa_1\kappa_2')F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} \\
&-\kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v} - 4\kappa_1'F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\
&-\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}. \qquad (3.23)
\end{aligned}
$$

Since $\tau_3(\alpha) = 0$, we can write $F_*\tau_3(\alpha) = 0$. Then, using this equation in $\tau_3(\gamma)$, we have

$$
\begin{aligned}
&\tau_3(\gamma) = -c\kappa_1^2 g_N(F_*\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{2h}))F_*E_{1h} + (9\kappa_1^2\kappa_1' - 4\kappa_1''' + 4\kappa_1'\kappa_2^2 \\
&+5\kappa_1\kappa_2\kappa_2' - 2c\kappa_1')F_*\mathcal{A}_{E_{1h}}E_{2v} + (-6\kappa_1''\kappa_2 + \kappa_1^3\kappa_2 + \kappa_1\kappa_2^3 - 8\kappa_1'\kappa_2' \\
&-3\kappa_1\kappa_2'' + \kappa_1\kappa_2\kappa_3^2 - c\kappa_1\kappa_2)F_*\mathcal{A}_{E_{1h}}E_{3v} - (4\kappa_1'\kappa_2\kappa_3 + 3\kappa_1\kappa_2'\kappa_3 \\
&+2\kappa_1\kappa_2\kappa_3')F_*\mathcal{A}_{E_{1h}}E_{4v} - \kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{A}_{E_{1h}}E_{5v} + (-6\kappa_1'' + \kappa_1^3 + \kappa_1\kappa_2^2 \\
&-c\kappa_1)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - (4\kappa_1'\kappa_2 + 3\kappa_1\kappa_2')F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} \\
&-\kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v} - 4\kappa_1'F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\
&-\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}. \qquad (3.24)
\end{aligned}
$$

which completes proof. □

For a horizontal Frenet curve, we have the following result. In this case, the steps given in the proof of the previous theorem are mainly followed although the computations are tedious. Therefore, we omit its proof.

Theorem 3.2. *Let $F : (M(c), g_M) \to (N, g_N)$ be a Riemannian submersion from a real space form $(M(c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(c), g_M)$ be a horizontal Frenet curve and $F \circ \alpha : \gamma : I \to (N, g_N)$ a Frenet curve. Then, $F \circ \alpha : \gamma : I \to (N, g_N)$ is a triharmonic curve if and only if*

$$
\begin{aligned}
&-10\tilde{\kappa}_1{}'\tilde{\kappa}_1{}'' - 5\tilde{\kappa}_1\tilde{\kappa}_1{}''' + 10\tilde{\kappa}_1{}^3\tilde{\kappa}_1{}' + 5\tilde{\kappa}_1\tilde{\kappa}_1{}'\tilde{\kappa}_2{}^2 + 5\tilde{\kappa}_1{}^2\tilde{\kappa}_2\tilde{\kappa}_2{}' = 0, \\
&-15\tilde{\kappa}_1(\tilde{\kappa}_1{}')^2 - 10\tilde{\kappa}_1{}^2\tilde{\kappa}_1{}'' + \tilde{\kappa}_1{}^5 + 2\tilde{\kappa}_1{}^3\tilde{\kappa}_2{}^2 + \tilde{\kappa}_1{}'''' - 6\tilde{\kappa}_1{}''\tilde{\kappa}_2{}^2 - 12\tilde{\kappa}_1{}'\tilde{\kappa}_2\tilde{\kappa}_2{}' \\
&-3\tilde{\kappa}_1(\tilde{\kappa}_2{}')^2 - 4\tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_2{}'' + \tilde{\kappa}_1\tilde{\kappa}_2{}^4 + \tilde{\kappa}_1\tilde{\kappa}_2{}^2\tilde{\kappa}_3{}^2 + c(\tilde{\kappa}_1{}'' - 2\tilde{\kappa}_1{}^3 - \tilde{\kappa}_1\tilde{\kappa}_2{}^2) = 0, \\
&-9\tilde{\kappa}_1{}^2\tilde{\kappa}_1{}'\tilde{\kappa}_2 + 4\tilde{\kappa}_1{}'''\tilde{\kappa}_2 - 4\tilde{\kappa}_1{}'\tilde{\kappa}_2{}^3 - 6\tilde{\kappa}_1\tilde{\kappa}_2{}^2\tilde{\kappa}_2{}' + 6\tilde{\kappa}_1{}''\tilde{\kappa}_2{}' - \tilde{\kappa}_1{}^3\tilde{\kappa}_2{}' + 4\tilde{\kappa}_1{}'\tilde{\kappa}_1{}'' \\
&\tilde{\kappa}_1\tilde{\kappa}_2{}''' - 4\tilde{\kappa}_1{}'\tilde{\kappa}_2\tilde{\kappa}_3{}^2 - 3\tilde{\kappa}_1\tilde{\kappa}_2{}'\tilde{\kappa}_3{}^2 - 3\tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3\tilde{\kappa}_3{}' + c(2\tilde{\kappa}_1{}'\tilde{\kappa}_2 + \tilde{\kappa}_1\tilde{\kappa}_2{}') = 0, \\
&6\tilde{\kappa}_1{}''\tilde{\kappa}_2\tilde{\kappa}_3 - \tilde{\kappa}_1{}^3\tilde{\kappa}_2\tilde{\kappa}_3 - \tilde{\kappa}_1\tilde{\kappa}_2{}^3\tilde{\kappa}_3 + 8\tilde{\kappa}_1{}'\tilde{\kappa}_2{}'\tilde{\kappa}_3 + 3\tilde{\kappa}_1\tilde{\kappa}_2{}''\tilde{\kappa}_3 - \tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3{}^3 \\
&+4\tilde{\kappa}_1{}'\tilde{\kappa}_2\tilde{\kappa}_3{}' + 3\tilde{\kappa}_1\tilde{\kappa}_2{}'\tilde{\kappa}_3{}' + \tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3{}'' - \tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3\tilde{\kappa}_4{}^2 + c\tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3 = 0, \\
&4\tilde{\kappa}_1{}'\tilde{\kappa}_2\tilde{\kappa}_3\tilde{\kappa}_4 + 3\tilde{\kappa}_1\tilde{\kappa}_2{}'\tilde{\kappa}_3\tilde{\kappa}_4 + 2\tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3{}'\tilde{\kappa}_4 + \tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3\tilde{\kappa}_4{}' = 0, \\
&\tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3\tilde{\kappa}_4\tilde{\kappa}_5 = 0. \qquad (3.25)
\end{aligned}
$$

In particular cases, we have the following result.

Theorem 3.3. *Let $F : (M(c), g_M) \to (N, g_N)$ be a Riemannian submersion from a real space form $(M(c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(c), g_M)$ be a horizontal*

Frenet curve and $F\circ\alpha:\gamma:I\to(N,g_N)$ *be a Frenet curve such that* $\tilde{\kappa_1}=constant\neq 0$*. Then,* $F\circ\alpha:\gamma:I\to(N,g_N)$ *is a triharmonic curve if and only if*

$$
\begin{aligned}
&\tilde{\kappa_2}=constant,\\
&\tilde{\kappa_1}^4+2\tilde{\kappa_1}^2\tilde{\kappa_2}^2+\tilde{\kappa_2}^4+\tilde{\kappa_2}^2\tilde{\kappa_3}^2-c(2\tilde{\kappa_1}^2+\tilde{\kappa_2}^2)=0,\\
&\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_3}'=0,\\
&\tilde{\kappa_1}^2\tilde{\kappa_2}\tilde{\kappa_3}+\tilde{\kappa_2}^3\tilde{\kappa_3}+\tilde{\kappa_2}\tilde{\kappa_3}^3-\tilde{\kappa_2}\tilde{\kappa_3}''+\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_4}^2-c\tilde{\kappa_2}\tilde{\kappa_3}=0,\\
&2\tilde{\kappa_2}\tilde{\kappa_3}'\tilde{\kappa_4}+\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_4}'=0,\\
&\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_4}\tilde{\kappa_5}=0.
\end{aligned}
\tag{3.26}
$$

Proof. The assertion follows from Theorem 3.2. □

Corollary 3.4. *Let* $F:(M^m(c),g_M)\to(N^n,g_N)$ *be a Riemannian submersion from a real space form* $(M(c),g_M)$ *to a Riemannian manifold* (N,g_N)*. Let* $\alpha:I\to(M(c),g_M)$ *be a horizontal Frenet curve and* $F\circ\alpha:\gamma:I\to(N,g_N)$ *be a Frenet curve such that* $\tilde{\kappa_1}=constant\neq 0,\tilde{\kappa_2}=0$*. Then,* $F\circ\alpha:\gamma:I\to(N,g_N)$ *is a triharmonic curve if and only if* γ *is a circle with* $\tilde{\kappa_1}=\sqrt{2c}$ *such that* $n\geq 2$*.*

In Theorem 3.1, we considered the curve as a horizontal curve. If the curve is considered as a general curve, the following result is obtained. As can be seen, it seems quite complicated to control the resulting equation in this case.

Theorem 3.5. *Let* $F:(M(c),g_M)\to(N,g_N)$ *be a Riemannian submersion from a real space form* $(M(c),g_M)$ *to a Riemannian manifold* (N,g_N) *. Let* $\alpha:I\to(M(c),g_M)$ *be a triharmonic curve. Then* $F\circ\alpha:\gamma:I\to(N,g_N)$ *is a triharmonic curve if and only if*

$$
\begin{aligned}
&(8(\kappa_1')^2+9\kappa_1\kappa_1''-\kappa_1^4-\kappa_1^2\kappa_2^2)F_*(\mathcal{T}_{E_{1v}}E_{1v}+\mathcal{A}_{E_{1h}}E_{1v})+(9\kappa_1^2\kappa_1'-4\kappa_1'''+4\kappa_1'\kappa_2^2\\
&+5\kappa_1\kappa_2\kappa_2')F_*(\mathcal{T}_{E_{1v}}E_{2v}+\mathcal{A}_{E_{1h}}E_{2v})+(-6\kappa_1''\kappa_2+\kappa_1^3\kappa_2+\kappa_1\kappa_2^3-6\kappa_1'\kappa_2'-3\kappa_1\kappa_2''\\
&-2\kappa_1'\kappa_2'+\kappa_1\kappa_2\kappa_3^2)F_*(\mathcal{T}_{E_{1v}}E_{3v}+\mathcal{A}_{E_{1h}}E_{3v})-(4\kappa_1'\kappa_2\kappa_3+3\kappa_1\kappa_2'\kappa_3\\
&+2\kappa_1\kappa_2\kappa_3')F_*(\mathcal{T}_{E_{1v}}E_{4v}+\mathcal{A}_{E_{1h}}E_{4v})-\kappa_1\kappa_2\kappa_3\kappa_4F_*(\mathcal{T}_{E_{1v}}E_{5v}+\mathcal{A}_{E_{1h}}E_{5v})\\
&+(8(\kappa_1')^2+9\kappa_1\kappa_1''-\kappa_1^4-\kappa_1^2\kappa_2^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{1h}+(9\kappa_1^2\kappa_1'-4\kappa_1'''+4\kappa_1'\kappa_2^2\\
&+5\kappa_1\kappa_2\kappa_2')F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{2h}+(-6\kappa_1''\kappa_2+\kappa_1^3\kappa_2+\kappa_1\kappa_2^3-6\kappa_1'\kappa_2'-3\kappa_1\kappa_2''\\
&-2\kappa_1'\kappa_2'+\kappa_1\kappa_2\kappa_3^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{3h}-(4\kappa_1'\kappa_2\kappa_3+3\kappa_1\kappa_2'\kappa_3+2\kappa_1\kappa_2\kappa_3')F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{4h}\\
&-\kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{5h}+7\kappa_1\kappa_1'F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{1v}+\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{1v})\\
&+(-5\kappa_1''+\kappa_1^3+\kappa_1\kappa_2^2)F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{2v}+\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v})-(4\kappa_1'\kappa_2\\
&+3\kappa_1\kappa_2')F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{3v}+\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v})-\kappa_1\kappa_2\kappa_3F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{4v}\\
&+\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v})+7\kappa_1\kappa_1'F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{1h}-(6\kappa_1''-\kappa_1^3\\
&-\kappa_1\kappa_2^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{2h}-(4\kappa_1'\kappa_2+3\kappa_1\kappa_2')F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{3h}\\
&-\kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{4h}+\kappa_1^2F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{1v}\\
&+2\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{1v})-2\kappa_1'F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{2v}\\
&-\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{3v}.
\end{aligned}
\tag{3.27}
$$

4 Triharmonic Curves along Riemannian Submersions from Complex Space Forms

In this section, first, we obtain necessary and sufficient conditions for a curve on a complex space form to be a triharmonic curve. Then we study triharmonic curves along Riemannian submersions from complex space forms and investigate necessary and sufficient conditions for the curves on complex space forms along Riemannian submersions to be triharmonic.

Let $(\bar{M}, g)$ be a Kaehler manifold. This means [16] that $\bar{M}$ admits a tensor field J of type (1,1) on $\bar{M}$ such that, $\forall X, Y \in \Gamma(T\bar{M})$, we have

$$J^2 = -I, \quad g(X,Y) = g(JX, JY), \quad (\bar{\nabla}_X J)Y = 0, \tag{4.1}$$

where g is the Riemannian metric and $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$.

Let $M^m(4c)$ be a complex space form with holomorphic sectional curvature $4c$. Then its curvature tensor field is given by

$$\begin{aligned} R^{M^m(4c)}(X,Y)Z = \quad & c\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ & +2g(X,JY)JZ\} \end{aligned} \tag{4.2}$$

for $X, Y, Z \in \chi(M)$. We first give the following general result for a curve on a complex space form to be triharmonic curve.

Theorem 4.1. *Let $\alpha : I \to (N(4c), g_N)$ be a horizontal Frenet curve on a complex space form $(N(4c), g_N)$. Then α is a triharmonic curve if and only if*

$$\begin{aligned} &(-10\kappa_1^{'}\kappa_1^{''} - 5\kappa_1\kappa_1^{'''} + 10\kappa_1^3\kappa_1^{'} + 5\kappa_1\kappa_1^{'}\kappa_2^2 + 5\kappa_1^2\kappa_2\kappa_2^{'})E_{1h} \\ &+(-15\kappa_1(\kappa_1^{'})^2 - 10\kappa_1^2\kappa_1^{''} + \kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1^{''''} - 6\kappa_1^{''}\kappa_2^2 - 12\kappa_1^{'}\kappa_2\kappa_2^{'} \\ &-3\kappa_1(\kappa_2^{'})^2 - 4\kappa_1\kappa_2\kappa_2^{''} + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2 + c(\kappa_1^{''} - 2\kappa_1^3 - \kappa_1\kappa_2))E_2 \\ &+(-9\kappa_1^2\kappa_1^{'}\kappa_2 + 4\kappa_1^{'''}\kappa_2 - 4\kappa_1^{'}\kappa_2^3 - 6\kappa_1\kappa_2^2\kappa_2^{'} + 6\kappa_1^{''}\kappa_2^{'} - \kappa_1^3\kappa_2^{'} + 4\kappa_1^{'}\kappa_1^{''} \\ &+\kappa_1\kappa_2^{'''} - 4\kappa_1^{'}\kappa_2\kappa_3^2 - 3\kappa_1\kappa_2^{'}\kappa_3^2 - 3\kappa_1\kappa_2\kappa_3\kappa_3^{'} + c(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'}))E_3 \\ &+(6\kappa_1^{''}\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 + 8\kappa_1^{'}\kappa_2^{'}\kappa_3 + 3\kappa_1\kappa_2^{''}\kappa_3 - \kappa_1\kappa_2\kappa_3^3 + 4\kappa_1^{'}\kappa_2\kappa_3^{'} \\ &+3\kappa_1\kappa_2^{'}\kappa_3^{'} + \kappa_1\kappa_2\kappa_3^{''} - \kappa_1\kappa_2\kappa_3\kappa_4^2 + c\kappa_1\kappa_2\kappa_3)E_4 + (4\kappa_1^{'}\kappa_2\kappa_3\kappa_4 + 3\kappa_1\kappa_2^{'}\kappa_3\kappa_4 \\ &+2\kappa_1\kappa_2\kappa_3^{'}\kappa_4 + \kappa_1\kappa_2\kappa_3\kappa_4^{'})E_5 + \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5E_6 - c(3(\kappa_1^{''} - 2\kappa_1^3 - \kappa_1\kappa_2^2)\tau_{12} \\ &+3(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})\tau_{13} + 3\kappa_1\kappa_2\kappa_3\tau_{14} - 2\kappa_1^2\kappa_2\tau_{23})JE_{1h} - c\kappa_1^2\kappa_2\tau_{13}JE_2 \\ &+c\kappa_1^2\kappa_2\tau_{12}JE_3 = 0. \end{aligned} \tag{4.3}$$

Proof. Let $\alpha : I \to (N(4c), g_N)$ be a horizontal Frenet curve on complex space form $(N(4c), g_N)$.We have the following equation.

$$\begin{aligned} \overset{N}{\nabla}{}^3_T T = -3\kappa_1\kappa_1^{'}E_{1h} + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)E_2 + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})E_3 \\ +\kappa_1\kappa_2\kappa_3E_4. \end{aligned} \tag{4.4}$$

We calculate $\overset{N}{\nabla}{}^4_T T$ as follows

$$\overset{N}{\nabla}{}^4_T T = -3(\kappa_1^{'})^2E_{1h} - 3\kappa_1\kappa_1^{''}E_{1h} - 3\kappa_1\kappa_1^{'}\overset{N}{\nabla}_{E_{1h}}E_{1h} + (\kappa_1^{'''} - 3\kappa_1^2\kappa_1^{'}$$

$$-\kappa_1^{'}\kappa_2^2 - 2\kappa_1\kappa_2\kappa_2^{'})E_2 + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)\overset{N}{\nabla}_{E_{1h}}E_2 + (2\kappa_1^{''}\kappa_2 + 3\kappa_1^{'}\kappa_2^{'}$$
$$+\kappa_1\kappa_2^{''})E_3 + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})\overset{N}{\nabla}_{E_{1h}}E_3 + (\kappa_1^{'}\kappa_2\kappa_3 + \kappa_1\kappa_2^{'}\kappa_3 + \kappa_1\kappa_2\kappa_3^{'})E_4$$
$$+\kappa_1\kappa_2\kappa_3\overset{N}{\nabla}_{E_{1h}}E_4. \tag{4.5}$$

Using the Frenet formula of α, we have

$$\overset{N}{\nabla}{}^4_T T = (-3(\kappa_1^{'})^2 - 4\kappa_1\kappa_1^{''} + \kappa_1^4 + \kappa_1^2\kappa_2^2)E_{1h} + (-6\kappa_1^2\kappa_1^{'} + \kappa_1^{'''} - 3\kappa_1^{'}\kappa_2^2$$
$$-3\kappa_1\kappa_2\kappa_2^{'})E_2 + (3\kappa_1^{''}\kappa_2 - \kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + 3\kappa_1^{'}\kappa_2^{'} + \kappa_1\kappa_2^{''} - \kappa_1\kappa_2\kappa_3^2)E_3$$
$$+(3\kappa_1^{'}\kappa_2\kappa_3 + 2\kappa_1\kappa_2^{'}\kappa_3 + \kappa_1\kappa_2\kappa_3)E_4 + \kappa_1\kappa_2\kappa_3\kappa_4E_5. \tag{4.6}$$

Similarly, using the Frenet formula of α, we calculate $\overset{N}{\nabla}{}^5_T T$ as follows.

$$\overset{N}{\nabla}{}^5_T T = (-10\kappa_1^{'}\kappa_1^{''} - 5\kappa_1\tilde{\kappa}_1^{'''} + 10\kappa_1^3\kappa_1^{'} + 5\kappa_1\kappa_1^{'}\kappa_2^2 + 5\kappa_1^2\kappa_2\kappa_2^{'})E_{1h}$$
$$+(-15\kappa_1(\kappa_1^{'})^2 - 10\kappa_1^2\kappa_1^{''} + \kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1^{''''} - 6\kappa_1^{''}\kappa_2^2$$
$$-12\kappa_1^{'}\kappa_2\kappa_2^{'} - 3\kappa_1(\kappa_2^{'})^2 - 4\kappa_1\kappa_2\kappa_2^{''} + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2)E_2$$
$$+(-9\kappa_1^2\kappa_1^{'}\kappa_2 + 4\kappa_1^{'''}\kappa_2 - 4\kappa_1^{'}\kappa_2^3 - 6\kappa_1\kappa_2^2\kappa_2^{'} + 6\kappa_1^{''}\kappa_2^{'} - \kappa_1^3\kappa_2^{'}$$
$$+4\kappa_1^{'}\kappa_2^{''} + \kappa_1\kappa_2^{'''} - 4\kappa_1^{'}\kappa_2\kappa_3^2 - 3\kappa_1\kappa_2^{'}\kappa_3^2 - 3\kappa_1\kappa_2\kappa_3\kappa_3^{'})E_3$$
$$+(6\kappa_1^{''}\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 + 8\kappa_1^{'}\kappa_2^{'}\kappa_3 + 3\kappa_1\kappa_2^{''}\kappa_3 - \kappa_1\kappa_2\kappa_3^3$$
$$+4\kappa_1^{'}\kappa_2\kappa_3^{'} + 3\kappa_1\kappa_2^{'}\kappa_3^{'} + \kappa_1\kappa_2\kappa_3^{''} - \kappa_1\kappa_2\kappa_3\kappa_4^2)E_4 + (4\kappa_1^{'}\kappa_2\kappa_3\kappa_4$$
$$+3\kappa_1\kappa_2^{'}\kappa_3\kappa_4 + 2\kappa_1\kappa_2\kappa_3^{'}\kappa_4 + \kappa_1\kappa_2\kappa_3\kappa_4^{'})E_5 + \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5E_6. \tag{4.7}$$

Using (4.2) and (4.4), we have

$$\overset{N}{R}(\overset{N}{\nabla}{}^3_T T, T)T = c(\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)E_2 + c(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})E_3 + c\kappa_1\kappa_2\kappa_3E_4$$
$$+[3c(\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)g_M(E_2, JE_{1h}) + 3c(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})g_M(E_3, JE_{1h})$$
$$+3c\kappa_1\kappa_2\kappa_3g_M(E_4, JE_{1h})]JE_{1h}. \tag{4.8}$$

Similarly , we have

$$\overset{N}{R}(\overset{N}{\nabla}{}^2_T T, \overset{N}{\nabla}_T T)T = c\kappa_1^3E_2 + [-3c\kappa_1^3g_M(E_{1h}, JE_2) + 2c\kappa_1^2\kappa_2g_M(E_3, JE_2)]JE_{1h}$$
$$-c\kappa_1^2\kappa_2g_M(E_{1h}, JE_3)JE_2 + c\kappa_1^2\kappa_2g_M(E_{1h}, JE_2)JE_3. \tag{4.9}$$

Thus putting (4.7), (4.8) and (4.9) in (2.9), we get

$$\tau_3(\alpha) = (-10\kappa_1^{'}\kappa_1^{''} - 5\kappa_1\kappa_1^{'''} + 10\kappa_1^3\kappa_1^{'} + 5\kappa_1\kappa_1^{'}\kappa_2^2 + 5\kappa_1^2\kappa_2\kappa_2^{'})E_{1h}$$
$$+(-15\kappa_1(\kappa_1^{'})^2 - 10\kappa_1^2\kappa_1^{''} + \kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1^{''''} - 6\kappa_1^{''}\kappa_2^2 - 12\kappa_1^{'}\kappa_2\kappa_2^{'}$$
$$-3\kappa_1(\kappa_2^{'})^2 - 4\kappa_1\kappa_2\kappa_2^{''} + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2 + c(\kappa_1^{''} - 2\kappa_1^3 - \kappa_1\kappa_2))E_2$$
$$+(-9\kappa_1^2\kappa_1^{'}\kappa_2 + 4\kappa_1^{'''}\kappa_2 - 4\kappa_1^{'}\kappa_2^3 - 6\kappa_1\kappa_2^2\kappa_2^{'} + 6\kappa_1^{''}\kappa_2^{'} - \kappa_1^3\kappa_2^{'} + 4\kappa_1^{'}\kappa_1^{''}$$
$$+\kappa_1\kappa_2^{'''} - 4\kappa_1^{'}\kappa_2\kappa_3^2 - 3\kappa_1\kappa_2^{'}\kappa_3^2 - 3\kappa_1\kappa_2\kappa_3\kappa_3^{'} + c(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'}))E_3$$
$$+(6\kappa_1^{''}\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 + 8\kappa_1^{'}\kappa_2^{'}\kappa_3 + 3\kappa_1\kappa_2^{''}\kappa_3 - \kappa_1\kappa_2\kappa_3^3$$

$$+4\kappa_1'\kappa_2\kappa_3' + 3\kappa_1\kappa_2'\kappa_3' + \kappa_1\kappa_2\kappa_3'' - \kappa_1\kappa_2\kappa_3\kappa_4^2 + c\kappa_1\kappa_2\kappa_3)E_4 + (4\kappa_1'\kappa_2\kappa_3\kappa_4$$
$$+3\kappa_1\kappa_2'\kappa_3\kappa_4 + 2\kappa_1\kappa_2\kappa_3'\kappa_4 + \kappa_1\kappa_2\kappa_3\kappa_4')E_5 + \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5E_6$$
$$+c(3(\kappa_1'' - 2\kappa_1^3 - \kappa_1\kappa_2^2)\tau_{12} + 3(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')\tau_{13} + 3\kappa_1\kappa_2\kappa_3\tau_{14}$$
$$-2\kappa_1^2\kappa_2\tau_{23})JE_{1h} + c\kappa_1^2\kappa_2\tau_{13}JE_2 - c\kappa_1^2\kappa_2\tau_{12}JE_3. \tag{4.10}$$

The assertion folllows from (4.10). □

In the following theorem, the conditions for being triharmonic on the base manifold of the curve, which is triharmonic on the total manifold of the Riemannian submersion from a complex space form to a Riemannian manifold, are obtained.

Theorem 4.2. *Let $F : (M(4c), g_M) \to (N, g_N)$ be a Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(4c), g_M)$ be a triharmonic horizontal curve. Then $F\circ\alpha : \gamma : I \to (N, g_N)$ is a triharmonic curve if and only if*

$$c\kappa_1^2 g_N(F_*\mathcal{A}_{E_{1h}}E_{2v}, E_{2h})F_*E_{1h} + (9\kappa_1^2\kappa_1' - 4\kappa_1''' + 4\kappa_1'\kappa_2^2 + 5\kappa_1\kappa_2\kappa_2'$$
$$-2c\kappa_1')F_*\mathcal{A}_{E_{1h}}E_{2v} + (-6\kappa_1''\kappa_2 + \kappa_1^3\kappa_2 + \kappa_1\kappa_2^3 - 8\kappa_1'\kappa_2' - 3\kappa_1\kappa_2'' + \kappa_1\kappa_2\kappa_3^2$$
$$-c\kappa_1\kappa_2)F_*\mathcal{A}_{E_{1h}}E_{3v} - (4\kappa_1'\kappa_2\kappa_3 + 3\kappa_1\kappa_2'\kappa_3 + 2\kappa_1\kappa_2\kappa_3')F_*\mathcal{A}_{E_{1h}}E_{4v}$$
$$-\kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{A}_{E_{1h}}E_{5v} + (-6\kappa_1'' + \kappa_1^3 + \kappa_1\kappa_2^2 + c\kappa_1)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}$$
$$-(4\kappa_1'\kappa_2 + 3\kappa_1\kappa_2')F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v}$$
$$-4\kappa_1'F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v}$$
$$-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} + c[3(\kappa_1'' - 2\kappa_1^3 - \kappa_1\kappa_2^2)\tau_{12mix}$$

$$+3(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')\tau_{13mix} + 3\kappa_1\kappa_2\kappa_3\tau_{14mix} - 2\kappa_1^2\kappa_2\tau_{23mix}$$
$$-2\kappa_1^2 g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{2h}) - 6\kappa_1' g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})$$
$$-3\kappa_1\kappa_2 g_M(\mathcal{A}_{E_{1h}}E_{3v}, JE_{1h}) - 3\kappa_1 g_M(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})]F_*JE_{1h}$$
$$-c[\kappa_1^2\kappa_2\tau_{13mix} + \kappa_1^2 g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h}]F_*JE_{2h} + c\kappa_1^2\kappa_2\tau_{12mix}F_*JE_{3h}$$
$$+c\kappa_1^2\kappa_2\tau_{13}F_*JE_{2v} - c\kappa_1^2\kappa_2\tau_{12}F_*JE_{3v} + c\kappa_1^2\tau_{12\mathcal{H}}F_*J\mathcal{A}_{E_{1h}}E_{2v} = 0. \tag{4.11}$$

where $\tau_{ijmix} = g_M(E_{ih}, JE_{jv})$, $\tau_{ij\mathcal{H}} = g_M(E_{ih}, JE_{jh})$.

Proof. Let $F : (M(4c), g_M) \to (N, g_N)$ be a Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(4c), g_M)$ be a triharmonic horizontal curve. By direct computations, we get

$$\overset{N}{\nabla}{}^4_{\tilde{T}}\tilde{T} = -3(\kappa_1')^2F_*E_{1h} - 3\kappa_1\kappa_1''F_*E_{1h} - 3\kappa_1\kappa_1'\overset{N}{\nabla}_{F_*E_{1h}}F_*E_{1h} + (\kappa_1'''$$
$$-3\kappa_1^2\kappa_1' - \kappa_1'\kappa_2^2 - 2\kappa_1\kappa_2\kappa_2')F_*E_{2h} + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)\overset{N}{\nabla}_{F_*E_{1h}}F_*E_{2h}$$
$$+(2\kappa_1''\kappa_2 + 3\kappa_1'\kappa_2' + \kappa_1\kappa_2'')F_*E_{3h} + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')\overset{N}{\nabla}_{F_*E_{1h}}F_*E_{3h}$$
$$+(\kappa_1'\kappa_2\kappa_3 + \kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3')F_*E_{4h} + \kappa_1\kappa_2\kappa_3\overset{N}{\nabla}_{F_*E_{1h}}F_*E_{4h}$$
$$-2\kappa_1''F_*\mathcal{A}_{E_{1h}}E_{2v} - 2\kappa_1'\overset{N}{\nabla}_{F_*E_{1h}}F_*\mathcal{A}_{E_{1h}}E_{2v} - (\kappa_1'\kappa_2 + \kappa_1\kappa_2')F_*\mathcal{A}_{E_{1h}}E_{3v}$$

$$-\kappa_1\kappa_2\overset{N}{\nabla}_{F_*E_{1h}}F_*\mathcal{A}_{E_{1h}}E_{3v}-\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}$$
$$-\kappa_1\overset{N}{\nabla}_{F_*E_{1h}}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}. \tag{4.12}$$

On the other hand, from (2.3),(2.4) and (2.6), we get

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{4h}=-\kappa_3E_{3h}+\kappa_4E_{5h}-\mathcal{A}_{E_{1h}}E_{4v}. \tag{4.13}$$

Using (3.2), (3.4), (3.7) and (4.13), we derive

$$\overset{N}{\nabla}{}^4_{\tilde{T}}\tilde{T}=(-3(\kappa_1^{'})^2-4\kappa_1\kappa_1^{''}+\kappa_1^4+\kappa_1^2\kappa_2^2)F_*E_{1h}+(-6\kappa_1^2\kappa_1^{'}+\kappa_1^{'''}-3\kappa_1^{'}\kappa_2^2$$
$$-3\kappa_1\kappa_2\kappa_2^{'})F_*E_{2h}+(3\kappa_1^{''}\kappa_2-\kappa_1^3\kappa_2-\kappa_1\kappa_2^3+3\kappa_1^{'}\kappa_2^{'}+\kappa_1\kappa_2^{''}-\kappa_1\kappa_2\kappa_3^2)F_*E_{3h}$$
$$+(3\kappa_1^{'}\kappa_2\kappa_3+2\kappa_1\kappa_2^{'}\kappa_3+\kappa_1\kappa_2\kappa_3^{'})F_*E_{4h}+\kappa_1\kappa_2\kappa_3\kappa_4F_*E_{5h}-(3\kappa_1^{''}-\kappa_1^3$$
$$-\kappa_1\kappa_2^2)F_*\mathcal{A}_{E_{1h}}E_{2v}-(3\kappa_1^{'}\kappa_2+2\kappa_1\kappa_2^{'})F_*\mathcal{A}_{E_{1h}}E_{3v}-\kappa_1\kappa_2\kappa_3F_*\mathcal{A}_{E_{1h}}E_{4v}$$
$$-2\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}-\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v}-\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}$$
$$-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}. \tag{4.14}$$

Using (3.7), we have

$$\overset{N}{\nabla}{}^5_{\tilde{T}}\tilde{T}=(-10\kappa_1^{'}\kappa_1^{''}-5\kappa_1\kappa_1^{'''}+10\kappa_1^3\kappa_1^{'}+5\kappa_1\kappa_1^{'}\kappa_2^2+5\kappa_1^2\kappa_2\kappa_2^{'})F_*E_{1h}$$
$$+(-15\kappa_1(\kappa_1^{'})^2-10\kappa_1^2\kappa_1^{''}+\kappa_1^5+2\kappa_1^3\kappa_2^2+\kappa_1^{''''}-6\kappa_1^{''}\kappa_2^2-12\kappa_1^{'}\kappa_2\kappa_2^{'}$$
$$-3\kappa_1(\kappa_2^{'})^2-4\kappa_1\kappa_2\kappa_2^{''}+\kappa_1\kappa_2^4+\kappa_1\kappa_2^2\kappa_3^2)F_*E_{2h}+(-9\kappa_1^2\kappa_1^{'}\kappa_2+4\kappa_1^{'''}\kappa_2$$
$$-4\kappa_1^{'}\kappa_2^3-6\kappa_1\kappa_2^2\kappa_2^{'}+6\kappa_1^{''}\kappa_2^{'}-\kappa_1^3\kappa_2^{'}+4\kappa_1^{'}\kappa_2^{''}+\kappa_1\kappa_2^{'''}-4\kappa_1^{'}\kappa_2\kappa_3^2$$
$$-3\kappa_1\kappa_2^{'}\kappa_3^2-3\kappa_1\kappa_2\kappa_3\kappa_3^{'})F_*E_{3h}+(6\kappa_1^{''}\kappa_2\kappa_3-\kappa_1^3\kappa_2\kappa_3-\kappa_1\kappa_2^3\kappa_3$$
$$+8\kappa_1^{'}\kappa_2^{'}\kappa_3+3\kappa_1\kappa_2^{''}\kappa_3-\kappa_1\kappa_2\kappa_3^3+4\kappa_1^{'}\kappa_2\kappa_3^{'}+3\kappa_1\kappa_2^{'}\kappa_3^{'}+\kappa_1\kappa_2\kappa_3^{''}$$
$$+\kappa_1\kappa_2\kappa_3\kappa_4^2)F_*E_{4h}+(4\kappa_1^{'}\kappa_2\kappa_3\kappa_4+3\kappa_1\kappa_2^{'}\kappa_3\kappa_4+2\kappa_1\kappa_2\kappa_3^{'}\kappa_4$$
$$+\kappa_1\kappa_2\kappa_3\kappa_4^{'})F_*E_{5h}+\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5F_*E_{6h}+(9\kappa_1^2\kappa_1^{'}-4\kappa_1^{'''}+4\kappa_1^{'}\kappa_2^2$$
$$+5\kappa_1\kappa_2\kappa_2^{'})F_*\mathcal{A}_{E_{1h}}E_{2v}+(-6\kappa_1^{''}\kappa_2+\kappa_1^3\kappa_2+\kappa_1\kappa_2^3-8\kappa_1^{'}\kappa_2^{'}-3\kappa_1\kappa_2^{''}$$
$$+\kappa_1\kappa_2\kappa_3^2)F_*\mathcal{A}_{E_{1h}}E_{3v}-(4\kappa_1^{'}\kappa_2\kappa_3+3\kappa_1\kappa_2^{'}\kappa_3+2\kappa_1\kappa_2\kappa_3^{'})F_*\mathcal{A}_{E_{1h}}E_{4v}$$
$$-\kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{A}_{E_{1h}}E_{5v}+(-6\kappa_1^{''}+\kappa_1^3+\kappa_1\kappa_2^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}$$
$$-(4\kappa_1^{'}\kappa_2+3\kappa_1\kappa_2^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v}-\kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v}$$
$$-4\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}-\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v}$$
$$-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}. \tag{4.15}$$

Then, using (3.8) we have

$$R^N(\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T},\tilde{T})\tilde{T}=-3\kappa_1\kappa_1^{'}R^N(F_*E_{1h},F_*E_{1h})F_*E_{1h}$$
$$+(\kappa_1^{''}-\kappa_1^3-\kappa_1\kappa_2^2)R^N(F_*E_{2h},F_*E_{1h})F_*E_{1h}$$
$$+(2\kappa_1^{'}\kappa_2+\kappa_1\kappa_2^{'})R^N(F_*E_{3h},F_*E_{1h})F_*E_{1h}+\kappa_1\kappa_2\kappa_3R^N(F_*E_{4h},F_*E_{1h})F_*E_{1h}$$

$$-2\kappa^{'} R^N(F_*\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{1h})F_*E_{1h} - \kappa_1\kappa_2 R^N(F_*\mathcal{A}_{E_{1h}}E_{3v}, F_*E_{1h})F_*E_{1h}$$
$$-\kappa_1 R^N(F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{1h})F_*E_{1h}. \tag{4.16}$$

Using horizontal and vertical part of E_2, we can write

$$R^M(E_2, E_{1h})E_{1h} = R^M(E_{2v}, E_{1h})E_{1h} + R^M(E_{2h}, E_{1h})E_{1h}. \tag{4.17}$$

Applying F_* two sides of equation (4.17), we get

$$F_*(R^M(E_2, E_{1h})E_{1h}) = F_*(R^M(E_{2v}, E_{1h})E_{1h}) + F_*(R^M(E_{2h}, E_{1h})E_{1h}). \tag{4.18}$$

Since F is a Riemannian submersion, we get

$$R^N(F_*E_{2h}, F_*E_{1h})F_*E_{1h} = F_*(R^M(E_2, E_{1h})E_{1h}) - F_*(R^M(E_{2v}, E_{1h})E_{1h}). \tag{4.19}$$

Using equation (4.2), we have

$$R^N(F_*E_{2h}, F_*E_{1h})F_*E_{1h} = cF_*E_{2h} - 3c\tau_{12\mathcal{H}}F_*JE_{1h}. \tag{4.20}$$

Similarly, we have

$$R^N(F_*E_{3h}, F_*E_{1h})F_*E_{1h} = cF_*E_{3h} - 3c\tau_{13\mathcal{H}}F_*JE_{1h}. \tag{4.21}$$
$$R^N(F_*E_{4h}, F_*E_{1h})F_*E_{1h} = cF_*E_{4h} - 3c\tau_{14\mathcal{H}}F_*JE_{1h}. \tag{4.22}$$

$$R^N(F_*\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{1h})F_*E_{1h} = cF_*\mathcal{A}_{E_{1h}}E_{2v}$$
$$+3cg_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})F_*JE_{1h}. \tag{4.23}$$
$$R^N(F_*\mathcal{A}_{E_{1h}}E_{3v}, F_*E_{1h})F_*E_{1h} = cF_*\mathcal{A}_{E_{1h}}E_{3v}$$
$$+3cg_M(\mathcal{A}_{E_{1h}}E_{3v}, JE_{1h})F_*JE_{1h}. \tag{4.24}$$
$$R^N(F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{1h})F_*E_{1h} = cF_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}$$
$$-c\kappa_1 g_M(\mathcal{A}_{E_{1h}}E_{2v}, E_{2h})F_*E_{1h} + 3cg_M(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})F_*JE_{1h}. \tag{4.25}$$

Thus putting (4.20), (4.21), (4.22),(4.23), (4.24), (4.25) in (4.16), we have

$$R^N(\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T}, \tilde{T})\tilde{T} = c\kappa_1^2 g_N(F_*\mathcal{A}_{E_{1h}}E_{2v}, F_*E_{2h})F_*E_{1h} + c(\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)F_*E_{2h}$$
$$+c(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h} + c\kappa_1\kappa_2\kappa_3 F_*E_{4h} - 2c\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - c\kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v}$$
$$+c\kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - c[3(\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)\tau_{12\mathcal{H}} + 3(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})\tau_{13\mathcal{H}}$$
$$+3\kappa_1\kappa_2\kappa_3\tau_{14\mathcal{H}} + 6\kappa_1^{'}g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h}) + 3\kappa_1\kappa_2 g_M(\mathcal{A}_{E_{1h}}E_{3v}, JE_{1h})$$
$$+3\kappa_1 g_M(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})]F_*JE_{1h}. \tag{4.26}$$

Similarly, we have

$$R^N(\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = c\kappa_1^3 F_*E_{2h} - c[3\kappa_1^3\tau_{12\mathcal{H}} + 2\kappa_1^2\kappa_2\tau_{23\mathcal{H}}$$
$$-2\kappa_1^2 g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{2h})]F_*JE_{1h} - c[\kappa_1^2\kappa_2\tau_{13\mathcal{H}}$$
$$-\kappa_1^2 g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})]F_*JE_{2h} + c\kappa_1^2\kappa_2\tau_{12\mathcal{H}}F_*JE_{3h}$$

$$-c\kappa_1^2\tau_{12\mathcal{H}}F_*J\mathcal{A}_{E_{1h}}E_{2v}. \tag{4.27}$$

Thus, putting (4.15), (4.26) and (4.27) in (2.9), we get

$$\begin{aligned}
\tau_3(\gamma) &= (-10\kappa_1^{'}\kappa_1^{''} - 5\kappa_1\kappa_1^{'''} + 10\kappa_1^3\kappa_1^{'} + 5\kappa_1\kappa_1^{'}\kappa_2^2 + 5\kappa_1^2\kappa_2\kappa_2^{'} \\
&+c\kappa_1^2 g_N(F_*\mathcal{A}_{E_{1h}}E_{2v}, E_{2h}))F_*E_{1h} + (-15\kappa_1(\kappa_1^{'})^2 - 10\kappa_1^2\kappa_1^{''} + \kappa_1^5 + 2\kappa_1^3\kappa_2^2 \\
&+\kappa_1^{''''} - 6\kappa_1^{''}\kappa_2^2 - 12\kappa_1^{'}\kappa_2\kappa_2^{'} - 3\kappa_1(\kappa_2^{'})^2 - 4\kappa_1\kappa_2\kappa_2^{''} + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2 \\
&+c(\kappa_1^{''} - 2\kappa_1^3 - \kappa_1\kappa_2^2))F_*E_{2h} + (-9\kappa_1^2\kappa_1^{'}\kappa_2 + 4\kappa_1^{'''}\kappa_2 - 4\kappa_1^{'}\kappa_2^3 - 6\kappa_1\kappa_2^2\kappa_2^{'} \\
&+6\kappa_1^{''}\kappa_2^{'} - \kappa_1^3\kappa_2^{'} + 4\kappa_1^{'}\kappa_2^{''} + \kappa_1\kappa_2^{'''} - 4\kappa_1^{'}\kappa_2\kappa_3^2 - 3\kappa_1\kappa_2^{'}\kappa_3^2 - 3\kappa_1\kappa_2\kappa_3\kappa_3^{'} \\
&+c(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'}))F_*E_{3h} + (6\kappa_1^{''}\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 + 8\kappa_1^{'}\kappa_2^{'}\kappa_3 \\
&+3\kappa_1\kappa_2^{''}\kappa_3 - \kappa_1\kappa_2\kappa_3^3 + 4\kappa_1^{'}\kappa_2\kappa_3^{'} + 3\kappa_1\kappa_2^{'}\kappa_3^{'} + \kappa_1\kappa_2\kappa_3^{''} - \kappa_1\kappa_2\kappa_3\kappa_4^2 + c\kappa_1\kappa_2\kappa_3)F_*E_{4h} \\
&+(4\kappa_1^{'}\kappa_2\kappa_3\kappa_4 + 3\kappa_1\kappa_2^{'}\kappa_3\kappa_4 + 2\kappa_1\kappa_2\kappa_3^{'}\kappa_4 + \kappa_1\kappa_2\kappa_3\kappa_4^{'})F_*E_{5h} + \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5F_*E_{6h} \\
&+(9\kappa_1^2\kappa_1^{'} - 4\kappa_1^{'''} + 4\kappa_1^{'}\kappa_2^2 + 5\kappa_1\kappa_2\kappa_2^{'} - 2c\kappa_1^{'})F_*\mathcal{A}_{E_{1h}}E_{2v} + (-6\kappa_1^{''}\kappa_2 + \kappa_1^3\kappa_2 \\
&+\kappa_1\kappa_2^3 - 8\kappa_1^{'}\kappa_2^{'} - 3\kappa_1\kappa_2^{''} + \kappa_1\kappa_2\kappa_3^2 - c\kappa_1\kappa_2)F_*\mathcal{A}_{E_{1h}}E_{3v} - (4\kappa_1^{'}\kappa_2\kappa_3 + 3\kappa_1\kappa_2^{'}\kappa_3 \\
&+2\kappa_1\kappa_2\kappa_3^{'})F_*\mathcal{A}_{E_{1h}}E_{4v} - \kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{A}_{E_{1h}}E_{5v} + (-6\kappa_1^{''} + \kappa_1^3 + \kappa_1\kappa_2^2 \\
&+c\kappa_1)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - (4\kappa_1^{'}\kappa_2 + 3\kappa_1\kappa_2^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} \\
&-\kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v} - 4\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\
&-\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\
&-c[3(\kappa_1^{''} - 2\kappa_1^3 - \kappa_1\kappa_2^2)\tau_{12\mathcal{H}} + 3(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})\tau_{13\mathcal{H}} + 3\kappa_1\kappa_2\kappa_3\tau_{14\mathcal{H}} \\
&-2\kappa_1^2\kappa_2\tau_{23\mathcal{H}} + 2\kappa_1^2 g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{2h}) + 6\kappa_1^{'}g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h}) \\
&+3\kappa_1\kappa_2 g_M(\mathcal{A}_{E_{1h}}E_{3v}, JE_{1h}) + 3\kappa_1 g_M(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})]F_*JE_{1h} \\
&+c[\kappa_1^2\kappa_2\tau_{13\mathcal{H}} - \kappa_1^2 g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})]F_*JE_{2h} + c\kappa_1^2\kappa_2\tau_{12}F_*JE_{3h} \\
&-c\kappa_1^2\tau_{12\mathcal{H}}F_*J\mathcal{A}_{E_{1h}}E_{2v}.
\end{aligned} \tag{4.28}$$

Since $\tau_3(\alpha) = 0$, we can write $F_*\tau_3(\alpha) = 0$. Then, using this equation in $\tau_3(\gamma)$, we have

$$\begin{aligned}
\tau_3(\gamma) &= c\kappa_1^2 g_N(F_*\mathcal{A}_{E_{1h}}E_{2v}, E_{2h})F_*E_{1h} + (9\kappa_1^2\kappa_1^{'} - 4\kappa_1^{'''} + 4\kappa_1^{'}\kappa_2^2 + 5\kappa_1\kappa_2\kappa_2^{'} \\
&-2c\kappa_1^{'})F_*\mathcal{A}_{E_{1h}}E_{2v} + (-6\kappa_1^{''}\kappa_2 + \kappa_1^3\kappa_2 + \kappa_1\kappa_2^3 - 8\kappa_1^{'}\kappa_2^{'} - 3\kappa_1\kappa_2^{''} + \kappa_1\kappa_2\kappa_3^2 \\
&-c\kappa_1\kappa_2)F_*\mathcal{A}_{E_{1h}}E_{3v} - (4\kappa_1^{'}\kappa_2\kappa_3 + 3\kappa_1\kappa_2^{'}\kappa_3 + 2\kappa_1\kappa_2\kappa_3^{'})F_*\mathcal{A}_{E_{1h}}E_{4v} \\
&-\kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{A}_{E_{1h}}E_{5v} + (-6\kappa_1^{''} + \kappa_1^3 + \kappa_1\kappa_2^2 + c\kappa_1)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\
&-(4\kappa_1^{'}\kappa_2 + 3\kappa_1\kappa_2^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v} \\
&-4\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v} \\
&-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} + c[3(\kappa_1^{''} - 2\kappa_1^3 - \kappa_1\kappa_2^2)\tau_{12mix} \\
&+3(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})\tau_{13mix} + 3\kappa_1\kappa_2\kappa_3\tau_{14mix} - 2\kappa_1^2\kappa_2\tau_{23mix} \\
&-2\kappa_1^2 g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{2h}) - 6\kappa_1^{'}g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h}) \\
&-3\kappa_1\kappa_2 g_M(\mathcal{A}_{E_{1h}}E_{3v}, JE_{1h}) - 3\kappa_1 g_M(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h})]F_*JE_{1h} \\
&-c[\kappa_1^2\kappa_2\tau_{13mix} + \kappa_1^2 g_M(\mathcal{A}_{E_{1h}}E_{2v}, JE_{1h}]F_*JE_{2h} + c\kappa_1^2\kappa_2\tau_{12mix}F_*JE_{3h}
\end{aligned}$$

$$+c\kappa_1^2\kappa_2\tau_{13}F_*JE_{2v} - c\kappa_1^2\kappa_2\tau_{12}F_*JE_{3v} + c\kappa_1^2\tau_{12\mathcal{H}}F_*J\mathcal{A}_{E_{1h}}E_{2v}. \tag{4.29}$$

Thus $F \circ \alpha : \gamma : I \to (N, g_N)$ is a triharmonic curve if and only if (4.11) is satisfied. □

Theorem 4.3. *Let $F : (M(4c), g_M) \to (N, g_N)$ be a Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(4c), g_M)$ be a triharmonic horizontal curve with parallel horizontal tensor field $\mathcal{A}$. Then $F \circ \alpha : \gamma : I \to (N, g_N)$ is a triharmonic curve if and only if*

$$\begin{aligned} &c[3(\kappa_1^{''} - 2\kappa_1^3 - \kappa_1\kappa_2^2)\tau_{12mix} + 3(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})\tau_{13mix} + 3\kappa_1\kappa_2\kappa_3\tau_{14mix} \\ &-2\kappa_1^2\kappa_2\tau_{23mix}]F_*JE_{1h} - c\kappa_1^2\kappa_2\tau_{13mix}F_*JE_{2h} + c\kappa_1^2\kappa_2\tau_{12mix}F_*JE_{3h} \\ &+c\kappa_1^2\kappa_2\tau_{13}F_*JE_{2v} - c\kappa_1^2\kappa_2\tau_{12}F_*JE_{3v} = 0. \end{aligned} \tag{4.30}$$

Proof. Since horizontal vector field $\mathcal{A}$ is parallel, we have $\mathcal{A} = 0$. The assertion follows from Theorem 4.2. □

Theorem 4.4. *Let $F : (M(4c), g_M) \to (N, g_N)$ be a Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(4c), g_M)$ be a triharmonic horizontal curve, horizontal tensor field A be a parallel, $\kappa_1, \kappa_2 = constant \neq 0$, $c \neq 0$. Then $F \circ \alpha : \gamma : I \to (N, g_N)$ is a triharmonic curve if and only if*

$$\begin{aligned} &c[3(-2\kappa_1^2 - \kappa_2^2)\tau_{12mix} + 3\kappa_2\kappa_3\tau_{14mix} - 2\kappa_1\kappa_2\tau_{23mix}]F_*JE_{1h} \\ &-c\kappa_1\kappa_2\tau_{13mix}F_*JE_{2h} + c\kappa_1\kappa_2\tau_{12mix}F_*JE_{3h} + c\kappa_1\kappa_2\tau_{13}F_*JE_{2v} \\ &-c\kappa_1\kappa_2\tau_{12}F_*JE_{3v} = 0. \end{aligned} \tag{4.31}$$

Proof. Since horizontal vector field $\mathcal{A}$ is parallel, we have $\mathcal{A} = 0$. Then, we have $\kappa_1, \kappa_2 = constant \neq 0$ and $c \neq 0$. The assertion follows from Theorem 4.3. □

Theorem 4.5. *Let $F : (M(4c), g_M) \to (N, g_N)$ be a Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(4c), g_M)$ be a triharmonic horizontal curve, horizontal tensor field A be a parallel, $\kappa_1 = constant \neq 0$, $\kappa_2 = 0$, $c \neq 0$. Then $F \circ \alpha : \gamma : I \to (N, g_N)$ is a triharmonic circle if and only if $\tau_{12mix} = 0$ or F is an anti-invariant.*

Proof. Since horizontal vector field $\mathcal{A}$ is parallel, we have $\mathcal{A} = 0$. Then, we have $\kappa_1 = constant \neq 0$, $\kappa_2 = 0$ and $c \neq 0$. The assertion follows from Theorem 4.4. □

Theorem 4.6. *Let $F : (M(4c), g_M) \to (N, g_N)$ be an invariant Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N) . Let $\alpha : I \to (M(4c), g_M)$ be a triharmonic horizontal curve, horizontal tensor field A be a parallel. Then $F \circ \alpha : \gamma : I \to (N, g_N)$ is a triharmonic curve.*

Proof. Since horizontal vector field $\mathcal{A}$ is parallel, we have $\mathcal{A} = 0$. The assertion follows from Theorem 4.5. □

Theorem 4.7. *Let $F : (M(4c), g_M) \to (N, g_N)$ be a Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(4c), g_M)$ be a horizontal Frenet curve and $F \circ \alpha : \gamma : I \to (N, g_N)$ be a Frenet curve. Then, $F \circ \alpha : \gamma : I \to (N, g_N)$ is a triharmonic curve if and only if*

$$(-10\tilde{\kappa_1}^{'}\tilde{\kappa_1}^{''} - 5\tilde{\kappa_1}\tilde{\kappa_1}^{'''} + 10\tilde{\kappa_1}^3\tilde{\kappa_1}^{'} + 5\tilde{\kappa_1}\tilde{\kappa_1}^{'}\tilde{\kappa_2}^2 + 5\tilde{\kappa_1}^2\tilde{\kappa_2}\tilde{\kappa_2}^{'})F_*E_{1h}$$

$$
\begin{aligned}
&+(-15\tilde{\kappa_1}(\tilde{\kappa_1}^{'})^2-10\tilde{\kappa_1}^2\tilde{\kappa_1}^{''}+\tilde{\kappa_1}^5+2\tilde{\kappa_1}^3\tilde{\kappa_2}^2+\tilde{\kappa_1}^{''''}-6\tilde{\kappa_1}^{''}\tilde{\kappa_2}^2\\
&-12\tilde{\kappa_1}^{'}\tilde{\kappa_2}\tilde{\kappa_2}^{'}-3\tilde{\kappa_1}(\tilde{\kappa_2}^{'})^2-4\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_2}^{''}+\tilde{\kappa_1}\tilde{\kappa_2}^4+\tilde{\kappa_1}\tilde{\kappa_2}^2\tilde{\kappa_3}^2+c(\tilde{\kappa_1}^{''}\\
&-2\tilde{\kappa_1}^3-\tilde{\kappa_1}\tilde{\kappa_2})^2)F_*E_{2h}+(-9\tilde{\kappa_1}^2\tilde{\kappa_1}^{'}\tilde{\kappa_2}+4\tilde{\kappa_1}^{'''}\tilde{\kappa_2}-4\tilde{\kappa_1}^{'}\tilde{\kappa_2}^3\\
&-6\tilde{\kappa_1}\tilde{\kappa_2}^2\tilde{\kappa_2}^{'}+6\tilde{\kappa_1}^{''}\tilde{\kappa_2}^{'}-\tilde{\kappa_1}^3\tilde{\kappa_2}^{'}+4\tilde{\kappa_1}^{'}\tilde{\kappa_1}^{''}+\tilde{\kappa_1}\tilde{\kappa_2}^{'''}-4\tilde{\kappa_1}^{'}\tilde{\kappa_2}\tilde{\kappa_3}^2\\
&-3\tilde{\kappa_1}\tilde{\kappa_2}^{'}\tilde{\kappa_3}^2-3\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_3}^{'}+c(2\tilde{\kappa_1}^{'}\tilde{\kappa_2}+\tilde{\kappa_1}\tilde{\kappa_2}^{'}))F_*E_{3h}+(6\tilde{\kappa_1}^{''}\tilde{\kappa_2}\tilde{\kappa_3}\\
&-\tilde{\kappa_1}^3\tilde{\kappa_2}\tilde{\kappa_3}-\tilde{\kappa_1}\tilde{\kappa_2}^3\tilde{\kappa_3}+8\tilde{\kappa_1}^{'}\tilde{\kappa_2}^{'}\tilde{\kappa_3}+3\tilde{\kappa_1}\tilde{\kappa_2}^{''}\tilde{\kappa_3}-\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}^3+4\tilde{\kappa_1}^{'}\tilde{\kappa_2}\tilde{\kappa_3}^{'}\\
&+3\tilde{\kappa_1}\tilde{\kappa_2}^{'}\tilde{\kappa_3}^{'}+\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}^{''}-\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_4}^2+c\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3})F_*E_{4h}+(4\tilde{\kappa_1}^{'}\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_4}\\
&+3\tilde{\kappa_1}\tilde{\kappa_2}^{'}\tilde{\kappa_3}\tilde{\kappa_4}+2\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}^{'}\tilde{\kappa_4}+\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_4}^{'})F_*E_{5h}+\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}\tilde{\kappa_4}\tilde{\kappa_5}F_*E_{6h}\\
&+[3c(\tilde{\kappa_1}^{''}-2\tilde{\kappa_1}^3-\tilde{\kappa_1}\tilde{\kappa_2}^2)g_M(E_{2h},JE_{1h})+3c(2\tilde{\kappa_1}^{'}\tilde{\kappa_2}+\tilde{\kappa_1}\tilde{\kappa_2}^{'})g_M(E_{3h},JE_{1h})\\
&+3c\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}g_M(E_{4h},JE_{1h})-2c\tilde{\kappa_1}^2\tilde{\kappa_2}g_M(E_{3h},JE_{2h})]F_*JE_{1h}\\
&+c\tilde{\kappa_1}^2\tilde{\kappa_2}g_M(JE_{3h},E_{1h})F_*JE_{2h}-c\tilde{\kappa_1}\tilde{\kappa_2}g_M(JE_{2h},E_{1h})F_*JE_{3h}. \qquad (4.32)
\end{aligned}
$$

Proof. Let $F:(M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha:I\to(M(4c),g_M)$ be a horizontal Frenet curve and $F\circ\alpha:\gamma:I\to(N,g_N)$ be a Frenet curve. We have the equation (3.10). Then, we calculate the Riemannian curvature tensors, as follows.

$$
\begin{aligned}
&\overset{N}{R}(\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T},\tilde{T})\tilde{T}=-3\tilde{\kappa_1}\tilde{\kappa_1}^{'}\overset{N}{R}(F_*E_{1h},F_*E_{1h})F_*E_{1h}\\
&+(\tilde{\kappa}_1^{''}-\tilde{\kappa_1}^3-\tilde{\kappa_1}\tilde{\kappa_2}^2)\overset{N}{R}(F_*E_{2h},F_*E_{1h})F_*E_{1h}\\
&+(2\tilde{\kappa_1}^{'}\tilde{\kappa_2}+\tilde{\kappa_1}\tilde{\kappa_2}^{'})\overset{N}{R}(F_*E_{3h},F_*E_{1h})F_*E_{1h}\\
&+\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}\overset{N}{R}(F_*E_{4h},F_*E_{1h})F_*E_{1h}. \qquad (4.33)
\end{aligned}
$$

Since F is a Riemannian submersion, then we have

$$
\begin{aligned}
&\overset{N}{R}(\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T},\tilde{T})\tilde{T}=-3\tilde{\kappa_1}\tilde{\kappa_1}^{'}F_*(\overset{M}{R}(E_{1h},E_{1h})E_{1h})\\
&+(\tilde{\kappa}_1^{''}-\tilde{\kappa_1}^3-\tilde{\kappa_1}\tilde{\kappa_2}^2)F_*(\overset{M}{R}(E_{2h},E_{1h})E_{1h})\\
&+(2\tilde{\kappa_1}^{'}\tilde{\kappa_2}+\tilde{\kappa_1}\tilde{\kappa_2}^{'})F_*(\overset{M}{R}(E_{3h},E_{1h})E_{1h})+\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*(\overset{M}{R}(E_{4h},E_{1h})E_{1h}). \qquad (4.34)
\end{aligned}
$$

Using (4.2) in (4.34) we get

$$
\begin{aligned}
&\overset{N}{R}(\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T},\tilde{T})\tilde{T}=c(\tilde{\kappa}_1^{''}-\tilde{\kappa_1}^3-\tilde{\kappa_1}\tilde{\kappa_2}^2)F_*E_{2h}+c(2\tilde{\kappa_1}^{'}\tilde{\kappa_2}+\tilde{\kappa_1}\tilde{\kappa_2}^{'})F_*E_{3h}\\
&+c\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*E_{4h}+[3c(\tilde{\kappa}_1^{''}-\tilde{\kappa_1}^3-\tilde{\kappa_1}\tilde{\kappa_2}^2)g_M(E_{2h},JE_{1h})\\
&+3c(2\tilde{\kappa_1}^{'}\tilde{\kappa_2}+\tilde{\kappa_1}\tilde{\kappa_2}^{'})g_M(E_{3h},JE_{1h})+3c\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}g_M(E_{4h},JE_{1h})]F_*JE_{1h}. \qquad (4.35)
\end{aligned}
$$

Similarly , we have

$$
\begin{aligned}
&\overset{N}{R}(\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T}=c\tilde{\kappa}_1^3F_*E_{2h}+[-3c\tilde{\kappa}_1^3g_M(E_{1h},JE_{2h})\\
&+2c\tilde{\kappa}_1^2\tilde{\kappa}_2g_M(E_{3h},JE_{2h})]F_*JE_{1h}-c\tilde{\kappa}_1^2\tilde{\kappa}_2g_M(E_{1h},JE_{3h})F_*JE_{2h}\\
&+c\tilde{\kappa}_1^2\tilde{\kappa}_2g_M(E_{1h},JE_{2h})F_*JE_{3h}. \qquad (4.36)
\end{aligned}
$$

Thus putting (3.10), (4.35) and (4.36) in (2.9), we get the assertion. □

Theorem 4.8. *Let $F:(M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) . Let $\alpha:I\to(M(4c),g_M)$ be a triharmonic curve. Then $F\circ\alpha:\gamma:I\to(N,g_N)$ is a triharmonic curve if and only if*

$$\begin{aligned}
&(8(\kappa_1^{'})^2+9\kappa_1\kappa_1^{''}-\kappa_1^4-\kappa_1^2\kappa_2^2)F_*(\mathcal{T}_{E_{1v}}E_{1v}+\mathcal{A}_{E_{1h}}E_{1v})+(9\kappa_1^2\kappa_1^{'}-4\kappa_1^{'''}+4\kappa_1^{'}\kappa_2^2\\
&+5\kappa_1\kappa_2\kappa_2^{'})F_*(\mathcal{T}_{E_{1v}}E_{2v}+\mathcal{A}_{E_{1h}}E_{2v})+(-6\kappa_1^{''}\kappa_2+\kappa_1^3\kappa_2+\kappa_1\kappa_2^3-6\kappa_1^{'}\kappa_2^{'}-3\kappa_1\kappa_2^{''}\\
&-2\kappa_1^{'}\kappa_2^{'}+\kappa_1\kappa_2\kappa_3^2)F_*(\mathcal{T}_{E_{1v}}E_{3v}+\mathcal{A}_{E_{1h}}E_{3v})-(4\kappa_1^{'}\kappa_2\kappa_3+3\kappa_1\kappa_2^{'}\kappa_3\\
&+2\kappa_1\kappa_2\kappa_3^{'})F_*(\mathcal{T}_{E_{1v}}E_{4v}+\mathcal{A}_{E_{1h}}E_{4v})-\kappa_1\kappa_2\kappa_3\kappa_4F_*(\mathcal{T}_{E_{1v}}E_{5v}+\mathcal{A}_{E_{1h}}E_{5v})\\
&+(8(\kappa_1^{'})^2+9\kappa_1\kappa_1^{''}-\kappa_1^4-\kappa_1^2\kappa_2^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{1h}+(9\kappa_1^2\kappa_1^{'}-4\kappa_1^{'''}+4\kappa_1^{'}\kappa_2^2\\
&+5\kappa_1\kappa_2\kappa_2^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{2h}+(-6\kappa_1^{''}\kappa_2+\kappa_1^3\kappa_2+\kappa_1\kappa_2^3-6\kappa_1^{'}\kappa_2^{'}-3\kappa_1\kappa_2^{''}\\
&-2\kappa_1^{'}\kappa_2^{'}+\kappa_1\kappa_2\kappa_3^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{3h}-(4\kappa_1^{'}\kappa_2\kappa_3+3\kappa_1\kappa_2^{'}\kappa_3+2\kappa_1\kappa_2\kappa_3^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{4h}\\
&-\kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{5h}+7\kappa_1\kappa_1^{'}F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{1v}+\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{1v})\\
&+(-5\kappa_1^{''}+\kappa_1^3+\kappa_1\kappa_2^2)F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{2v}+\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v})\\
&-(4\kappa_1^{'}\kappa_2+3\kappa_1\kappa_2^{'})F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{3v}+\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{3v})\\
&-\kappa_1\kappa_2\kappa_3F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{4v}+\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{4v})+7\kappa_1\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{1h}\\
&-(6\kappa_1^{''}-\kappa_1^3-\kappa_1\kappa_2^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{2h}-(4\kappa_1^{'}\kappa_2+3\kappa_1\kappa_2^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{3h}\\
&-\kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{4h}+\kappa_1^2F_*(\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{1v}\\
&+2\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{1v})-2\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{2v}\\
&-\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{T}_{E_{1v}}E_{3v}.
\end{aligned}\tag{4.37}$$

Theorem 4.9. *Let $F:(M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) . Let $\alpha:I\to(M(4c),g_M)$ be a triharmonic curve with totally geodesic fibers and horizontal tensor field $\mathcal{A}$ be a parallel. Then $F\circ\alpha:\gamma:I\to(N,g_N)$ is a triharmonic curve if and only if*

$$\begin{aligned}
&(8(\kappa_1^{'})^2+9\kappa_1\kappa_1^{''}-\kappa_1^4-\kappa_1^2\kappa_2^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{1h}+(9\kappa_1^2\kappa_1^{'}-4\kappa_1^{'''}+4\kappa_1^{'}\kappa_2^2\\
&+5\kappa_1\kappa_2\kappa_2^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{2h}+(-6\kappa_1^{''}\kappa_2+\kappa_1^3\kappa_2+\kappa_1\kappa_2^3-6\kappa_1^{'}\kappa_2^{'}-3\kappa_1\kappa_2^{''}\\
&-2\kappa_1^{'}\kappa_2^{'}+\kappa_1\kappa_2\kappa_3^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{3h}-(4\kappa_1^{'}\kappa_2\kappa_3+3\kappa_1\kappa_2^{'}\kappa_3+2\kappa_1\kappa_2\kappa_3^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{4h}\\
&-\kappa_1\kappa_2\kappa_3\kappa_4F_*\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{5h}+7\kappa_1\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{1h}-(6\kappa_1^{''}-\kappa_1^3\\
&-\kappa_1\kappa_2^2)F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{2h}-(4\kappa_1^{'}\kappa_2+3\kappa_1\kappa_2^{'})F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{3h}\\
&-\kappa_1\kappa_2\kappa_3F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{H}\overset{M}{\nabla}_{E_{1v}}E_{4h}=0.
\end{aligned}\tag{4.38}$$

Proof. Since fibers are totally geodesic, we have $\mathcal{T}=0$. Since horizontal vector field $\mathcal{A}$ is parallel, we have $\mathcal{A}=0$. The assertion follows from Theorem 4.8 . □

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