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ON RELAXATION NORMALITY IN THE FUGLEDE-PUTNAM THEOREM FOR A QUASI-CLASS A OPERATORS

M. H. M. RASHID AND M. S. M. NOORANI

Abstract. Let T be a bounded linear operator acting on a complex Hilbert space $\mathcal{H}$. In this paper, we show that if A is quasi-class A, B^* is invertible quasi-class A, X is a Hilbert-Schmidt operator, $AX = XB$ and $\||A^*|\| \, \||B|^{-1}\| \le 1$, then $A^*X = XB^*$.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbf{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathbf{B}(\mathcal{H})$, we shall write $\ker(T)$, $ran(T)$ for the null space and range of T, respectively. An operator T is said to be *positive* (denoted by $T \ge 0$) if $\langle Tx, x\rangle \ge 0$ for all $x \in \mathcal{H}$ and also T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

Recall ([1, 10, 12]) that an operator T is called *p-quasihyponormal* if $T^*((T^*T)^p - (TT^*)^pT) \ge 0$ for $p \in (0, 1]$, and T is called *paranormal* if $\|T^2x\| \ge \|Tx\|^2$ for all unit vector $x \in \mathcal{H}$. Following [7, 9, 13] we say that $T \in \mathbf{B}(\mathcal{H})$ belongs to *class* A if $|T^2| \ge |T|^2$ and T is called *normaloid* if $\|T^n\| = \|T\|^n$, for $n \in \mathbb{N}$ (equivalently, $\|T\| = r(T)$, the spectral radius of T). Recall [2], an operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *ω-hyponormal* if $|\widetilde{T}| \ge |T| \ge |\widetilde{T^*}|$. We remark that ω-hyponormal operator is defined by using Aluthge transformation $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. An operator T is said to be *quasi-class* A if

$$T^* \left|T^2\right| T \ge T^* |T|^2 T.$$

The *quasi-class* A operators were introduced , and their properties were studied in [11]. (see also [6]). In particular, it was shown in [11] that the class of *quasi-class* A operators contains properly classes of *class* A and *p-quasihyponormal* operators. *quasi-class* A operators were independently introduced by Jeon and Kim [11]. They gave an example of a *quasi-class* A operator which is not *paranormal* nor *normaloid*. Jeon and Kim example show that neither the class *paranormal* operators nor the class of *quasi-class* A contains the other. We shall denote classes of ω-hyponormal, p-quasihyponormal operators, paranormal operators, normaloid operators, class A operators, and quasi-class A operators by $\omega\mathcal{H}, \mathcal{QH}(p), \mathcal{PN}, \mathcal{N}, \mathcal{A}$, and $\mathcal{QA}$, respectively. It is well known that

$$\omega\mathcal{H} \subset \mathcal{A} \subset \mathcal{PN} \subset \mathcal{N} \qquad \text{and} \qquad \mathcal{QH}(p) \subset \mathcal{PN} \subset \mathcal{N},$$

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also, the following inclusions holds;

$$\mathcal{A} \subset \mathcal{QA} \qquad \text{and} \qquad \mathcal{QH}(p) \subset \mathcal{QA}.$$

2. Properties of quasi-class A operators

We begin this section by the following famous lemma.

Lemma 2.1.([11]) *Let $T \in \mathbf{B}(\mathcal{H})$. Suppose that T is quasi-class A with not have dense range. Then*

$$T = \begin{pmatrix} B & S \\ 0 & 0 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{ran(T)} \oplus \ker(T^*),$$

where $B = T|_{\overline{ran(T)}}$ is the restriction of T to $\overline{ran(T)}$, and B is belongs to class A. Moreover, $\sigma(T) = \sigma(B) \cup \{0\}$.

It is a slight generalization of [1, Corollary 3] we have

Theorem 2.2. *If $T \in \mathbf{B}(\mathcal{H})$ is a quasi-class A and T^* is ω-hyponormal, then T is normal.*

Proof. Since T^* is ω-hyponormal, it follows from [1, Theorem 4] that $|T^{*2}| \geq |T^*|^2$ and $|T^2| \leq |T|^2$. Hence $T^*|T^2|T = T^*|T|^2T$ because T is quasi-class A. Now $T^*|T|^2T = T^{*2}T^2 = |T|^4 = (T^*T)^2$ and $P(T^*T - TT^*)P = 0$, where P is the orthogonal projection onto $\overline{ran(T)}$. Let

$$T = \begin{pmatrix} B & S \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{ran(T)} \oplus \ker(T^*)$$

be the matrix representation of T. Then $P(T^*T - TT^*)P = 0$ implies that $B^*B = BB^* + SS^*$. Since T^* is ω-hyponormal, then T^* belongs to class A, so we have

$$\begin{aligned} B^*B = BB^* + SS^* = PTT^*P \leq P|T^{*2}|P &= (B^2B^{*2} + BSS^*B^*)^{\frac{1}{2}} \\ &= [B(BB^* + SS^*)B^*]^{\frac{1}{2}} \\ &= BB^*. \end{aligned}$$

Hence $S = 0$ and B is normal. Therefore T is normal.

3. Generalized Fuglede-Putnam Theorem

Let $\mathcal{C}_2$ denote the Hilbert-Schmidt class. Let $T \in \mathcal{C}_2$. Suppose $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$. We define the Hilbert-Schmidt norm of T to be

$$\|T\|_2 = \left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{\frac{1}{2}}.$$

This definition is independent of the choice of basis (see[5]). If $\|T\|_2 < \infty$, T is said to be a Hilbert-Schmidt operator.
Let $\mathcal{C}_1$ be the set $\{C = AB | A, B \in \mathcal{C}_2\}$. Then operators belonging to $\mathcal{C}_1$ are called trace class operators.
We define a linear functional

$$tr : \mathcal{C}_1 \to \mathbb{C} \quad \text{by} \quad tr(T) = \sum_{n=1}^{\infty} \langle Ce_n, e_n \rangle$$

for an orthonormal basis $\{e_n\}$ for $\mathcal{H}$.
In this case, the definition of $tr(C)$ does not depend on the choice of an orthonormal basis and $tr(C)$ is called the trace of C.

Theorem 3.1.([5]) *We have the following properties.*

(a) *The set $\mathcal{C}_2$ is self adjoint ideal of $\mathbf{B}(\mathcal{H})$.*

(b) *If $\langle A, B \rangle = \sum_{n=1}^{\infty} \langle Ae_n, Be_n \rangle = tr(B^*A) = tr(AB^*)$ for A and B in $\mathcal{C}_2$ and for any orthonormal basis $\{e_n\}$ for $\mathcal{H}$, then $\langle ., . \rangle$ is an inner product on $\mathcal{C}_2$ and $\mathcal{C}_2$ is a Hilbert-Schmidt space with respect to this inner product.*

Theorem 3.2.([5])*If $T \in \mathbf{B}(\mathcal{H})$ and $A \in \mathcal{C}_2$, then*

(i) $\|A\| \leq \|A\|_2$,

(ii) $\|TA\|_2 \leq \|T\| \|A\|_2$,

(iii) $\|AT\|_2 \leq \|A\|_2 \|T\|$.

For each pair of operators $A, B \in \mathbf{B}(\mathcal{H})$, there is an operator Γ defined on $\mathcal{C}_2$ via the formula $\Gamma X = AXB$, which due to [3]. Evidently, by Theorem 3.1 and Theorem 3.2, $\|\Gamma\| \leq \|A\| \|B\|$ and the adjoint of Γ is given by the formula $\Gamma^* X = A^* X B^*$, its easily to see that from the calculation $\langle \Gamma^* X, Y \rangle = \langle X, \Gamma Y \rangle = \langle X, AYB \rangle = tr((AYB)^* X) = tr(XB^*Y^*A^*) = tr(A^*XB^*Y^*) = \langle A^*XB^*, Y \rangle$. If $A \geq 0$ and $B \geq 0$, then also $\Gamma \geq 0$ and $\Gamma^{\frac{1}{2}} X = A^{\frac{1}{2}} X B^{\frac{1}{2}}$ because of

$$\begin{aligned} \langle AX, X \rangle &= tr(AXBX^*) = tr(A^{\frac{1}{2}} XBX^* A^{\frac{1}{2}}) \\ &= tr((A^{\frac{1}{2}} X B^{\frac{1}{2}})(A^{\frac{1}{2}} X B^{\frac{1}{2}})^*) \geq 0. \end{aligned}$$

The classical Fuglede-Putnam theorem asserts that if $A, B, X \in \mathbf{B}(\mathcal{H})$ such that $AX = XB$, and if A and B are normal, then also $A^*X = XB^*$ (see [8, Problem 192]).

Theorem 3.3. *Let $A, B^*, X \in \mathbf{B}(\mathcal{H})$. Suppose that A is quasi-class A, B^* is invertible quasi-class A and X is a Hilbert-Schmidt operator. Assume that $AX = XB$, then the operator Γ defined by $\Gamma X = AXB$ is a quasi-class A operator.*

Proof. Since A and B^* are quasi-class A, we have

$$\begin{aligned}(\Gamma^* \left|\Gamma^2\right| \Gamma - \Gamma^* \left|\Gamma\right|^2 \Gamma)X &= A^*|A^2|AXB|B^{*2}|B^* - A^*|A|^2AXB|B^*|^2B^* \\ &= A^*(|A^2| - |A|^2)AXB|B^{*2}|B^* \\ &+ A^*|A|^2AX(B|B^{*2}|B^* - B|B^*|^2B^*) \\ &\geq 0\end{aligned}$$

this show that Γ is quasi-class A.

Lemma 3.4.(Hölder-McCarthy Inequality) *Let B be a positive operator. Then the following inequalities hold for all $x \in \mathcal{H}$;*
(1) $\langle B^\alpha x, x\rangle \leq \langle Bx, x\rangle^\alpha \|x\|^{2(1-\alpha)}$, for $0 < \alpha \leq 1$.
(2) $\langle B^\alpha x, x\rangle \geq \langle Bx, x\rangle \|x\|^{2(1-\alpha)}$ for $\alpha \geq 1$.

Lemma 3.5. *Let $B \in \mathbf{B}(\mathcal{H})$. If B^* is quasi-class A and invertible, then $(B^*)^{-1}$ is quasi-class A.*

Proof. We cite the following obvious result (see [9]): Let S be an invertible operator. Then

$$(S^*S)^\lambda = S^*(SS^*)^{\lambda-1}S \qquad \text{holds for any real number} \lambda. \tag{3.1}$$

Suppose that B^* is quasi-class A and invertible. Then

$$\begin{aligned}BBB^*B^* = B|B^*|^2B^* &\leq B|B^{*2}|B^* \qquad (B^* \text{ is quasi-class } A) \\ &= B(B^2B^{*2})^{\frac{1}{2}}B^* \\ &= B^3(B^{*2}B^2)^{\frac{-1}{2}}B^{*3} \qquad (\text{by Equation } 3.1)\end{aligned} \tag{3.2}$$

(3.2) holds if and only if

$$B^{-1}B^{*-1} \leq (B^{*2}B^2)^{\frac{-1}{2}} = (B^{-2}B^{*-2})^{\frac{1}{2}} = |B^{*-2}|.$$

Hence,
$|B^{*-1}|^2 \leq |B^{*-2}|$ and so $B^{-1}|B^{*-1}|^2B^{*-1} \leq B^{-1}|B^{*-2}|B^{*-1}$. This end the proof.

Theorem 3.6. *Let $A, B, X \in \mathbf{B}(\mathcal{H})$. If A is quasi-class A, B^* is invertible quasi-class A and X is a Hilbert-Schmidt operator. Suppose that $AX = XB$ and $|||A^*||| \left\||B|^{-1}\right\| \leq 1$, then $A^*X = XB^*$.*

Proof. Let Γ be a Hilbert-Schmidt operator defined by $\Gamma X = AXB^{-1}$ for all $X \in \mathcal{C}_2$. Since $(B^*)^{-1} = (B^{-1})^*$ is quasi-class A then Theorem 3.3 implies Γ is quasi-class A. Since $\Gamma X = X$ and since Γ is quasi-class A, we have

$$\left\langle|\Gamma^2|X, X\right\rangle \geq \left\langle|\Gamma|^2X, X\right\rangle, \tag{3.3}$$

by Hölder-McCarthy inequality, we have

$$\begin{aligned}
\||\Gamma|X\|^2 &= \left\langle |\Gamma|^2 X, X \right\rangle \\
&\leq \left\langle (\Gamma^{*2}\Gamma^2)^{\frac{1}{2}} X, X \right\rangle && \text{(by Equation 3.3)} \\
&\leq \|X\| \left\langle \Gamma^{*2}\Gamma^2 X, X \right\rangle^{\frac{1}{2}} && \text{(by Hölder McCarthy inequality)} \\
&= \|X\| \left\langle \Gamma^2 X, \Gamma^2 X \right\rangle^{\frac{1}{2}} \\
&= \|X\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
\|\Gamma^* X\| &= \||\Gamma^*|X\| \\
&\leq \||A^*|\| \left\||B|^{-1}\right\| \|X\| \\
&\leq \|X\|.
\end{aligned}$$

Thus $\|\Gamma^* X - X\|^2 \leq \|\Gamma^* X\|^2 - 2\|X\|^2 + \|X\|^2 \leq 0$. So, $A^* X (B^*)^{-1} = X$ which ends the proof.

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References

[1] A. Aluthge and D. Wang, *An operator inequality which implies paranormality*, Math. Ineq. Appl., **2**(1999), 113–119.

[2] A. Aluthge and D. Wang, *ω-hyponormal operators*, Integral Equations Operator Theory, **36**(2000), 1–10.

[3] S. K. Berberian, *An extension of a theorem of Fuglede-Putnam*, Proc. Amer. Math. Soc., **71**(1978), 113–114.

[4] M. Chō and T. Yamazaki, *An operator transform from class A to the class of hyponormal operators and its application*, Integral Equations Operator Theory, **53**(2005), 497–508.

[5] J. B. Conway, A Course in Functional Analysis, Second Edition, Springer-Verlag, New york, 1990.

[6] B. P. Duggal, I. H. Jeon and I. H. Kim, *On Weyl's theorem for quasi-class A operators*, J. Korean Math. Soc., **43**(2006), 899–909.

[7] T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math., **1**(1998), 389–403.

[8] Halmos P. R., A Hilbert Space Problem Book, Second Edition, Springer-Verlag, New York , 1982.

[9] M. Ito, *Several properties on class A including p-hyponormal and log-hyponormal operators*, Math. Ineq. Appl., **2**(1999), 569–578.

[10] I. H. Jeon, J. I. Lee and A. Uchiyama, *On p-quasihyponormal operators and quasisimilarity*, Math. Ineq. App., **6**(2003) 309–315.

[11] I. H. Jeon and I. H. Kim, *On operators satisfying* $T^*|T^2|T \geq T^*|T|^2T^*$, Linear alg. Appl., **418**(2006), 854–862.

[12] I. H. Kim, *On* (p,k)*-quasihyponormal operators*, Math. Ineq. Appl., **4**(2004), 169–178.

[13] A. Uchiyama, *Weyl's theorem for class A operators*, Math. Ineq. App., **4**(2001), 143–150.

Department of Mathematics and Statistics, Faculty of Science P.O. Box(7), Mu'tah University, Mu'tah-Jordan.

E-mail: malik_okasha@yahoo.com

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM, Selangor Darul Ehsan, Malsysia.

E-mail: msn@pkrisc.cc.ukm.my