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REPRESENTATION OF L-FUZZY BINARY RELATIONS VIA A GALOIS CONNECTION

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Abstract. Our aim in this Paper is to establish Galois connections between various types of fuzzy binary relations and fuzzy I-ary relations on a crisp set, that take their truth values in a complete lattice, and same type of crisp binary and I-ary relations on the associated fuzzy-point-set.

0. Introduction

Fuzzy relations were first introduced by Lotfi Zadeh in his pioneering work Zadeh [6] on Fuzzy sets. Later on these $[0,1]$ valued fuzzy relations were extensively studied by several Mathematicians as well as Computer Scientists.

Our aim in this Paper is to establish Galois connections between various types of fuzzy binary (I-ary) relations on a crisp set, which take their truth values in a complete lattice, and similar/same type of crisp binary (I-ary) relations on the associated fuzzy-point-set.

A primitive version of this paper appeared in Nistala and Peruru [4].

We assume the following notions from Lattice Theory: (sub)poset, order preserving map between posets, (least) upper bound, (greatest) lower bound, least element, greatest element in a poset, (complete) (semi) lattice, (complete) sub (semi) lattice, (complete) homomorphism of (semi) lattices, ideal, filter and Galois connection etc.. One can refer to any standard text book on Lattice Theory for them. Observe that by a complete lattice we mean a poset in which every *nonempty* subset has both infimum and supremum, a subset of a complete lattice is a complete sublattice if and only if it is closed under infimums and supremums for its *nonempty* subsets and by a complete homomorphism we mean any map between complete lattices which preserves infimums and supremums for *nonempty* sets. With these definitions, the least and the greatest elements of a complete sub lattice may or may not be the same as the corresponding ones of the parent complete lattice, the empty set is trivially a complete (sub) lattice and the *inclusion maps* become complete homomorphisms, which is *not* the case with the other definition of complete homomorphism which requires complete homomorphisms to preserve infimums and supremums for all subsets including the empty set, when the inclusion maps fail to

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be complete homomorphisms as they have to preserve 0 and 1 which may not happen as in: $i = \{(\alpha,\alpha),(\beta,\beta)\}$ from $\{\alpha,\beta\}$ to $\{0,\alpha,\beta,1 \mid 0<\alpha<\beta<1\}$ -our complete homomorphism. We define complete semi lattices and complete semi lattice homomorphisms in a similar way.

Proofs are omitted for two reasons: 1. to minimize the size of the document and 2. in most cases, they are either easy or, straight forward and a little involving.

In this section, along with some standard notions of L-fuzzy set theory, a few other notions and theorems involving these notions were recalled from Nistala [3].

Note. Throughout this Paper L is an arbitrary but fixed complete Brouwerian lattice. Henceforwards, we drop the word "fuzzy" in all the phrases "L-fuzzy ......". In other words, we write L-subset, L-point, L-union, L-intersection, etc. instead of L-fuzzy subset, L-fuzzy point, L-fuzzy union, L-fuzzy intersection, etc..

An L-*subset* A of a set X is a mapping $A : X \longrightarrow L$ where L is a complete lattice. The L-subset A is said to be a $\vee$-1-L-*subset* iff $\vee AX = 1_L$.

The constant map assuming the value 0 of L, for each x in X is the *empty* L-*subset* of X and is denoted by $\bar{0}$. The constant map assuming the value 1 of L, for each x in X is the *whole* L-*subset* of X and is denoted by $\bar{1}$.

An L-subset $A : X \longrightarrow L$ is an L-*point* of X, denoted by x_α, sometimes also by (x,α), iff

$$A(y) = \begin{cases} \alpha & \text{if } \; y = x \\ 0 & \text{otherwise.} \end{cases}$$

The set of all L-points of X is denoted by X_L. Thus $X_L = \{x_\alpha \mid x \in X, \alpha \in L\}$. Note that X_L is a proper quotient set on $X \times L$ and *not* $X \times L$ itself. An L-point is empty or non-empty according as it is the empty or non-empty L-subset. Further, it is easy to see that (1) x_0 is the empty L-subset $\bar{0}$ for each $x \in X$. (2) $x_\alpha \neq \bar{0}$ iff $\alpha \neq 0$. (3) For any pair of L-points $x_\alpha \neq \bar{0}$ and $y_\beta \neq \bar{0}$, $x_\alpha = y_\beta$ iff $x = y$ and $\alpha = \beta$.

For any pair of L-subsets A, B of X, A is L-*contained* in B, denoted by $A \leq B$, iff for each $x \in X$, $Ax \leq Bx$.

For any family $(A_i)_{i\in I}$ of L-subsets of X, the L-*union* of $(A_i)_{i\in I}$, denoted by $\vee_{i\in I} A_i$, is defined by

$$(\vee_{i\in I} A_i)x = \vee_{i\in I} A_i x, \quad \text{for each } \; x \in X$$

and the L-*intersection* of $(A_i)_{i\in I}$, denoted by $\wedge_{i\in I} A_i$, is defined by

$$(\wedge_{i\in I} A_i)x = \wedge_{i\in I} A_i x, \quad \text{for each } \; x \in X.$$

For any L-subset A of X, the *associated crisp set* of A, denoted by A', is defined by

$$A' = \{x_\alpha \mid Ax \geq \alpha\}.$$

For any subset B of X_L, the *associated* L-*subset* of B, denoted by $\bar{B} : X \longrightarrow L$, is defined by

$$\bar{B}(x) = \vee\{\alpha \mid x_\alpha \in B\}.$$

Clearly, for any L-point x_α and for any L-subset A,

$$x_\alpha \leq A \Leftrightarrow Ax \geq \alpha \Leftrightarrow x_\alpha \in A^{'}$$

For any $x \in X$ and $\alpha \in L$, the α-*stalk* on x, denoted by $stalk_x(\alpha)$, is defined by

$$stalk_x(\alpha) = \{x_\delta \mid 0 \leq \delta \leq \alpha\}.$$

For any subset B of X_L, (1) B is *closed under stalks* iff for each $x_\alpha \in B$, $\mathrm{stalk}_x(\alpha) \subseteq B$. (2) B is *closed under supremums* iff for any subset M of L, if $x_\alpha \in B$ for each $\alpha \in M$ then $x_{\vee M} \in B$. (3) B is *s-closed* iff it is closed under supremums and stalks.

It ia easy to see that (1) for any subset B of X_L, B is s-closed iff for any subset M of L, for each $\alpha \in M$ $x_\alpha \in B \Leftrightarrow x_{\vee M} \in B$. (2) For any L-subset A of X, $A^{'}$ is always s-closed. (3) For any subset B of X_L, the following are equivalent: (a) B is s-closed. (b) $B = \bar{B}^{'}$. (c) For each $x \in X$, $x_{\bar{B}x} \in B$ and $stalk_x(\bar{B}x) \subseteq B$. (4) For any set X and for any complete lattice L, the set of all subsets of X_L, $P(X_L)$, is a complete lattice $(P(X_L), \cap, \cup)$ where, for any $(A_i)_{i\in I} \subseteq P(X_L)$, $\cap_{i\in I} A_i$ is the ordinary intersection of $(A_i)_{i\in I}$ and $\cup_{i\in I} A_i$ is the ordinary union of $(A_i)_{i\in I}$. The least and the greatest elements of $P(X_L)$ are $0_{X_L} = \phi 1_{X_L} = X_L$ respectively. (5) For any set X and for any complete lattice L, the set of all L-subsets of X, denoted by L^X, is a complete lattice $(L^X, \wedge, \vee)$ where, for any $(B_j)_{j\in J} \subseteq L^X$, $\wedge_{j\in J} B_j$ is the L-intersection of $(B_j)_{j\in J}$ and $\vee_{j\in J} B_j$ is the L-union of $(B_j)_{j\in J}$. The least and the greatest elements of L^X are $0_{L^X} = \bar{0}$, $1_{L^X} = \bar{1}$ respectively.

Representation of *L*-Subsets via Galois Connection

0.1. Theorem. *For any set X and for any complete lattice L, define $\phi : L^X \longrightarrow P(X_L)$ by*

$$\phi A = A^{'}, \quad \text{the associated crisp set of } A$$

and $\psi : P(X_L) \longrightarrow L^X$ by

$$\psi B = \bar{B}, \quad \text{the associated } L\text{-subset of } B.$$

Then the following are true:

(1) $A_1 \leq A_2 \Rightarrow \phi A_1 \subseteq \phi A_2$.
(2) $B_1 \subseteq B_2 \Rightarrow \psi B_1 \leq \psi B_2$.
(3) ϕ *is a $\wedge$-complete homomorphism of the complete lattices.*
(4) ψ *is a $\vee$-complete homomorphism of the complete lattices.*
(5) $\cup_{i\in I}\phi(A_i) \subseteq \phi(\vee_{i\in I} A_i)$ *holds in general.*
(6) $\psi(\cap_{i\in I} B_i) \leq \wedge_{i\in I}\psi(B_i)$ *holds in general.*
(7) $\psi \circ \phi = 1$.

(8) $\phi \circ \psi \supseteq 1$

(9) $A = 0 \Leftrightarrow (\phi A)^{\star} = \phi$.

(10) $B^{\star} = \phi \Leftrightarrow \psi B = 0$, *where* $B^{\star} = B - \{0\}$.

(11) *Image*$(\phi) = \{C \in P(X_L) \mid C$ *is s-closed.*

In connection to the above result, the first author would like to draw the attention of the reader to the result: Theorem 3.25*, Page 89 of Belohlavek* [7] *and Belohlavek* [8].

In what follows we briefly recall some standard notions and results from the theory of binary relations only to make the document more self contained.

0.2. Definitions and statements. For any pair of sets X and Y, a *relation* from X to Y is any subset of $X \times Y$. If R is a relation from a set X to itself, then R is a *binary relation* on X. For any set X, the relation $\{(x, x) \mid x \in X\}$ is the *identity relation* of X and is denoted by Δ_X. The set of all binary relations on X denoted by $R^2(X)$, is a complete meet semi lattice where the $\wedge$ is given by *the usual set intersection*. The least element is ϕ and the largest element is $X \times X$. In fact, For any set X, $R^2(X)$ is a complete lattice with the infimum and the supremum given by $\inf A = \cap_{i \in I} A_i$, $\sup A = \cup_{i \in I} A_i$, where $A = (A_i)_{i \in I}$ is any subset of $R^2(X)$. The least and the greatest elements of $R^2(X)$ are ϕ, $X \times X$ respectively.

Let X be a set and R be a binary relation on X. Then (a) R is *reflexive* iff for each $x \in X$, $(x, x) \in R$. The set of all reflexive relations on X, denoted by $R^r(X)$ is a sub poset of $R^2(X)$. The least element is Δ_X and the largest element is $X \times X$. Clearly, $\phi \subseteq X \times X$ is reflexive iff $X = \phi$. (b) R is *irreflexive* iff for each $x \in X$, $(x, x) \notin R$. The set of all irreflexive relations on X, denoted by $R^i(X)$, is a sub poset of $R^2(X)$. The least element is ϕ and the largest element is $\nabla_X = X \times X - \Delta_X$. (c) R is *symmetric* iff for each $x, y \in X$, $(x, y) \in R$ implies $(y, x) \in R$. The set of all symmetric relations on X, denoted by $R^s(X)$, is a sub poset of $R^2(X)$. The least element is ϕ and the largest element is $X \times X$. (d) R is *antisymmetric* iff $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$. The set of all antisymmetric relations on X, denoted by $R^a(X)$, is a sub poset of $R^2(X)$. The least element is ϕ. (e) R is *transitive* iff whenever $(x, y) \in A$ and $(y, z) \in A$, $(x, z) \in A$. The set of all transitive relations on X, denoted by $R^t(X)$, is a sub poset of $R^2(X)$. The least element is ϕ and the largest element is $X \times X$. (f) R is an *equivalence relation* iff it is reflexive, symmetric and transitive. The set of all equivalence relations on X, denoted by $R^e(X)$, is a sub poset of $R^r(X)$, $(R^s(X)$, $R^t(X)$, and $R^2(X))$. The least element is Δ_X and the largest element is $X \times X$. (g) R is a *partial order* iff it is reflexive, antisymmetric and transitive. The set of all partial orders on X, denoted by $R^p(X)$, is a sub poset of $R^r(X)$ $(R^a(X)$, $R^t(X)$ and $R^2(X))$. The least element is ϕ.

(h) It is easy to see that for any family of binary relations $(A_i)_{i \in I}$ on a set X, the following are true:

(1) A_i is a reflexive relation, for each $i \in I \implies \cap_{i \in I} A_i$ is a reflexive relation.

(2) A_i is an irreflexive relation, for each $i \in I \implies \cap_{i\in I} A_i$ is an irreflexive relation (In fact, more generally, if A is an irreflexive relation and B is any other binary relation such that $B \subseteq A$ then B is an irreflexive relation. Consequently, if some A_{i_0} is an irreflexive relation then $\cap_{i\in I} A_i$ is an irreflexive relation).

(3) A_i is symmetric relation, for each $i \in I \implies \cap_{i\in I} A_i$ is a symmetric relation.

(4) A_i is an antisymmetric relation, for each $i \in I \implies \cap_{i\in I} A_i$ is an antisymmetric relation (In fact, more generally, if A is an antisymmetric relation and B is any other binary relation such that $B \subseteq A$ then B is an antisymmetric relation. Consequently, if some A_{i_0} is an antisymmetric relation then $\cap_{i\in I} A_i$ is an antisymmetric relation.

(5) A_i is a transitive relation, for each $i \in I \implies \cap_{i\in I} A_i$ is a transitive relation.

(6) A_i is an equivalence relation, for each $i \in I \implies \cap_{i\in I} A_i$ is an equivalence relation.

(7) A_{i_0} is a poset, for some $i_0 \in I \implies \cap_{i\in I} A_i$ is a poset.

(i) Also, it is easy to see that for any family of binary relations $(A_i)_{i\in I}$ on a set X, the following are true:

(1) if A_{i_0} is a reflexive relation for some $i_0 \in I$ then $\cup_{i\in I} A_i$ is a reflexive relation.

(2) if A_i is an irreflexive relation for each $i \in I$ then $\cup_{i\in I} A_i$ is also an irreflexive relation.

(3) if A_i is an symmetric relation for each $i \in I$ then $\cup_{i\in I} A_i$ is also a symmetric relation.

(j) For any set X, the following are true:

(1) $R^r(X)$ is the principal filter generated by Δ_X in $R^2(X)$ with the least element Δ_X and the greatest element $X \times X$.

(2) $R^s(X)$ is a complete sublattice of $R^2(X)$ with the least element ϕ and the greatest element $X \times X$.

(3) $R^i(X)$ is the principal ideal generated by ∇_X in $R^2(X)$ with the least element ϕ and the greatest element ∇_X.

(4) $R^t(X)$ is a complete meet sub semi lattice of $R^2(X)$ with the least element ϕ and the greatest element $X \times X$.

(5) $R^a(X)$ is a complete meet sub semi lattice of $R^2(X)$ with the least element ϕ.

(6) $R^p(X)$ is a complete meet sub semi lattice of $R^2(X)$ with the least element Δ_X.

(7) $R^e(X)$ is a complete meet sub semi lattice of $R^2(X)$ with the least element Δ_X and the greatest element $X \times X$.

(k) For any set X, the following are true:

(1) $R^t(X)$ is a complete lattice with the meet extended join.
(2) $R^e(X)$ is a complete sublattice of $R^t(X)$.

(l) The union of transitive relations need *not* be a transitive relation as shown in the following example:

Example 1. Let $X = \{x, y, z\}$, $A_1 = \{(x, x), (y, y), (x, y)\}$ and $A_2 = \{(y, y), (z, z), (y, z)\}$. Then A_1, A_2 are transitive relations on X. But $A_1 \cup A_2$ is *not* a transitive relation, because $(x, y), (y, z) \in A_1 \cup A_2$ and $(x, z) \notin A_1 \cup A_2$.
Thus $R^t(X)$ is *not* a complete sub lattice of $R^2(X)$.

(m) The union of antisymmetric relations need *not* be an antisymmetric relation as shown in the following example:

Example 2. Let $X = \{x, y\}$, $A_1 = \{(x, y)\}$ and $A_2 = \{(y, x)\}$. Then A_1, A_2 are antisymmetric relations on X. But $A_1 \cup A_2$ is *not* an antisymmetric relation.
Thus $R^a(X)$ is a *not* a complete sub lattice of $R^2(X)$.

(n) The union of equivalence relations need *not* be an equivalence relation. Thus $R^e(X)$ is *not* a complete sub lattice of $R^2(X)$.

(o) The union of partial orders need *not* be a partial order as shown in the following example:

Example 3. Let $X = \{x, y\}$, $A_1 = \{(x, x), (x, y), (y, y)\}$ and $A_2 = \{(x, x), (y, x), (y, y)\}$. Then A_1, A_2 are partial ordering relations on X. But $A_1 \cup A_2$ is *not* a partial ordering relation, because $x \neq y$.
Thus $R^p(X)$ is *not* a complete sub lattice of $R^2(X)$.

Crisp I-ary relations

0.3. Definitions and Statements. For any set X and for any index set I, *X power I*, denoted by X^I, is defined by

$$\{f : I \longrightarrow X \mid f(i) \in X \ \forall\, i \in I\}$$

An *I-ary relation* on X is any subset of X^I. It is easy to see that (1) The set of all $I - ary$ relations on X denoted by $R^I(X)$ is a poset with $\leq$ given by *the set inclusion.* The least element is ϕ and the largest element is X^I. In fact,

(2) For any set X and for any index set I, $R^I(X)$ is a complete lattice with the infimum and the supremum given by $\inf A = \cap_{i \in I} A_i$, $\sup A = \cup_{i \in I} A_i$, where $A = (A_i)_{i \in I}$ is any subset of $R^I(X)$. The least and the greatest elements of $R^I(X)$ are $0_{R^I(X)} = \phi$, $1_{R^I(X)} = X^I$ respectively.

1. Crisp binary relations on an L-fuzzy-point-set

In this section, on an L-point-set we introduce the notion of (crisp) binary relations and for these binary relations, we introduce the notions of stalk closedness, strongly reflexivity, weakly transitivity, equivalence, weak equivalence and study some required properties involving them. These results will be used later in establishing a Galois connection between L-fuzzy binary (equivalence) relations on a set and crisp binary (equivalence) relations on the L-point-set.

Let us recall that X_L denotes the set of all L-points on the set X. Then the set of all binary relations on X_L, denoted by $R^2(X_L)$ is a complete lattice with the infimum and the supremum given by $\inf T = \cap_{i\in I} T_i$, $\sup T = \cup_{i\in I} T_i$, where $T = (T_i)_{i\in I}$ is any subset of $R^2(X_L)$. The least and the greatest elements of $R^2(X_L)$ are $0_{R^2(X_L)} = \phi$, $1_{R^2(X_L)} = X_L \times X_L$ respectively.

1.1. Definitions and Statements. Let R be a binary relation on X_L.

(a) The notion of equivalence relation is defined as usual.

(b) R is said to be *stalk closed* iff $(x_\alpha, y_\beta) \in R$, $\gamma \leq \alpha$, $\delta \leq \beta$ implies $(x_\gamma, y_\delta) \in R$.

(c) R is said to be a *weakly transitive relation* iff $(x_\alpha, y_\beta), (y_\beta, z_\gamma) \in R$ implies $(x_{\alpha\wedge\beta\wedge\gamma}, z_{\alpha\wedge\beta\wedge\gamma}) \in R$.

It is natural to expect some relation between transitivity and weak transitivity. However, there is no relation between them as will be seen in Examples 6 and 7 later.

(d) R is said to be a *weak equivalence relation* iff it is reflexive, symmetric and weak transitive.

Since there is *no* relation between weak transitivity and transitivity, there is no relation between weak equivalence and equivalence.

(e) The set of all weak equivalence relations on X_L, denoted by $R^{we}(X_L)$, is a meet complete sub semi lattice of $R^2(X_L)$, where the meet is given by the usual set intersection. Also $1_{R^{we}(X_L)} = X_L \times X_L$.

(f) $R^{we}(X_L)$ is a complete lattice where $\vee$ is the meet extended join.

(g) $R^{we}(X_L)$ is *not* a complete sublattice of $R^2(X_L)$ because the union of weak equivalence relations is *not* necessarily a weak equivalence relation as shown in the following example:

Example 4. Let $X = \{x, y, z\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha < \beta < 1\}$, $E_1 = \{(x_\delta, x_\delta), (y_\delta, y_\delta), (z_\delta, z_\delta), (x_\alpha, y_\alpha), (y_\alpha, x_\alpha), (x_\alpha, y_1), (y_1, x_\alpha) \mid \delta = 0, \alpha, \beta, 1\}$ and $E_2 = \{(x_\delta, x_\delta), (y_\delta, y_\delta), (z_\delta, z_\delta), (y_\beta, z_\beta), (z_\beta, y_\beta), (y_1, z_\beta), (z_\beta, y_1) \mid \delta = 0, \alpha, \beta, 1\}$. Then E_1, E_2 are weak equivalence relations. But $E_1 \cup E_2$ is *not* a weak equivalence relation, because $(x_\alpha, y_1), (y_1, z_\beta) \in E_1 \cup E_2$ and $(x_\alpha, z_\alpha) \notin E_1 \cup E_2$.

(h) The *stalk closure* of R, denoted by $Cl_{st}(R)$, is defined by $Cl_{st}(R) = \{(x_\gamma, y_\delta) \mid (x_\alpha, y_\beta) \in R, \gamma \leq \alpha, \delta \leq \beta\}$.

(i) A weak equivalence relation which is also stalk closed is called a *stalk closed weak equivalence relation.*

The set of all stalk closed weak equivalence relations on X_L, denoted by $R_s^{we}(X_L)$, is sub poset of $R^{we}(X_L)$. In fact,

(j) For any set X, $R_s^{we}(X_L)$ is a complete sub lattice of $R^{we}(X_L)$.

(k) R is said to be *strongly reflexive* iff $(x_\alpha, x_\beta) \in R$, whenever $\alpha \leq \beta$ or $\beta \leq \alpha$.

(l) Every strongly reflexive relation R on X_L is always a reflexive relation.

The converse of the above statement is *not* true as shown in the following example:

Example 5. Let $X = \{x, y, z\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha < \beta < 1\}$ and $R = \{(\phi, \phi), (x_\delta, x_\delta), (y_\delta, y_\delta), (z_\delta, z_\delta), (x_\alpha, y_\alpha), (y_\alpha, x_\alpha), (y_1, z_1), (z_1, y_1) \mid \delta = \alpha, \beta, 1\}$. Then R is a reflexive relation, but *not* a strongly-reflexive relation, because $(x_\alpha, x_1) \notin R$ and $(x_1, x_\alpha) \notin R$ though $\alpha < 1$.

(m) Every stalk closed transitive relation R on X_L is a weakly transitive relation.

The converse of the above statement is *not* true, that is, a weakly transitive relation need *not* be a transitive relation, as shown in the following example.

Example 6. Let $X = \{x, y\}$, $L = \{0, 1 \mid 0 < 1\}$ and $T = \{(\phi, \phi), (x_0, y_1), (y_1, x_0), (x_1, y_1), (y_1, x_1), (x_1, x_1), (y_1, y_1)\}$. Then T is a weakly transitive relation, but *not* transitive relation, because $(x_0, y_1), (y_1, x_1) \in T$ and $(x_0, x_1) \notin T$.

In fact, T is a weak equivalence relation, but *not* an equivalence relation.

(n) A mere transitive relation without being stalk closed need *not* be a weakly transitive relation, as shown in the following example:

Example 7. Let $X = \{x, y, z\}$, $L = \{0, \alpha, 1 \mid 0 < \alpha < 1\}$ and $T = \{(x_\delta, x_\delta), (y_\delta, y_\delta), (z_\delta, z_\delta), (x_\alpha, y_\alpha), (y_\alpha, x_\alpha), (y_\alpha, z_1), (z_1, y_\alpha) \mid \delta = 0, \alpha, 1\}$. Then T is a transitive relation, but *not* weakly transitive relation, because $(x_\alpha, y_\alpha), (y_\alpha, z_1) \in T$ and $(x_\alpha, z_\alpha) \notin T$. Note that T is *not* stalk closed because $(y_\alpha, z_\alpha) \notin T$.

In fact, T is an equivalence relation, but *not* a weak equivalence relation.

(o) For any reflexive, symmetric relation R on X_L, $Cl_{st}(R)$ is also a reflexive, symmetric relation.

(p) For an equivalence relation R on X_L, $Cl_{st}(R)$ need *not* be an equivalence relation, as shown in the following example:

Example 8. Let $X = \{x, y, z\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha, \beta < 1; \alpha \parallel \beta\}$. Then $R = \{(\phi, \phi), (x_\delta, x_\delta), (y_\delta, y_\delta), (z_\delta, z_\delta), (x_\alpha, y_\beta), (y_\beta, x_\alpha), (y_1, z_1), (z_1, y_1) \mid \delta = 0, \alpha, \beta, 1\}$ is an equivalence relation and $Cl_{st}(R) = \{(\phi, \phi), (x_\delta, x_\delta), (y_\delta, y_\delta), (z_\delta, z_\delta), (x_\alpha, y_\beta), (y_\beta, x_\alpha), (x_0, y_\beta), (y_\beta, x_0), (x_\alpha, y_0), (y_0, x_\alpha), (y_\delta, z_\delta), (z_\delta, y_\delta) \mid \delta = 0, \alpha, \beta, 1\}$. Further, $Cl_{st}(R)$ is *not* transitive relation, because $(x_\alpha, y_\beta), (y_\beta, z_\beta) \in Cl_{st}(R)$ and $(x_\alpha, z_\beta) \notin Cl_{st}(R)$. Hence $Cl_{st}(R)$ is *not* an equivalence relation.

(q) For any binary relation R on X_L, the following are true:

(1) $R \subseteq Cl_{st}(R)$.
(2) $R \subseteq S \Rightarrow Cl_{st}(R) \subseteq Cl_{st}(S)$.
(3) $Cl_{st}Cl_{st}(R) = Cl_{st}(R)$.
(4) R is stalk closed $\Leftrightarrow R = Cl_{st}(R)$.

Thus (1) for any binary relation R on X_L, $Cl_{st}(R)$ is a closure operator. (2) for any binary relation R on X_L, $Cl_{st}(R)$ is always stalk closed. (3) For any binary relations R, S on X_L such that $R \subseteq S$ and S is stalk closed, $Cl_{st}(R) \subseteq S$.

(r) For any equivalence relation R on X_L such that R is strongly reflexive, $Cl_{st}(R)$ is an equivalence relation.

2. L-fuzzy-binary relations

In this section we introduce the notions of L-I-ary relation, the associated (crisp) I-ary relation for any L-I-ary relation, the associated L-I-ary relation for any (crisp) I-ary relation on the L-point-set, (crisp) equivalence relation, weak equivalence relation, L-equivalence relation, the associated (crisp) binary relation for any L-binary relation on a set and the associated L-binary relation for any (crisp) binary relation on an L-fuzzy-point-set, and represent the L-I-ary relations (L-equivalence relations) on a *crisp set* as a $\wedge$-complete sublattice of the $\wedge$-complete lattice of crisp I-ary relations (weak equivalence relations) via a Galois connection.

Through out the following, L is assumed to be an arbitrary but fixed complete Brouwerian lattice. However, L being mere complete lattice will be enough in most of the cases. Hence, whenever the proof uses the fact that L is a Brouwerian lattice, in the hypothesis of its statement we specifically mention that L is a complete Brouwerian lattice.

2.1. Definitions and Statements

(a) An L-binary relation on a set X is any map $R : X \times X \longrightarrow L$.
The set of all L-binary relations on X, denoted by $R^2_L(X)$, is a poset with $\leq$ given by: for $R, S \in R^2_L(X)$, $R \leq S$ iff $R(\mathbf{x}) \leq S(\mathbf{x})\ \forall \mathbf{x} \in X \times X$. In fact,

(b) For any set X and for any complete lattice L, $R^2_L(X)$ is a complete lattice with the infimum and the supremum given by: $\inf S = \wedge_{j \in J} S_j$, $\sup S = \vee_{j \in J} S_j$, where

$S = (S_j)_{j\in J}$ is any subset of $R^2_L(X)$, $(\wedge_{j\in J} S_j)(\mathbf{x}) = \wedge_{j\in J} S_j(\mathbf{x})\ \forall \mathbf{x} \in X \times X$ and $(\vee_{j\in J} S_j)(\mathbf{x}) = \vee_{j\in J} S_j(\mathbf{x})\ \forall \mathbf{x} \in X \times X$. The least and the greatest elements of $R^2_L(X)$ are $0_{R^2_L(X)} = \bar{0}$, $1_{R^2_L(X)} = \bar{1}$ respectively, where $\bar{0}$ is the constant map assuming the value 0 of L on $X \times X$ and $\bar{1}$ is the constant map assuming the value 1 of L on $X \times X$.

Let R be an L-binary relation on a set X. Then

(c) R is said to be an L-*reflexive relation* iff $R \neq \bar{0}$ and $R(x, x) \geq R(y, z)$, for each $x, y, z \in X$.

(d) The set of all L-reflexive relations on X, denoted by $R^r_L(X)$, is a sub poset of $R^2_L(X)$

(e) R is said to be an L-*irreflexive relation* iff for each $x \in X$, $R(x, x) = 0_L$.

(f) The set of all L-irreflexive relations on X, denoted by $R^i_L(X)$, is a sub poset of $R^2_L(X)$.

(g) R is said to be an L-*symmetric relation* iff $R(x, y) = R(y, x)$, for each $x, y \in X$.

(h) The set of all L-symmetric relations on X, denoted by $R^s_L(X)$, is a sub poset of $R^2_L(X)$.

(i) R is said to be an L-antisymmetric relation iff $R(x, y) = R(y, x)$ implies $x = y$.

(j) The set of all L-antisymmetric relations on X, denoted by $R^a_L(X)$, is a sub poset of $R^2_L(X)$.

(k) R is said to be an L-*transitive relation* iff $R(x, z) \geq R(x, y) \wedge R(y, z)$, for each $x, y, z \in X$.

(l) The set of all L-transitive relations on X, denoted by $R^t_L(X)$, is a sub poset of $R^2_L(X)$.

(m) R is said to be an L-*equivalence relation* iff it is an L-reflexive, L-symmetric and L-transitive.

(n) The set of all L-equivalence relations on X, denoted by $R^e_L(X)$, is a sub poset of $R^2_L(X)$.

(o) R is said to be an L-*partial order* iff it is an L-reflexive, L-antisymmetric and L-transitive.

(p) The set of all L-partial orders on X, denoted by $R^p_L(X)$, is a sub poset of $R^2_L(X)$.

The following is a motivation for the preceeding definitions.

(q) For any set X and for any binary relation A on X, the following are true:

(1) A is a reflexive relation iff χ_A is an L-reflexive relation.

(2) A is an irreflexive relation iff χ_A is an L-irreflexive relation.

(3) A is a symmetric relation iff χ_A is an L-symmetric relation.

(4) A is a transitive relation iff χ_A is an L-transitive relation.

(5) χ_A is an L-antisymmetric relation $\Rightarrow$ A is an antisymmetric relation.

(r) Whenever A is an antisymmetric relation, χ_A need *not* be an L-antisymmetric relation as shown in the following example:

Example 9. Let $X = \{x, y\}$ and $A = \{(x, x), (y, y)\}$. Then A is an antisymmetric relation. But χ_A is *not* an L-antisymmetric relation, because $\chi_A(x, y) = \chi_A(y, x) = 0$ and $x \neq y$.

(s) For any binary relation A on X, A is an equivalence relation iff χ_A is an L-equivalence relation.

Now we begin studying some lattice theoretic properties of the above mentioned L-fuzzy relations.

2.2. Proposition *For any family of L-binary relations $(A_i)_{i\in I}$ on a set X and for any complete lattice L, the following are true:*

(1) *A_i is an L-reflexive relation, for each $i \in I \implies \wedge_{i\in I} A_i$ is an L-reflexive relation, whenever it is not the L-empty set $\bar{0}$ or whenever L is unique atomed.*

(2) *A_{i_0} is an L-irreflexive relation, for some $i \in I \Rightarrow \wedge_{i\in I} A_i$ is an L-irreflexive relation.*

(3) *A_i is an L-symmetric relation, for each $i \in I \Rightarrow \wedge_{i\in I} A_i$ is an L-symmetric relation.*

(4) *A_i is an L-transitive relation, for each $i \in I \Rightarrow \wedge_{i\in I} A_i$ is an L-transitive relation.*

(5) *A_i is an L-equivalence relation, for each $i \in I \Rightarrow \wedge_{i\in I} A_i$ is an L-equivalence relation, whenever it is not the L-empty set $\bar{0}$ or whenever L is unique atomed.*

Proof. It is straight forward and follows from (c), (e), (g), (k) and (m) of 2.1.

L-intersection of L-reflexive relations may become L-empty set as shown in the following example:

Example 10. Let $X = \{x, y\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha, \beta < 1; \alpha \parallel \beta\}$ $A = \{((x, x), \alpha), ((y, y), \alpha), ((x, y), 0), ((y, x), 0)\}$ and $B = \{((x, x), \beta), ((y, y), \beta), ((x, y), 0), ((y, x), 0)\}$. Then A and B are L-reflexive relations but $A \wedge B$ is *not* an L-reflexive relation because it is the L-empty set.

L-intersection of L-antisymmetric relations is *not* necessarily an L-antisymmetric relation even when L is the two element lattice, as shown in the following example:

Example 11. Let $X = \{x, y\}$, $L = \{0, 1 \mid 0 < 1\}$. Define $A_1 = \{((x,x),1), ((y,y),1), ((x,y),0), ((y,x),1)\}$ and $A_2 = \{((x,x),1), ((y,y),1), ((x,y),1), ((y,x), 0)\}$. Then A_1, A_2 are L-antisymmetric relations. But the L-intersection $A_1 \wedge A_2$ is *not* an L-antisymmetric relation, because $(A_1 \wedge A_2)(x,y) = 0 = (A_1 \wedge A_2)(y,x)$ and $x \neq y$.

Also, the same example shows that (1) the L-intersection $A_1 \wedge A_2$ of L-posets A_1, A_2 is *not* necessarily an L-poset (2) the L-union $(A_1 \vee A_2)$ of L-antisymmetric relations is *not* necessarily an L-antisymmetric relation (3) L-union $(A_1 \vee A_2)$ of L-posets is *not* necessarily an L-poset.

Thus, the posets $R^a_L(X)$ and $R^p_L(X)$ are at best sub posets of $R^2_L(X)$! (Please refer to the discussion after Corollary 2.5)

2.3. Proposition *For any set X, for any complete lattice L and for any family of L-binary relations $(A_i)_{i\in I}$ on X, the following are true:*

(1) *A_i is an L-reflexive relation, for each $i \in I \implies$ the L-union $\vee_{i\in I} A_i$ is an L-reflexive relation.*
(2) *A_i is an L-irreflexive relation, for each $i \in I \implies$ the L-union $\vee_{i\in I} A_i$ is an L-irreflexive relation.*
(3) *A_i is an L-symmetric relation, for each $i \in I \implies$ the L-union $\vee_{i\in I} A_i$ is an L-symmetric relation.*

Proof. It is straight forward and follows from (c), (e) and (g) of 2.1.

2.4. Corollary. *For any set X and for any complete lattice L, the following are true:*

(1) *$R^r_L(X)$ is always a $\vee$-complete sublattice of $R^2_L(X)$. However, it is also a $\wedge$-complete whenever L is uniquely atomed. Thus $R^r_L(X)$ is a complete sub lattice of $R^2_L(X)$ whenever L is unique atomed.*
(2) *$R^i_L(X)$ is a complete sublattice of $R^2_L(X)$.*
(3) *$R^s_L(X)$ is a complete sublattice of $R^2_L(X)$.*
(4) *$R^t_L(X)$ is a complete sub semi lattice of $R^2_L(X)$.*
(5) *$R^e_L(X)$ is a complete sub semi lattice of $R^2_L(X)$.*

Proof. (1) follows from the Propositions 2.2(1) and 2.3(1), (2) follows from the Propositions 2.2(2) and 2.3(2), (3) follows from the Propositions 2.2(3) and 2.3(3), (4) follows from the Proposition 2.2(4) and (5) follows from the Proposition 2.2(5).

2.5. Corollary. *For any set X and for any complete lattice L, the following are true:*

(1) *$R^t_L(X)$ is a complete lattice with the meet extended join.*
(2) *$R^e_L(X)$ is a complete sublattice of $R^t_L(X)$, whenever L is unique atomed.*

Proof. (1) follows from the Corollary 2.4(4)and (2) follows from the Corollary 2.4(1).

L-union of L-antisymmetric relations is *not* necessarily an L-antisymmetric relation, as mentioned earlier with a counter example.

L-union of an L-posets is *not* necessarily an L-poset as mentioned earlier with a counter example.

L-union of L-transitive relations is *not* necessarily an L-transitive relation, as shown in the following example:

Example 12. Let $X = \{x, y, z\}$, $L = \{0, 1 \mid 0 < 1\}$. Define $A_1 = \{((x, x), 1), ((y, y), 1), ((z, z), 0), ((x, y), 1), ((y, x), 0), ((x, z), 0), ((y, z), 0), ((z, x), 0), ((z, y), 0)\}$ and $A_2 = \{((x, x), 0), ((y, y), 1), ((z, z), 1), ((x, y), 0), ((x, z), 0), ((y, x), 0), ((y, z), 1), ((z, x), 0), ((z, y), 0)\}$. Then A_1, A_2 are L-transitive relations. But $A_1 \vee A_2$ is *not* an L-transitive relation, because $(A_1 \vee A_2)(x, z) = 0 \neq (A_1 \vee A_2)(x, y) \wedge (A_1 \vee A_2)(y, z) = 1 \wedge 1 = 1$.

Thus $R^t_L(X)$ is *not* a complete sub lattice of $R^2_L(X)$.

L-union of L-equivalence relations is *not* necessarily an L-equivalence relation.

Thus $R^e_L(X)$ is *not* a complete sub lattice of $R^2_L(X)$.

2.6. Definitions

(a) For any L-binary relation S on X, the *associated binary relation for S on X_L*, denoted by S', is defined by:

$$S' = \{(x_\alpha, y_\beta) \mid S(x, y) \geq \alpha \wedge \beta\}.$$

(b) For any binary relation R on X_L, the *associated L-binary relation for R on X*, denoted by $\bar{R}$, is defined by:

$$\bar{R}(x, y) = \vee\{\alpha \wedge \beta \mid (x_\alpha, y_\beta) \in R\}.$$

2.7. Proposition. *For any L-binary relation S on X, $\overline{S'} = S$.*

Proof. It is straight forward and follows from (a) and (b) of 2.6.

2.8. Lemma. *For any L-binary relation S on X, S' is always stalk closed.*

Proof. It is straight forward and follows from 2.6(a) and 1.1(b).

In general the above S' is *not* an equivalence relation on X_L, even if L is a two element chain and S is an L-equivalence relation on X as shown in the following example:

Example 13. Let $X = \{x, y, z\}$, $L = \{0, 1 \mid 0 < 1\}$ and $A = \{(x, x), (y, y), (z, z), (x, y), (y, x)\}$. Then A is an equivalence relation. Since A is an equivalence relation iff χ_A is an L-equivalence relation, $S = \chi_A$ is an L-equivalence relation. However,

$S^{'} = \{(x_\alpha, y_\beta) \mid S(x,y) \geq \alpha \wedge \beta\}$ is *not* an equivalence relation on X_L, because $(y_1, x_0), (x_0, z_1) \in S^{'}$ but $(y_1, z_1) \notin S^{'}$.

2.9. Proposition. *For any L-equivalence relation S on X, $S^{'}$ is always a weak-equivalence relation on X_L.*

Proof. It is straight forward and follows from 2.6(a) and 1.1(d).

2.10. Proposition. *For any stalk closed equivalence relation R on X_L, the associated L-binary relation $\bar{R}$ on X is always an L-equivalence relation.*

Proof. It is straight forward and follows from 2.6(b) and 2.1(m).

In the above Proposition, R being stalk closed is necessary as shown in the following example:

Example 14. Let $X = \{x, y, z\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha < \beta < 1\}$, $X_L = \{x_0 = y_o = z_0, x_\alpha, y_\alpha, z_\alpha, x_\beta, y_\beta, z_\beta, x_1, y_1, z_1\}$ and $R = \{(\phi, \phi), (x_\delta, x_\delta), (y_\delta, y_\delta), (z_\delta, z_\delta), (x_\alpha, y_\alpha), (y_\alpha, x_\alpha), (y_1, z_1), (z_1, y_1) \mid \delta = \alpha, \beta, 1\}$. Then R is an equivalence relation on X_L. Note that R is *not* stalk closed, because $(x_0, y_\alpha) \notin R$. Also $\bar{R}(x,x) = \bar{R}(y,y) = \bar{R}(z,z) = 1$, $\bar{R}(x,y) = \bar{R}(y,x) = \alpha$, $\bar{R}(y,z) = \bar{R}(z,y) = 1$ and $\bar{R}(z,x) = \bar{R}(x,z) = 0$, because $(x,z) \notin R$. If $\bar{R}(x,z) \geq \bar{R}(x,y) \wedge \bar{R}(y,z)$ then $0 \geq \alpha \wedge 1 = \alpha$, which is a contradiction. Therefore $\bar{R}$ is *not* an L-transitive relation. Hence $\bar{R}$ is *not* an L-equivalence relation on X.

2.11. Proposition. *For any stalk closed weak-equivalence relation T on X_L, the associated L-binary relation $\bar{T}$ on X is an L-equivalence relation.*

Proof. It is straight forward and follows from (b), (i) of 1.1, 2.1(m) and 2.6(b).

It appears as if a strongly reflexive, symmetric and transitive binary relation is all of $X_L \times X_L$. But this is *not* the case as shown in the following example:

Example 15. Let $X = \{x, y, z\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha, \beta < 1, \alpha \parallel \beta\}$ and $R = \{(a_\gamma, a_\delta) \mid \gamma \leq \delta \text{ or } \delta \leq \gamma, a = x, y, z\}$. Then R is strongly-reflexive, symmetric and transitive relation. But $R \neq X_L \times X_L$, because $(x_\alpha, y_\beta) \in X_L \times X_L$ and $(x_\alpha, y_\beta) \notin R$.

3. *L*-fuzzy *I*-ary relations

In this section, we recall the notion of L-I-ary relation and introduce the notions of stalk closedness of an I-ary relation on an L-fuzzy-point-set, the associated L-subset for a (crisp) I-ary relation on an L-fuzzy-point-set and the associated (crisp) subset for an L-fuzzy I-ary relation on a set, and prove some properties involving these notions.

3.1. Definitions and Statements. (a) For any set X and for any complete Brouwerian lattice L, an L-I-ary relation on X is any map $A : X^I \longrightarrow L$. The set of all L-I-ary relations on X, denoted by $R_L^I(X)$.

(b) An I-ary relation R on X_L is said to be *stalk closed* iff for each $i \in I$, $(x_i, \beta_i) \in R$ and $\alpha_i \leq \beta_i$ implies $(x_i, \alpha_i) \in R$.

(c) For any L-I-ary relation S on X, the *associated crisp subset* for S denoted by $S^{'}$, is defined by $S^{'} = \{(x_i, \alpha_i)_{i\in I} \in {X_L}^I \mid S(x_i)_{i\in I} \geq \wedge_{i\in I}\alpha_i\}$.

(d) For any subset T of ${X_L}^I$, the *associated L-subset* for T, denoted by $\bar{T} : X^I \longrightarrow L$, is defined by $\bar{T}(x_i)_{i\in I} = \vee\{\wedge_{i\in I}\alpha_i \mid (x_i, \alpha_i)_{i\in I} \in T\}$.

3.2. Proposition. *For any L-I-ary relation S on X, $S^{'}$ is always stalk closed.*

Proof. It follows from (a) and (b) of 3.1.

3.3. Proposition. *For any L-I-ary relation S on X, $\overline{S'} = S$.*

Proof. It follows from (c) and (d) of 3.1.

3.4. Proposition. *For any set X and for any complete lattice L, $R^I(X_L)$ is a complete lattice with the infimum and the supremum given by* $\inf T = \cap_{j\in J}T_j$, $\sup T = \cup_{j\in J}T_j$, *where $T = (T_j)_{j\in J}$ is any subset of $R^I(X_L)$. The least and the greatest elements of $R^I(X_L)$ are $0_{R^I(X_L)} = \phi$, $1_{R^I(X_L)} = {X_L}^I$ respectively.*

3.5. Proposition. *For any set X and for any complete lattice L, $R_L^I(X)$ is a complete lattice with the infimum and the supremum given by:* $\inf S = \wedge_{j\in J}S_j$, $\sup S = \vee_{j\in J}S_j$, *where $S = (S_j)_{j\in J}$ is any subset of $R_L^I(X)$, $(\wedge_{j\in J}S_j)(\boldsymbol{x}) = \wedge_{j\in J}S_j(\boldsymbol{x})$ $\forall \boldsymbol{x} \in X^I$ and $(\vee_{j\in J}S_j)(\boldsymbol{x}) = \vee_{j\in J}S_j(\boldsymbol{x})$ $\forall \boldsymbol{x} \in X^I$. The least and the greatest elements of $R_L^I(X)$ are $0_{R_L^I(X)} = \bar{0}$ and $1_{R_L^I}(X) = \bar{1}$ respectively, where $\bar{0}$ is the constant map assuming the value 0 of L on X^I and $\bar{1}$ is the constant map assuming the value 1 of L on X^I.*

4. A Galois connection between L-fuzzy binary relations and (crisp) binary relations

In this section we establish a Galois connection between the complete lattice of all L-fuzzy binary relations on a (crisp) set X and the (crisp) binary relations on the L-fuzzy-point-set of the given set, X_L.

4.1. Proposition. *For any set X and for any complete lattice L, let $\phi : R_L^2(X) \longrightarrow R^2(X_L)$ be defined by $\phi S = S^{'}$ where $S^{'}$ is the associated crisp set of S and ψ :*

$R^2(X_L) \longrightarrow R^2_L(X)$ be defined by $\psi T = \bar{T}$ where $\bar{T}$ is the associated L-binary relation of T. Then the following are true:

(1) $S_1 \leq S_2 \Rightarrow \phi S_1 \subseteq \phi S_2$.
(2) $T_1 \subseteq T_2 \Rightarrow \psi T_1 \leq \psi T_2$.
(3) *ϕ is a $\wedge$-complete homomorphism of the complete lattices.*
(4) *ψ is a $\vee$-complete homomorphism of the complete lattices.*
(5) *$\cup_{i\in I}\phi S_i \subseteq \phi(\vee_{i\in I} S_i)$ holds in general.*
(6) *$\psi(\cap_{i\in I} T_i) \leq \wedge_{i\in I}\psi(T_i)$ hold in general.*
(7) $\psi \circ \phi = 1$.
(8) $\phi \circ \psi \supseteq 1$.

Proof. It is tedious but straight forward and similar to that of 0.1.

The above map $\phi : R^2_L(X) \longrightarrow R^2(X_L)$ is *not* a $\vee$-complete homomorphism as shown in the following example:

Example 16. Let $X = \{x\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha, \beta, 1; \alpha \parallel \beta\}$ and $S_1 = \{((x, x), \alpha)\}$, $S_2 = \{((x, x), \beta)\}$. Then $S_1^{'} = \{(x_0, x_0), (x_0, x_\alpha), (x_\alpha, x_0), (x_\alpha, x_\alpha)\}$, $S_2^{'} = \{(x_0, x_0), (x_0, x_\beta), (x_\beta, x_0), (x_\beta, x_\beta)\}$ and $(S_1 \vee S_2)^{'} = \{(x_0, x_0), (x_0, x_\alpha), (x_\alpha, x_0), (x_0, x_\beta), (x_\beta, x_0), (x_0, x_1), (x_1, x_0), (x_\alpha, x_\beta), (x_\beta, x_\alpha), (x_\alpha, x_\alpha), (x_\beta, x_\beta), (x_\alpha, x_1), (x_1, x_\alpha), (x_\beta, x_1), (x_1, x_\beta), (x_1, x_1)\}$. Hence $\phi S_1 \cup \phi S_2 \subset \phi(S_1 \vee S_2)$.

The above map $\psi : R^2(X_L) \longrightarrow R^2_L(X)$ is *not* a $\wedge$-complete homomorphism, as shown in the following example:

Example 17. Let $X = \{x\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha < \beta < 1\}$ and $T_1 = \{(x_\alpha, x_\alpha)\}$, $T_2 = \{(x_\beta, x_\beta)\}$. Then $\bar{T_1} = \{((x, x), \alpha)\}$, $\bar{T_2} = \{((x, x), \beta)\}$ and $\overline{T_1 \cap T_2} = \{((x, x), 0)\}$. Hence $\psi(T_1 \cap T_2) < \psi T_1 \wedge \psi T_2$.

In general for an arbitrary $T \in R^2_L(X)$, it may happen that $T \subset \phi \circ \psi(T)$ as shown in the following example:

Example 18. Let $X = \{x\}$, $L = \{0, 1 \mid 0 < 1\}$ and $T = \{(x_0, x_0)\}$. Then $\bar{T} = \{((x, x), 0)\}$ and $\bar{T}^{'} = \{(x_0, x_0), (x_0, x_1), (x_1, x_0)\}$. Therefore $T \subset \bar{T}^{'}$. Hence $T \subset \phi \circ \psi(T)$.

Representation of *L*-fuzzy binary relations

4.2. Corollary. *For any set X and for any complete lattice L, the complete semi lattice of all L-binary relations on X is isomorphic to a complete sub semi lattice of the complete semi lattice of crisp binary relations on the L-point-set X_L.*

Proof. It follows from the Proposition 4.1.

5. Galois connection between L-equivalence relations and (crisp) weak-equivalence relations

In this section we establish a Galois connection between the complete lattice of all L-fuzzy equivalence relations on a (crisp) set X and the s-closed (crisp) weak-equivalence relations on the L-fuzzy-point-set of the given set, X_L.

5.1. Proposition. *For any set X and for any complete lattice L, let $\bar{\phi} : R^e_L(X) \longrightarrow R^{we}_s(X_L)$ be defined by $\bar{\phi}S = S^{'}$ where $S^{'}$ is the associated crisp set of S and $\bar{\psi} : R^{we}_s(X_L) \longrightarrow R^e_L(X)$ be defined by $\bar{\psi}T = \bar{T}$ where $\bar{T}$ is the associated L-binary relation of T, $\bar{\phi} = \phi \mid R^e_L(X)$, $\bar{\psi} = \psi \mid R^{we}_s(X_L)$. Then the following are true:*

(1) $S_1 \leq S_2 \Rightarrow \bar{\phi}S_1 \subseteq \bar{\phi}S_2$.
(2) $T_1 \subseteq T_2 \Rightarrow \bar{\psi}T_1 \leq \bar{\psi}T_2$.
(3) $\bar{\phi}$ *is a* $\wedge$*-complete homomorphism of the complete lattices.*
(4) $\bar{\psi}$ *is a* $\vee$*-complete homomorphism of the complete lattices.*
(5) $\cup_{i\in I}\bar{\phi}S_i \subseteq \bar{\phi}(\vee_{i\in I}S_i)$ *holds in general.*
(6) $\bar{\psi}(\cap_{i\in I}T_i) \leq \wedge_{i\in I}\bar{\psi}(T_i)$ *holds in general.*
(7) $\bar{\psi} \circ \bar{\phi} = 1$.
(8) $\bar{\phi} \circ \bar{\psi} \supseteq 1$.

Proof. It is tedious but straight forward.

The above map $\bar{\phi} : R^e_L(X) \longrightarrow R^{we}_s(X_L)$ is *not* a $\vee$-complete homomorphism, as shown in the following example:

Example 19. Let $X = \{x\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha, \beta < 1; \alpha \parallel \beta\}$ and $S_1 = \{((x, x), \alpha)\}$, $S_2 = \{((x, x), \beta)\}$. Then $S_1^{'} = \{(x_0, x_0), (x_0, x_\alpha), (x_\alpha, x_0), (x_\alpha, x_\alpha)\}$, $S_2^{'} = \{(x_0, x_0), (x_0, x_\beta), (x_\beta, x_0), (x_\beta, x_\beta)\}$ and $(S_1 \vee S_2)^{'} = \{(x_0, x_0), (x_0, x_\alpha), (x_\alpha, x_0), (x_0, x_\beta), (x_\beta, x_0), (x_0, x_1), (x_1, x_0), (x_\alpha, x_\beta), (x_\beta, x_\alpha), (x_\alpha, x_\alpha), (x_\beta, x_\beta), (x_\alpha, x_1), (x_1, x_\alpha), (x_\beta, x_1), (x_1, x_\beta), (x_1, x_1)\}$. Hence $\bar{\phi}S_1 \cup \bar{\phi}S_2 \subset \bar{\phi}(S_1 \vee S_2)$.

The above map $\bar{\psi} : R^{we}_s(X_L) \longrightarrow R^e_L(X)$ is *not* a $\wedge$-complete homomorphism, as shown in the following example:

Example 20. Let $X = \{x\}$, $L = \{0, \alpha, \beta, 1 \mid 0 < \alpha < \beta < 1\}$ and $T_1 = \{(x_0, x_0), (x_0, x_\alpha), (x_\alpha, x_0), (x_\alpha, x_\alpha)\}$, $T_2 = \{(x_0, x_0), (x_0, x_\beta), (x_\beta, x_0), (x_\beta, x_\beta)\}$. Then $\bar{T_1} = \{((x, x), \alpha)\}$, $\bar{T_2} = \{((x, x), \beta)\}$ and $\overline{T_1 \cap T_2} = \{((x, x), 0)\}$. Hence $\bar{\psi}(T_1 \cap T_2) < \bar{\psi}T_1 \wedge \bar{\psi}T_2$.

In general for an arbitrary $T \in R^e_L(X)$, it may happen that $T \subset \bar{\phi} \circ \bar{\psi}(T)$, as shown in the following example:

Example 21. Let $X = \{x\}$, $L = \{0, 1 \mid 0 < 1\}$ and $T = \{(x_0, x_0)\}$. Then $\bar{T} = \{((x, x), 0)\}$ and $\bar{T}^{'} = \{(x_0, x_0), (x_0, x_1), (x_1, x_0)\}$. Therefore $T \subset \bar{T}^{'}$. Hence $T \subset \bar{\phi} \circ \bar{\psi}(T)$.

Representation of *L*-equivalence relations

5.2. Corollary. *For any set X and for any complete lattice L, the complete semi lattice of all L-equivalence relations on X is isomorphic to a complete sub semi lattice of the complete semi lattice of crisp weak equivalence relations on the L-point-set X_L.*

Proof. It follows from the Proposition 5.1.

6. Galois connection between *L*-*I*-ary relations and crisp *I*-ary relations valence relations

In this section we establish a Galois connection between the complete lattice of all L-fuzzy I-ary relations on a (crisp) set X and the (crisp) I-ary relations on the L-fuzzy-point-set of the given set, X_L.

6.1. Proposition. *For any set X and for any complete Brouwerian lattice L, let $\phi_I : {R_L}^I(X) \longrightarrow R^I(X_L)$ be defined by*

$$\phi_I S = S^{'} \text{ the associated crisp set for } S$$

and $\psi_I : R^I(X_L) \longrightarrow {R_L}^I(X)$ be defined by

$$\psi_I T = \bar{T} \text{ the associated } L - subsetfor\ T.$$

Then the following are true:

(1) $S_1 \leq S_2 \Rightarrow \phi_I(S_1) \subseteq \phi_I(S_2)$.
(2) $T_1 \subseteq T_2 \Rightarrow \psi_I(T_1) \leq \psi_I(T_2)$.
(3) ϕ_I *is a $\wedge$-complete homomorphism of the complete lattices.*
(4) ψ_I *is a $\vee$-complete homomorphism of the complete lattices.*
(5) $\cup_{j \in J} \phi_I(S_j) \subseteq \phi_I(\vee_{j \in J} S_j)$ *holds in general.*
(6) $\psi_I(\cap_{j \in J} T_j) \leq \wedge_{j \in J} \psi_I(T_j)$ *holds in general.*
(7) $\psi_I \circ \phi_I = 1$.
(8) $\phi_I \circ \psi_I \supseteq 1$.

Proof. It is tedious but straight forward and similar to that of 4.1.

Representation of *L*-*I*-ary relations

6.2. Corollary. *For any set X and for any complete lattice L, the complete semilattice of all L-I-ary relations on X is isomorphic to a complete sub semilattice of the complete semilattice of crisp I-ary relations on the L-point-set X_L.*

Proof. It follows from the Proposition 6.1.

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References

[1] J. A. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications, **18**(1967), 145–174.

[2] Nistala V.E.S. Murthy, *How fuzzy are the fuzzy ideals?*, Journal of Fuzzy Mathematics, **4**(1), 15–23.

[3] Nistala V.E.S. Murthy and Peruru G. Prasad, *Galois connections between fuzzy, binary and equivalence relations and the crisp, binary and equivalence relations*, Proceedings of The National Seminar on Topology, Category Theory and their applications to Computer Science, 120–129, March 11–13, 2004, Department of Mathematics, St Joseph's College, Irinjalaguda, Kerala (organized by the Kerala Mathematical Society.)

[4] Nistala V.E.S. Murthy and Peruru G. Prasad, Representation of L-Fuzzy Relations via a Galois Connection, Communicated.

[5] Nistala V.E.S. Murthy and Peruru G. Prasad, Some (Fuzzy) Binary Relations Are Fixed Points Of Certain Operators, Communicated.

[6] R. Belohlavek, Fuzzy Relational Systems, Kluwer Academic/Plenum Publishers, New York, 2002.

[7] R. Belohlavek, *Reduction and a simple proof of characterization of fuzzy concept lattices*, Fundamenta Informaticae, **46**(2001), 277–285.

[8] G. Szasz, An Introduction to Lattice Theory, Academic Press, New York.

[9] R. R. Yager, *Some properties of fuzzy relationships*, Cybernetics and systems, **12**(1981), 123–140.

[10] L. Zadeh, *Fuzzy sets*, Information and Control, **8**(1965), 338–353.

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