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On the numerical radius of an upper triangular operator matrix

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Abstract. The main purpose of this paper is to give an improvement of numerical radius inequality for an upper triangular operator matrix.

Keywords. Numerical range, numerical radius, essential numerical range, upper triangular operator matrix.

1 Introduction

Let H be infinite dimensional complex Hilbert space and let T be a bounded linear operator on H. The numerical range and the numerical radius of T are defined by

$$W(T) := \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$$

and

$$w(T) := \sup\{|z| : z \in W(T)\}$$

respectively. It is well known that $W(A)$ is a convex set in the complex plane $\mathbb{C}$, its closure contains the spectrum of A, see [5]. Also, [8],

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\| \tag{1.1}$$

and hence the numerical radius defines an equivalent norm in the Banach algebra $B(H)$, the set of all bounded operators in H. The first inequality becomes an equality if $T^2 = 0$, and the second inequality becomes an equality if T is normal.

Let H_1 and H_2 be Hilbert spaces. We consider the following upper triangular operator of the form

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \tag{1.2}$$

defined on the Hilbert space $H = H_1 \oplus H_2$ where $A \in B(H_1)$, $B \in B(H_2)$ and $C \in B(H_2, H_1)$, the set of all bounded linear operators from H_1 into H_2. A related, and seemingly more demanding,

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problem is the following. Let T be a bounded linear operator on H, and E be a T-invariant closed subspace of H. Then T takes the form

$$T = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} : E \oplus E^{\perp} \to E \oplus E^{\perp}$$

which motivated the interest in 2×2 upper-triangular operator matrices. It is well known that, [6],

$$w\left(M_C\right) \leq \max(w(A), w(B)) + \frac{\|C\|}{2}. \tag{1.3}$$

The main purpose of this paper is to give an improvement of the inequality (1.3). Among other applications, our results can be viewed as an extension of some earlier works, see [4] [6], and references therein.

2 Main Results

We start this section by recalling the following lemma, which will be used subsequently.

Lemma 2.1. *Let $A \in B(H_1)$, $B \in B(H_2)$ and $C \in B(H_2, H_1)$, then*

1. $W\left(M_O\right) = conv(W(A) \cup W(B))$.
2. $\max(w(A), w(B)) \leq w\left(M_C\right) \leq \max(w(A), w(B)) + \frac{\|C\|}{2}$.

Here $conv(E)$ denotes the convex hull of a subset E of $\mathbb{C}$.

Proof. (1) is obvious.

(2) The first inequality follows from the pinching inequality for the numerical radius, while the second inequality can be found in [6]. □

Recall that the essential numerical range $W_e(T)$ is (by definition) the numerical range of the coset $T + K(H)$ in the Calkin algebra $B(H)/K(H)$ where $K(H)$ is the ideal of all compact operators on H, see [1]. Equivalently,

$$W_e(T) = \cap \overline{W(T + K)},$$

where the intersection runs over the compact operators K. It follows that $W_e(T)$ is a compact convex and invariant under compact perturbation. Also,

$$W_e(T) = \{\lambda \in \mathbb{C} : \exists x_n \in H \text{ with } \|x_n\| = 1, \, x_n \xrightarrow{\text{weakly}} 0, \quad \text{and } \langle T x_n, x_n \rangle \longrightarrow \lambda\}.$$

Analogously, the essential numerical radius of T, is defined by

$$w_e(T) := \sup\{|\lambda| : \lambda \in W_e(T)\}.$$

Now, we are ready to give our main theorem.

Theorem 2.1. *Let H_1 and H_2 be two Hilbert spaces, and let $A \in B(H_1)$, $B \in B(H_2)$ and $C \in B(H_2, H_1)$ such that $w_e\left(\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}\right) < \frac{\|C\|}{2}$ and $W\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$ is an open set with $w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = r$. Then*

$$w\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) < r + \frac{\|C\|}{2} \text{ if and only if } C \neq 0.$$

Proof. It's obvious that if $C = 0$ then $w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = r$. Let assume now that $C \neq 0$ and

$$w(M_C) = r + \frac{\|C\|}{2}.$$

Let $x_n, n \geq 1$, be unit vectors in $H_1 \oplus H_2$ such that

$$|\langle M_C x_n, x_n \rangle| \longrightarrow r + \frac{\|C\|}{2}. \tag{2.1}$$

We have

$$|\langle M_C x_n, x_n \rangle| \leq \left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} x_n, x_n \right\rangle \right| + \left| \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_n, x_n \right\rangle \right| \longrightarrow r + \frac{\|C\|}{2}.$$

However, for all $x_n, n \geq 1$, we have

$$\left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} x_n, x_n \right\rangle \right| \leq r \qquad \text{and} \qquad \left| \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_n, x_n \right\rangle \right| \leq \frac{\|C\|}{2}.$$

By taking a subsequence (x_{n_k}), if it is necessary such that

$$\left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} x_{n_k}, x_{n_k} \right\rangle \right| \longrightarrow a \qquad \text{and} \qquad \left| \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_{n_k}, x_{n_k} \right\rangle \right| \longrightarrow b$$

it is easy to see that $a = r$ and $b = \frac{\|C\|}{2}$. So, we can assume that

$$\left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} x_n, x_n \right\rangle \right| \longrightarrow r \qquad \text{and} \qquad \left| \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_n, x_n \right\rangle \right| \longrightarrow \frac{\|C\|}{2}.$$

Now, the sequence $(x_n)_n$ contains a subsequence noted again by $(x_n)_n$ converges weakly in $H_1 \oplus H_2$ to x_0, with $\|x_0\| \leq 1$. We have $x_0 \neq 0$ because if $x_0 = 0$, then there exists $\lambda \in W_e\left(\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}\right)$ such that $|\lambda| = \frac{\|C\|}{2}$ which contradict our assumption. Also, if x_n converge in norm to x_0 then we find $\left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} x_0, x_0 \right\rangle \right| = r$ which also impossible, thus $x_n, n \geq 1$ doesn't converge in norm, so $0 < \|x_0\| < 1$. Therefore,

$$\begin{aligned}
\left| \left\langle \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} x_n, x_n \right\rangle \right| &= \left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} x_n, x_n \right\rangle + \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_n, x_n \right\rangle \right| \\
&= \left| \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} (x_n - x_0), (x_n - x_0) \right\rangle - \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_0, x_0 \right\rangle + \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_n, x_0 \right\rangle \right. \\
&\qquad \left. + \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_0, x_n \right\rangle + \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} x_n, x_n \right\rangle \right| \\
&= \left| \|x_n - x_0\|^2 \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \frac{(x_n - x_0)}{\|x_n - x_0\|}, \frac{(x_n - x_0)}{\|x_n - x_0\|} \right\rangle - \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_0, x_0 \right\rangle \right. \\
&\qquad \left. + \left\langle x_n, \begin{bmatrix} 0 & 0 \\ C^* & 0 \end{bmatrix} x_0 \right\rangle + \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_0, x_n \right\rangle + \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} x_n, x_n \right\rangle \right| \\
&\leq \left| \|x_n - x_0\|^2 \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \frac{(x_n - x_0)}{\|x_n - x_0\|}, \frac{(x_n - x_0)}{\|x_n - x_0\|} \right\rangle - \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_0, x_0 \right\rangle \right. \\
&\qquad \left. + \left\langle x_n, \begin{bmatrix} 0 & 0 \\ C^* & 0 \end{bmatrix} x_0 \right\rangle + \left\langle \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} x_0, x_n \right\rangle \right| + r.
\end{aligned}$$

From (2.1) and by letting n to $+\infty$, we obtain

$$r+\frac{\|C\|}{2}\leq\left|(1-\|x_0\|^2)\lambda_1-\left\langle\begin{bmatrix}0 & C\\ 0 & 0\end{bmatrix}x_0,x_0\right\rangle+\left\langle x_0,\begin{bmatrix}0 & 0\\ C^* & 0\end{bmatrix}x_0\right\rangle+\left\langle\begin{bmatrix}0 & C\\ 0 & 0\end{bmatrix}x_0,x_0\right\rangle\right|+r$$
$$\leq(1-\|x_0\|^2)|\lambda_1|+\|x_0\|^2\frac{\|C\|}{2}+r$$

where $\lambda_1\in W_e\left(\begin{bmatrix}0 & C\\ 0 & 0\end{bmatrix}\right)$. Hence

$$(1-\|x_0\|^2)|\lambda_1|+\|x_0\|^2\frac{\|C\|}{2}+r\geq\frac{\|C\|}{2}+r.$$

Thus

$$(1-\|x_0\|^2)|\lambda_1|\geq(1-\|x_0\|^2)\frac{\|C\|}{2}.$$

So $|\lambda_1|\geq\frac{\|C\|}{2}$, which is impossible. Consequently, $w\left(\begin{bmatrix}A & C\\ 0 & B\end{bmatrix}\right)<r+\frac{\|C\|}{2}$. □

Remark 1. The condition in Theorem 2.1 that $W\left(\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\right)$ be an open set is essential, we can easily construct an operator on $H_1\oplus H_2$ not satisfying this condition, let $C\in B(H_2,H_1)$ and

$$w\left(\begin{bmatrix}I_{H_1} & C\\ 0 & I_{H_2}\end{bmatrix}\right)=1+w\left(\begin{bmatrix}0 & C\\ 0 & 0\end{bmatrix}\right)=1+\frac{\|C\|}{2},$$

with $C\neq0$.

As a consequence of Theorem 2.1, we have the following result.

Corollary 2.2. *Let $C\in B(H_2,H_1)$ be a compact operator and $W\left(\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\right)$ is an open set with $w\left(\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\right)=r$. Then $w(M_C)<r+\frac{\|C\|}{2}$ if and only if $C\neq0$.*

Proof. Since the essential numerical range of the compact operator $\begin{bmatrix}0 & C\\ 0 & 0\end{bmatrix}$ on $H_1\oplus H_2$ is always equal to $\{0\}$, then the result follows from Theorem 2.1. □

An operator F_C in $l^2\times l^2$ is said to be Foguel operator if

$$F_C=\begin{bmatrix}S^* & C\\ 0 & S\end{bmatrix}$$

where $C\in B(l^2)$ and the unilateral shift S defined on l^2 by $Se_n=e_{n+1}$, for an orthonormal basis $\{e_n\}$ on l^2. This operator plays an important role in many applications, see [2], [9], [3] and reference therein. The following result gives an estimate of the numerical radius of the Foguel operator,

Corollary 2.3. *[4, Theorem 2.6.] Let C be a compact operator in $B(l^2)$. Then $w(F_C)<1+\frac{\|C\|}{2}$ if and only if $C\neq0$.*

Proof. It follows immediately from Corollary 2.2, if we take $A = S^*$ and $B = S$. Since $W(S) = \mathbb{D}$ (the open unit disc of $\mathbb{C}$) and the essential numerical range of a compact operator is always equal to $\{0\}$. ☐

Remark 2. Corollary 2.2 generalizes [4, Theorem 2.6.].

Example 1. One can easily construct an operator F_C which satisfies the conditions of Theorem 2.1, for that we take the diagonal operator $C = \mathrm{diag}(\sqrt{2}, 1, 1, ...)$ in l^2.

Clearly, we have

$$w_e\left(\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}\right) < \frac{\|C\|}{2} = \frac{\sqrt{2}}{2}.$$

Indeed,

$$\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|cccc} & \sqrt{2} & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ \hline & & & & \end{array}\right]$$

$$= \left[\begin{array}{c|cccc} & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ \hline & & & & \end{array}\right] + \left[\begin{array}{c|cccc} & \sqrt{2} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots \\ \hline & & & & \end{array}\right].$$

So

$$W_e\left(\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}\right) \subseteq \overline{W\left(\left[\begin{array}{c|cccc} & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ \hline & & & & \end{array}\right]\right)}.$$

Since

$$\left[\begin{array}{c|cccc} & \sqrt{2} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots \\ \hline & & & & \end{array}\right]$$

is compact and taking account of [8, Theorem 2], we obtain

$$w\left(\begin{bmatrix} S^* & C \\ 0 & S \end{bmatrix}\right) = \frac{1}{2}\left\| \left|\begin{bmatrix} S & 0 \\ 0 & S^* \end{bmatrix}\right|^{2(1-\alpha)} + \left|\begin{bmatrix} S^* & 0 \\ 0 & S \end{bmatrix}\right|^{2\alpha} + \left|\begin{bmatrix} 0 & 0 \\ C^* & 0 \end{bmatrix}\right|^{2(1-\alpha)} + \left|\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}\right|^{2\alpha} \right\|$$

with $\alpha = 1/2$. Hence,

$$w\left(\begin{bmatrix} S^* & C \\ 0 & S \end{bmatrix}\right) \leq \frac{1}{2}\left\| \begin{bmatrix} S^*S & 0 \\ 0 & SS^* \end{bmatrix} + \begin{bmatrix} SS^* & 0 \\ 0 & S^*S \end{bmatrix} + \begin{bmatrix} CC^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C^*C \end{bmatrix} \right\|$$

$$= \frac{1}{2}\left\| \left[\begin{array}{cccc|cccc} 1 &&&&&&& \\ & 2 &&&&&& \\ && 2 &&&&& \\ &&& \ddots &&&& \\ \hline &&&& 1 &&& \\ &&&&& 2 && \\ &&&&&& 2 & \\ &&&&&&& \ddots \end{array}\right] + \left[\begin{array}{cccc|cccc} 2 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& \ddots &&&& \\ \hline &&&& 2 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& \ddots \end{array}\right] \right\|$$

$$= \frac{3}{2} < 1 + \frac{\|C\|}{2}.$$

Example 2. [4], Let C be a weighted shift operator in $B(l^2)$ with weights $\{s_n\}$ and $\lim_{n\to\infty} s_n = k < \sup\{s_n\}$, then $w(F_C) < \frac{\|C\|}{2} + 1$.

Indeed, we note first that $\sup\{s_n\} = \|C\|$, and as $\lim_{n\to\infty} s_n = k < \sup\{s_n\}$, there exists $n_1 \in \mathbb{N}$ such that $s_n < \sup\{s_n\}$ for all $n \geq n_1$. We have

$$\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|cccc} & 0 &&& \\ & s_1 & 0 && \\ && s_2 & \ddots & \\ &&& \ddots & \ddots \\ \hline && && \end{array}\right]$$

$$= \left[\begin{array}{c|ccccc} & 0 &&&& \\ & s_1 & 0 &&& \\ && \ddots & 0 && \\ &&& s_{n_1} & \ddots & \\ &&&& 0 & \ddots \\ \hline &&&&& \end{array}\right] + \left[\begin{array}{c|ccccc} & 0 &&&& \\ & 0 & 0 &&& \\ && \ddots & 0 && \\ &&& s_{n_1+1} & \ddots & \\ &&&& \ddots & \ddots \\ \hline &&&&& \end{array}\right].$$

So

$$w_e\left(\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}\right) = w_e\left(\left[\begin{array}{c|ccccc} & 0 &&&& \\ & 0 & 0 &&& \\ && \ddots & 0 && \\ &&& s_{n_1+1} & \ddots & \\ &&&& \ddots & \ddots \\ \hline &&&&& \end{array}\right]\right) < \frac{\|C\|}{2}.$$

Therefore from Theorem 2.1 it follows that $w(F_C) < 1 + \dfrac{\|C\|}{2}$.

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