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On a class of Kirchhoff type problems with singular exponential nonlinearity

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Abstract. We study the following singular Kirchhoff type problem

$$(P)\begin{cases} -m\left(\int\limits_{\Omega}|\nabla u|^2\,dx\right)\Delta u = h(u)\frac{e^{\alpha u^2}}{|x|^\beta} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary and $0\in\Omega$, $\beta\in[0,2)$, $\alpha>0$ and m is a continuous function on $\mathbb{R}^+$. Here, h is a suitable preturbation of $e^{\alpha u^2}$ as $u\to\infty$. In this paper, we prove the existence of solutions of (P). Our tools are Trudinger-Moser inequality with a singular weight and the mountain pass theorem.

Keywords. Trudinger-Moser inequality, exponential critical growth, mountain pass theorem

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$ containing the origin. In this article, we study the existence of solutions to the following singular Kirchhoff problems with exponential nonlinearities

$$\begin{cases} -m\left(\int\limits_{\Omega}|\nabla u|^2\,dx\right)\Delta u = h(u)\frac{e^{\alpha u^2}}{|x|^\beta} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}\tag{1.1}$$

where $\beta\in[0,2)$, $\alpha>0$ and $m:\mathbb{R}^+\to\mathbb{R}^+$, is a continuous function that satisfies some conditions which will be stated later on, and h satisfies the following conditions:

$(H1)$ $h\in C(\mathbb{R})$, $h(t)\geq 0$ for all $t\in\mathbb{R}$, $h(t)=0$ if $t<0$;

$(H2)$ $\lim\limits_{t\to 0^+}\frac{h(t)}{t}=0$ and $\lim\limits_{t\to+\infty}h(t)=0$.

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$(H3)$ The map $t \mapsto \frac{h(t)e^{\alpha t^2}}{t^3}$ is increasing for $t > 0$.

$(H4)$ There exists $\gamma > \frac{(2-\beta)^2}{2\alpha d^{2-\beta}} m(\frac{4\pi}{\alpha}(1-\frac{\beta}{2}))$ such that $0 < \gamma = \liminf_{t\to+\infty} t h(t) < \infty$,

where d is the radius of the largest open ball contained in Ω. As examples of a function satisfying the above assumptions, we have

Example 1. For $\alpha \geq 1$;

$$h(t) = \begin{cases} \frac{\gamma t^3}{1+t^4} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Example 2. For $0 < \alpha < 1$;

$$h(t) = \begin{cases} \frac{\gamma t^3}{1+t^4} & \text{if } t \geq \sqrt{\frac{2}{\alpha}}, \\ \frac{\gamma \alpha^2}{\alpha^2+4} t^3 & \text{if } 0 \leq t < \sqrt{\frac{2}{\alpha}}, \\ 0 & \text{if } t < 0. \end{cases}$$

The hypotheses on the function $m : \mathbb{R}^+ \to \mathbb{R}^+$ are the following.

(M_1) There exist real numbers m_0, $m_1, m_2 > 0$ such that for some $\kappa \in \mathbb{R}$

$$m_0 \leq m(t) \leq m_1 t^\kappa + m_2, \quad \text{for all } t \geq 0$$

(M_2) $M(s) + M(t) \leq M(s+t) \qquad \forall s, t \geq 0$ where $M(t) = \int\limits_0^t m(x)\, dx$

(M_3) $\frac{m(t)}{t}$ is noninreasing for $t > 0$.

A typical example of a function satisfying the conditions $(M_1) - (M_3)$ is given by $m(t) = m_0 + bt$ with $b > 0$ and for all $t \geq 0$. As a consequence of (M_3), a straightforward computation shows that $\frac{1}{2}M(t) - \frac{1}{4}m(t)\, t$ is nondecreasing for $t \geq 0$, which implies that

$$\frac{1}{2}M(t) - \frac{1}{4}m(t)\, t \geq 0. \tag{1.2}$$

Problem (1.1) is related to the stationary version of a model established by Kirchhoff [10]. More precisely, Kirchhoff proposed the following model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int\limits_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

which extends D'Alembert's wave equation with free vibrations of elastic strings, where ρ denotes the mass density, P_0 denotes the initial tension, h denotes the area of the cross section, E denotes the Young modulus of the material, and L denotes the length of the string.

Many interesting results for the problem of Kirchhoff type were obtained, see for example [5], [6], [9], [16], [8] and the references therein. The authors have used the variational method and the topological method to get the existence of solutions. In [8], by a direct variational approach, the authors establish the existence of a positive ground state solution for a nonlocal Kirchhoff of the type

$$\begin{cases} -m\left(\int\limits_{\Omega} |\nabla u|^2\, dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies some appropriate conditions. Our paper is closely related to the works of de Figueiredo et al. [8]. Indeed, we extend the results in [8] from $\beta = 0$ to $\beta \in [0, 2)$. This limitation on β is due to Lemma 2.1.

Now, we are ready to state our main result

Theorem 1.1. *Under assumptions* $(M_1) - (M_3)$ *and* $(H_1) - (H_4)$, *problem (1.1) admits a non-tirivial solution* $u \in H_0^1(\Omega)$.

This work is organised as follows: In Section 2, we present the variational setting in which our problem will be treated , and some preliminary results. Section 3 is devoted to show that the energy functional has the mountain pass geometry and in section 4 we obtain an estimate for the minimax level associated to our functional. Finally, we prove our main result in section 5.

2 Preliminary results

It is natural to find solution of our problem by looking for critical points of the corresponding functional of problem (1.1) which we define next.

Let $g(u) = h(u)\, e^{\alpha u^2}$ and $G(u) = \int\limits_0^u g(s)\, ds$, the functional associated to (1.1) is given by

$$I(u) = \frac{1}{2} M\left(\|u\|^2\right) - \int_\Omega \frac{G(u)}{|x|^\beta} dx,$$

where $\|u\| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}$. Under our assumptions this functional is well defined on $H_0^1(\Omega)$. Moreover, by standard arguments, $I \in C^1\left(H_0^1(\Omega), \mathbb{R}\right)$ with

$$\langle I'(u), \varphi\rangle = m\left(\|u\|^2\right) \int_\Omega \nabla u \nabla \varphi dx - \int_\Omega \frac{g(u)}{|x|^\beta} \varphi dx, \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

Let consider the following eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

From classical theory of Hilbert Spaces we get the next classical result (see [7])

Proposition 2.1. *There exists an eigenvalue sequence* $\{\lambda_k(\beta)\} \subset \mathbb{R}^+$, *with* $\lambda_k(\beta) \to \infty$ *as* $k \to \infty$ *for which problem (2.1) has nontrivial solution. Furthermore, the first eigenvalue* $\lambda_1(\beta)$

is simple and isolated and the corresponding eigenfunctions don't change sign in Ω. The first eigenvalue is variationally characterized as

$$\lambda_1(\beta) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{|x|^\beta} dx}. \tag{2.2}$$

The exponential nature of the nonlinearity $g(u)$ is motivated by the following version of Trudinger-Moser inequality with a singular weight due to Adimurthi-Sandeep [3].

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ containing 0 and $u \in H_0^1(\Omega)$. Then for every $\alpha > 0$ and $\beta \in [0, 2)$*

$$\int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} dx < \infty$$

Moreover,

$$\sup_{\|u\| \leq 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} dx < \infty \tag{2.3}$$

if and only if $\frac{\alpha}{4\pi} + \frac{\beta}{2} \leq 1$.

We end this section with a singular version of the following theorem of P. L. Lions [3]

Lemma 2.2. *Let (u_n) be a sequence in $H_0^1(\Omega)$ such that $\|u_n\| = 1$, for all $n \in \mathbb{N}^*$ and $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ for some $u \neq 0$. Then, for $p < 4\pi \left(1 - \frac{\beta}{2}\right)\left(1 - \|u\|^2\right)^{-1}$,*

$$\sup_{n \geq 1} \int_\Omega \frac{e^{pu_n^2}}{|x|^\beta} dx < \infty.$$

3 The Mountain Pass Geometry

In the sequel, we prove that the functional I has the Mountain Pass Geometry. This fact is proved in the next lemmas:

Lemma 3.1. *Assume (M_1), (H_1) and (H_2), then there exist positive constants τ and ρ such that*

$$I(u) \geq \tau > 0, \ \forall u \in H_0^1(\Omega) : \|u\| = \rho.$$

Proof. It follows from (H_2) that, for each $\varepsilon > 0$, there exists a positive constant C such that

$$|G(u)| \leq \varepsilon u^2 + Cu^3 e^{\alpha u^2},$$

Let $2 < q < \frac{4}{\beta}$. By (2.2) and generalized Hölder's inequality, we have

$$\begin{aligned}\int_\Omega \frac{|G(u)|}{|x|^\beta} dx &= \varepsilon \int_\Omega \frac{|u|^2}{|x|^\beta} dx + C \int_\Omega \frac{|u|^3 e^{\alpha u^2}}{|x|^\beta} dx \\ &\leq \frac{\varepsilon}{\lambda_1(\beta)} \int_\Omega |\nabla u|^2 dx + C \int_\Omega |u|^3 \frac{1}{|x|^{\frac{\beta}{2}}} \frac{e^{\alpha u^2}}{|x|^{\frac{\beta}{2}}} dx\end{aligned}$$

$$\leq \frac{\varepsilon}{\lambda_1(\beta)}\|u\|^2 + C\left(\int_\Omega |u|^{3p}dx+\right)^{\frac{1}{p}}\left(\int_\Omega \frac{1}{|x|^{\frac{q\beta}{2}}}dx\right)^{\frac{1}{q}}\left(\int_\Omega \frac{e^{2\alpha u^2}}{|x|^\beta}dx\right)^{\frac{1}{2}},$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. So, Using the Sobolev embedding theorem, there is a positive constant C such that

$$\int_\Omega \frac{|G(u)|}{|x|^\beta}dx \leq \frac{\varepsilon}{\lambda_1(\beta)}\|u\|^2 + C\|u\|^3\left(\int_\Omega \frac{1}{|x|^{\frac{q\beta}{2}}}dx\right)^{\frac{1}{q}}\left(\int_\Omega \frac{e^{2\alpha u^2}}{|x|^\beta}dx\right)^{\frac{1}{2}}$$

$$\leq \frac{\varepsilon}{\lambda_1(\beta)}\|u\|^2 + C\|u\|^3\left(\int_\Omega \frac{1}{|x|^{\frac{q\beta}{2}}}dx\right)^{\frac{1}{q}}\left(\int_\Omega \frac{e^{2\alpha\rho^2\left(\frac{u}{\|u\|}\right)^2}}{|x|^\beta}dx\right)^{\frac{1}{2}}.$$

The first integral on the right-hand side is finite since $q\beta < 4$. If $\rho \leq \sqrt{\frac{2\pi\left(1-\frac{\beta}{2}\right)}{\alpha}}$, the second integral is bounded by lemma 2.1. Thus, using the condition (M_1) one has

$$I(u) \geq \left(\frac{m_0}{2} - \frac{\varepsilon}{\lambda_1(\beta)}\right)\|u\|^2 - C_1\|u\|^3.$$

Consequently

$$I(u) \geq \left(\frac{m_0}{2} - \frac{\varepsilon}{\lambda_1(\beta)}\right)\rho^2 - C_1\rho^3.$$

Now, we may fix $\varepsilon > 0$ such that $\frac{m_0}{2} - \frac{\varepsilon}{\lambda_1(\beta)} > 0$. Thus, for $\rho > 0$ sufficiently small there exists $\tau := \left(\frac{m_0}{2} - \frac{\varepsilon}{\lambda_1(\beta)}\right)\rho^2 - C_1\rho^3 > 0$ such that

$$I(u) \geq \tau > 0, \ \forall u \in H_0^1(\Omega) \text{ with } \|u\| = \rho$$

The proof of Lemma is complete. □

Lemma 3.2. *Assume that conditions (M_1), (H_1) and (H_4) hold. Then, there exists $e_1 \in H_0^1(\Omega)$ with $\|e_1\| > \rho$ such that $I(e_1) < 0$.*

Proof. First, by assumption (M_1), we obtain

$$M(t) \leq \frac{m_1}{\kappa+1}t^{\kappa+1} + m_2 t. \tag{3.1}$$

On the other hand, fix $\varepsilon > 0$ and by (H_4), we get

$$th(t)e^{\alpha t^2} \geq (\gamma - \varepsilon)e^{\alpha t^2}, \quad \text{for } t > A_\varepsilon \ \text{ with } A_\varepsilon > 0.$$

Since $e^{\alpha t^2} \geq \frac{1}{\theta!}\alpha^\theta t^{2\theta}$ for all t and $\theta \in \mathbb{N}$, then there exists a constant $C_\varepsilon > 0$ such that

$$th(t)e^{\alpha t^2} \geq \frac{1}{\theta!}(\gamma - \varepsilon)\alpha^\theta t^{2\theta} - C_\varepsilon t, \quad \text{for } \ t > 0$$

and consequently

$$G(u) \geq \frac{1}{2\theta\theta!}(\gamma - \varepsilon)\alpha^\theta t^{2\theta} - C_\varepsilon t \ \text{ for } \ t > 0. \tag{3.2}$$

Let $u_0 \in H_0^1(\Omega)$ with $u_0 > 0$ in Ω and $\|u_0\| = 1$. Thus, from (3.1) and (3.2), we obtain

$$I(tu_0) \leq \frac{m_2}{2}t^2 + \frac{m_1}{2\kappa+2}t^{2(\kappa+1)} - \frac{1}{2\theta\theta!}\left(\gamma-\varepsilon\right)\alpha^{\theta}t^{2\theta}\int_{\Omega}\frac{{u_0}^{2\theta}}{|x|^{\beta}}dx - C_{\varepsilon}t\int_{\Omega}\frac{u_0}{|x|^{\beta}}dx$$

for all $t > 0$, which yields $I(tu_0) \to -\infty$ as $t \to +\infty$, provided that $\theta > \max\{2, 2\kappa+2\}$. Setting $e_1 = \bar{t}u_0$ with $\bar{t} > 0$ large enough, the proof is complete. □

4 On the mini-max level

In view of Lemmas 3.1 and 3.2, we may apply a version of the Mountain Pass theorem without Palais-Smale condition to obtain a sequence $u_n \in H_0^1(\Omega)$ such that

$$I(u_n) \to c_* \quad \text{and} \quad I'(u_n) \to 0,$$

where

$$c_* = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I\left(\gamma\left(t\right)\right) \tag{4.1}$$

with

$$\Gamma = \left\{\gamma \in C\left([0,1], H_0^1(\Omega)\right) : \gamma\left(0\right) = 0, \quad \gamma\left(1\right) < 0\right\}.$$

Let $B_d(x_0) \subset \Omega$ be an open ball where d was given in (H_4). We may assume that $x_0 = 0$. In order to get more information about the minimax level, it was crucial in our argument to consider the following concentrating functions $\psi_n(x) = \tilde{\psi}_n(\frac{x}{d})$, $n \in \mathbb{N}$ where

$$\tilde{\psi}_n(x) = \begin{cases} \frac{1}{\sqrt{2\pi}}(\log n)^{1/2} & \text{for } 0 \leq |x| \leq \frac{1}{n}, \\ \frac{1}{\sqrt{2\pi}}\frac{\log\frac{1}{|x|}}{(\log n)^{1/2}} & \text{for } \frac{1}{n} \leq |x| \leq 1, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

Then, ψ_n has support in $B_d(0)$ and $\|\psi_n\| = 1$ $\forall n \in \mathbb{N}$.

To show that the desired estimate for the level c_*, we will use the following inequality

Lemma 4.1. *The following inequality holds:*

$$\liminf_{n\to+\infty}\int_{B_d(0)}\frac{exp\left(4\pi(1-\frac{\beta}{2})\psi_n^2\right)}{|x|^{\beta}}dx \geq \frac{6\pi d^{(2-\beta)}}{(2-\beta)}.$$

Proof. Using the definition of $\tilde{\psi}_n$ and by change of variable, we have

$$\begin{aligned}\int_{B_d(0)}\frac{exp\left(4\pi(1-\frac{\beta}{2})\psi_n^2\right)}{|x|^{\beta}}dx &= d^{(2-\beta)}\int_{B_{\frac{1}{n}}(0)}\frac{exp\left(4\pi(1-\frac{\beta}{2})\widetilde{\psi}_n^2\right)}{|x|^{\beta}}dx \\ &+ d^{(2-\beta)}\int_{\frac{1}{n}\leq|x|\leq 1}\frac{exp\left(4\pi(1-\frac{\beta}{2})\widetilde{\psi}_n^2\right)}{|x|^{\beta}}dx \\ &= \frac{2\pi d^{(2-\beta)}}{(2-\beta)} + 2\pi d^{(2-\beta)}\int_{\frac{1}{n}}^{1} r^{1-\beta} exp\left(\frac{(2-\beta)\left(\log(\frac{1}{r})\right)^2}{\log(n)}\right)dr.\end{aligned}$$

Next, by using the change of variable $t=\frac{\log(\frac{1}{r})}{\log(n)}$, we obtain

$$\int_{B_d(0)} \frac{exp\left(4\pi(1-\frac{\beta}{2})\psi_n^2\right)}{|x|^\beta}dx = \frac{2\pi d^{(2-\beta)}}{(2-\beta)} + 2\pi d^{(2-\beta)}\log(n)\int_0^1 n^{(2-\beta)(t^2-t)}dt.$$

On the other hand, since

$$\begin{cases} t^2-t \geq -t & \text{for } t\in\left[0,\frac{1}{2}\right], \\ t^2-t\geq t-1 & \text{for } t\in\left[\frac{1}{2},1\right], \end{cases}$$

we get

$$\begin{aligned}\int_{B_d(0)} \frac{exp\left(4\pi(1-\frac{\beta}{2})\psi_n^2\right)}{|x|^\beta}dx &\geq \frac{2\pi d^{(2-\beta)}}{(2-\beta)} + 2\pi d^{(2-\beta)}\log(n)\int_0^{\frac{1}{2}} n^{-(2-\beta)t}dt \\ &\quad + 2\pi d^{(2-\beta)}\log(n)\int_{\frac{1}{2}}^1 n^{(2-\beta)(t-1)}dt \\ &\geq \frac{2\pi d^{(2-\beta)}}{(2-\beta)} + \frac{2\pi d^{(2-\beta)}}{(2-\beta)}\left(2-\frac{2}{n^{\frac{2-\beta}{2}}}\right).\end{aligned} \tag{4.2}$$

Passing to limit in (4.2), then the proof of lemma 4.1 is complete. □

We can now prove the following upper bounded for c_*.

Lemma 4.2. *With c_* defined as in (4.1), we have $c_* < \frac{1}{2}M\left(\frac{4\pi}{\alpha}\left(1-\frac{\beta}{2}\right)\right)$.*

Proof. Since $\psi_n\geq 0$ and $\|\psi_n\|=1$, we can deduce that $I(t\psi_n)\to-\infty$ as $t\to+\infty$. From (4.1), we have

$$c_* \leq \max_{t>0} I(t\psi_n), \quad \forall n\in\mathbb{N}.$$

For the sake of contraduction, Suppose that for all $\forall n\in\mathbb{N}$, we have

$$\max_{t>0} I(t\psi_n) \geq \frac{1}{2}M\left(\frac{4\pi}{\alpha}(1-\frac{\beta}{2})\right).$$

Since I possesses the mountain pass geometry, for each n, $\max_{t>0} I(t\psi_n)$ is attained at some $t_n>0$, that is $I(t_n\psi_n)=\max_{t>0} I(t\psi_n)$. Thus,

$$I(t_n\psi_n) = \frac{1}{2}M(t_n^2) - \int_\Omega \frac{G(t_n\psi_n)}{|x|^\beta}dx.$$

Using $G(t)\geq 0$ for all $\forall t\in\mathbb{R}$, one can deduce that

$$\frac{1}{2}M(t_n^2) \geq \frac{1}{2}M\left(\frac{4\pi}{\alpha}(1-\frac{\beta}{2})\right).$$

Since $M:[0,+\infty)\to[0,+\infty)$ is a nondecreasing bijection function by (M_1) so

$$t_n^2 \geq \frac{4\pi}{\alpha}(1-\frac{\beta}{2}).$$

On the other hand, by using $\frac{d}{dt}I(t_n\psi_n)\mid_{t=t_n}=0$, we reach

$$\begin{aligned} m(t_n^2)t_n^2 &= \int_\Omega t_n\psi_n h(t_n\psi_n)\frac{e^{\alpha t_n^2\psi_n^2}}{|x|^\beta}dx \\ &\geq \int_{B_d(0)} t_n\psi_n h(t_n\psi_n)\frac{e^{\alpha t_n^2\psi_n^2}}{|x|^\beta}dx \\ &\geq \int_{B_{\frac{d}{n}}(0)} t_n\psi_n h(t_n\psi_n)\frac{e^{t_n^2\psi_n^2}}{|x|^\beta}dx, \end{aligned} \tag{4.3}$$

From (H_4) , we get

$$t_n\psi_n h(t_n\psi_n)\frac{e^{t_n^2\psi_n^2}}{|x|^\beta} \geq (\gamma-\varepsilon)\frac{e^{\alpha t_n^2 \log n/2\pi}}{|x|^\beta}, \tag{4.4}$$

and

$$\int_{B_{\frac{d}{n}}(0)} \frac{e^{\alpha t_n^2 \log n/2\pi}}{|x|^\beta}dx = \frac{2\pi d^{2-\beta}}{2-\beta}e^{\frac{\alpha t_n^2 \log n}{2\pi}-(2-\beta)\log n} \geq \frac{2\pi d^{2-\beta}}{2-\beta}e^{2\log n\left(\frac{\alpha t_n^2}{4\pi}-\left(1-\frac{\beta}{2}\right)\right)},$$

Hence

$$m(t_n^2)t_n^2 \geq \frac{2\pi d^{2-\beta}}{2-\beta}(\gamma-\varepsilon)e^{2\log n\left(\frac{\alpha t_n^2}{4\pi}-\left(1-\frac{\beta}{2}\right)\right)}. \tag{4.5}$$

Note that, by (M_1), we can see that

$$\frac{m(t_n^2)t_n^2}{e^{2\log n\left(\frac{\alpha t_n^2}{4\pi}-\left(1-\frac{\beta}{2}\right)\right)}} \to 0 \ \text{ if } \ t_n \to +\infty.$$

It follows from this and (4.5), we infer that

$$t_n^2 \to \frac{4\pi}{\alpha}(1-\frac{\beta}{2}). \tag{4.6}$$

Now, we are going to estimate (4.3) more exactly. For $0<\varepsilon<\gamma$ and $n\in\mathbb{N}$ we set

$$U_{n,\varepsilon} = \{x\in B_d(0) : t_n\psi_n > A_\varepsilon\} \ \text{ and } V_{n,\varepsilon} = B_d(0)\setminus U_{n,\varepsilon}.$$

So, by using (4.3) and (4.4) we obtain

$$\begin{aligned} m(t_n^2)t_n^2 \geq (\gamma-\varepsilon)&\int_{B_d(0)} \frac{e^{\alpha t_n^2\psi_n^2}}{|x|^\beta}dx - (\gamma-\varepsilon)\int_{V_{n,\varepsilon}} \frac{e^{\alpha t_n^2\psi_n^2}}{|x|^\beta}dx \\ &+ \int_{V_{n,\varepsilon}} t_n\psi_n h(t_n\psi_n)\frac{e^{\alpha t_n^2\psi_n^2}}{|x|^\beta}dx. \end{aligned} \tag{4.7}$$

Since $th(t)e^{\alpha t^2} \geq -C_\varepsilon t$ for all $t\geq 0$ and $\psi_n\to 0$ almost everywhere in $B_d(0)$, by using the Lebesgue dominated convergence theorem, we have

$$\int_{V_{n,\varepsilon}} t_n\psi_n h(t_n\psi_n)\frac{e^{\alpha t_n^2\psi_n^2}}{|x|^\beta}dx \geq -C_\varepsilon t_n\int_{V_{n,\varepsilon}} \frac{\psi_n}{|x|^\beta}dx \to 0 \text{ as } \ n\to+\infty$$

$$\int_{V_{n,\varepsilon}} \frac{e^{\alpha t_n^2 \psi_n^2}}{|x|^\beta} dx \to \int_{|x|\le d} \frac{1}{|x|^\beta} dx = \frac{2\pi d^{2-\beta}}{(2-\beta)}.$$

Then, from (4.6) and lemma4.1, passing to the limit in (4.7) we reach

$$m\left(\frac{4\pi}{\alpha}(1-\frac{\beta}{2})\right)\frac{4\pi}{\alpha}(1-\frac{\beta}{2}) \ge (\gamma-\varepsilon)\liminf_{n\to+\infty}\int_{B_d(0)} \frac{exp(4\pi(1-\frac{\beta}{2})\psi_n^2)}{|x|^\beta}dx - (\gamma-\varepsilon)\frac{2\pi d^{2-\beta}}{(2-\beta)},$$

and taking $\varepsilon \to 0$ we get $\frac{(2-\beta)^2}{2\alpha d^{2-\beta}} m(\frac{4\pi}{\alpha}(1-\frac{\beta}{2})) \ge \gamma$, which contradicts (H_4). Thus, the lemma is proved. □

Now, we consider the Nehari manifold associated to the functional I, namely

$$\mathcal{N} = \left\{u \in H_0^1(\Omega) : \langle I'(u), u\rangle = 0 \text{ and } u \neq 0\right\}$$

and let $b = \inf_{u\in\mathcal{N}} I(u)$. From the fact that $\frac{g(t)}{t^3}$ increasing, we deduce the following result (see [8])

Lemma 4.3. *If condition (H_3) holds, then for each $x \in \Omega$, $sg(s) - 4G(s)$ is increasing for $s > 0$. In particular*

$$sg(s) - 4G(s) \ge 0. \text{ for all } s \in [0,+\infty).$$

The next result gives a comparison between the minimax level c_* and b.

Lemma 4.4. *Assume that (M_3) and (H_3) are satisfied. Then $c_* \le b$.*

Proof. Given $u \in \mathcal{N}$, let us define $h(t) := I(tu)$ with $t \in (0,+\infty)$. The function h is differentiable and

$$h'(t) = \langle I'(tu), u\rangle = m\left(t^2\|u\|^2\right)t\|u\|^2 - \int_\Omega g(tu)u dx, \qquad \forall t > 0.$$

Since $\langle I'(u), u\rangle = 0$, for all $u \in \mathcal{N}$, we get

$$h'(t) = t^3\|u\|^4\left(\frac{m\left(t^2\|u\|_{1,2}^2\right)}{t^2\|u\|^2} - \frac{m\left(\|u\|^2\right)}{\|u\|^2}\right) + t^3\int_\Omega\left(\frac{g(u)}{|x|^\beta u^3} - \frac{g(tu)}{|x|^\beta (tu)^3}\right)u^4 dx.$$

Then $h'(1) = 0$ and from (M_3) and (H_3), we conclude that $h'(t) \ge 0$ for $0 < t < 1$ and $h'(t) \le 0$ for $t > 1$. Hence

$$I(u) = \max_{t\ge0} I(tu).$$

Now, defining $\gamma : [0,1] \to H_0^1(\Omega)$, $\gamma(t) = tt_0u$ we have $\gamma \in \Gamma$ and therfore

$$c_* \le \max_{t\in[0,1]} I(\gamma(t)) \le \max_{t\ge0} I(tu) = I(u),$$

which implies $c_* \le b$. □

5 Proof of main result

In this section we will give the proof of theorem 1.1. Thus we assume that the conditions $(M_1)-(M_3)$ and $(H_1)-(H_4)$ hold. First, we prove that I satisfies Palais-Smale condition. For this purpose, we will use the following convergence result due to M. de Souza and J. Marcos do Ò [17].

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $a \in [0,2)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then for any sequence (u_n) in $L^1(\Omega)$ such that*

$$u_n \to u \text{ in } L^1(\Omega), \quad \frac{f(x,u_n)}{|x|^a} \in L^1(\Omega) \quad \text{and} \quad \int_\Omega \frac{|f(x,u_n)u_n|}{|x|^a}dx \le C$$

up to a subsequence we have

$$\frac{f(x,u_n)}{|x|^a} \to \frac{f(x,u)}{|x|^a} \quad \text{in} \quad L^1(\Omega).$$

Proposition 5.1. *Assume that $\alpha > 0$ and $0 \le \beta < 2$ satisfy $\frac{\alpha}{4\pi} + \frac{\beta}{2} \le 1$. Then the functional E satisfies Palais-Smale condition for all $c_* < \frac{1}{2}M\left(\frac{4\pi}{\alpha}\right)$.*

Proof. Let $(u_n) \subset H_0^1(\Omega)$ be a sequence such that $I(u_n) \to c_*$ and $I'(u_n) \to 0$, that is, for any $\varphi \in H_0^1(\Omega)$

$$\frac{1}{2}M\left(\|u_n\|^2\right) - \int_\Omega \frac{G(u_n)}{|x|^\beta}dx = c_* + o(1), \tag{5.1}$$

$$m\left(\|u_n\|^2\right)\int_\Omega \nabla u_n \nabla\varphi dx - \int_\Omega h(u_n)\frac{e^{\alpha u_n^2}}{|x|^\beta}\varphi dx = o(\|\varphi\|). \tag{5.2}$$

It follows from (M_1) and (1.2), we obtian

$$\begin{aligned} C + \|u_n\| &\ge 8I(u_n) - \langle I'(u_n), u_n\rangle \\ &\ge m_0\|u_n\|^2 + \int_\Omega \left(u_n h(u_n) e^{\alpha u_n^2} - 8G(u_n)\right)\frac{1}{|x|^\beta}dx. \end{aligned} \tag{5.3}$$

So it suffices to prove that $th(t)e^{\alpha t^2} - 8G(t)$ is bounded from below. Here let us consider $0 < \varepsilon \le \frac{\gamma}{9}$. From (H_4) , for some constants $C_\varepsilon > 0$ and for all $t > 0$ we get

$$th(t)e^{\alpha t^2} \ge (\gamma - \varepsilon)e^{\alpha t^2} - C_\varepsilon,$$

and

$$G(t) \le \varepsilon e^{\alpha t^2} + C_\varepsilon. \tag{5.4}$$

Then, there exists a constant $C_\varepsilon(\Omega)$ such that

$$\int_\Omega \left(u_n h(u_n)e^{\alpha u_n^2} - 8G(u_n)\right)\frac{1}{|x|^\beta} \ge -C_\varepsilon(\Omega).$$

and therfore using (5.3), we obtain

$$C + \|u_n\| \ge m_0\|u_n\|^2 - C_\varepsilon(\Omega). \tag{5.5}$$

Hence (u_n) is bounded in $H_0^1(\Omega)$. Now we take a subsequence denoted again by u_n such that, for some $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_n &\to u \text{ strongly in } L^q(\Omega) \text{ for } \ 1 \leq q < +\infty, \\ u_n(x) &\to u(x) \text{ for almost every } \ x \in \Omega. \end{aligned}$$

In particular, $u_n \to u$ in $L^1(\Omega)$ and by (5.2), (H_2) and the Trudinger−Moser inequality, it also follows that $\int_\Omega \frac{u_n h(u_n) e^{\alpha u_n^2}}{|x|^\beta} dx$ is bounded and $\frac{h(u)e^{\alpha u^2}}{|x|^\beta} \in L^1(\Omega)$. Then, we can apply Lemma 5.1 to conclude that

$$\int_\Omega \frac{h(u_n)\, e^{\alpha u_n^2}}{|x|^\beta} dx \to \int_\Omega \frac{h(u)\, e^{\alpha u^2}}{|x|^\beta} dx \in L^1(\Omega).$$

It follows from (5.4) and (2.3), using the generalized Lebesgue dominated convergence, that

$$\int_\Omega \frac{G(u_n)}{|x|^\beta} dx \to \int_\Omega \frac{G(u)}{|x|^\beta} dx, \tag{5.6}$$

which implies

$$\frac{1}{2} M\left(\|u_n\|^2\right) \to c_* + \int_\Omega \frac{G(u)}{|x|^\beta} dx. \tag{5.7}$$

Next, we will make some assertions

Assertion 1. $m\Big(\|u\|^2\Big)\|u\|^2 \geq \int_\Omega \frac{g(u)u}{|x|^\beta} dx$

Proof : Suppose by contradiction $m\Big(\|u\|^2\Big)\|u\|^2 < \int_\Omega \frac{g(u)u}{|x|^\beta} dx$, so $\langle I'(u), u\rangle < 0$. Using (H_2) and (2.2), we can see that $\langle I'(tu), u\rangle > 0$ for t sufficiently small. Thus, there exists $\sigma \in (0,1)$ such that $\langle I'(\sigma u), u\rangle = 0$. That is, $\sigma u \in \mathcal{N}$.
Thus, according to lemma 4.3

$$\begin{aligned} c_* \leq b \leq I(\sigma u) &= I(\sigma u) - \frac{1}{4}\langle I'(\sigma u), u\rangle \\ &\leq \frac{1}{2} M\left(\|\sigma u\|^2\right) - \frac{1}{4} m\left(\|\sigma u\|^2\right)\|\sigma u\|^2 + \frac{1}{4}\int_\Omega \frac{g(\sigma u)\,\sigma u - 4G(\sigma u)}{|x|^\beta} dx \\ &< \frac{1}{2} M\left(\|u\|^2\right) - \frac{1}{4} m\left(\|u\|^2\right)\|u\|^2 + \frac{1}{4}\int_\Omega \frac{g(u)\,u - 4G(u)}{|x|^\beta} dx. \end{aligned}$$

By semicontinuity of norm and Fatou Lemma, we obtain

$$\begin{aligned} c_* &< \liminf_{n\to\infty}\Big(\frac{1}{2} M\left(\|u\|^2\right) - \frac{1}{4} m\left(\|u\|^2\right)\|u\|^2\Big) + \liminf_{n\to\infty} \frac{1}{4}\int_\Omega \frac{g(u)\,u - 4G(u)}{|x|^\beta} dx. \\ &\leq \lim_{n\to\infty}\Big(I(u_n) - \frac{1}{4} I'\langle u_n, u_n\rangle\Big) = c_*, \end{aligned}$$

which is a contradiction and the assertion is proved.

Assertion 2. $I(u) \geq 0$.

Proof : By Assertion 1, we have $I(u) \geq I(u) - \frac{1}{4}\langle I'(u), u\rangle$ which implies that

$$I(u) \geq \frac{1}{2}M\left(\|u\|^2\right) - \frac{1}{4}m\left(\|u\|^2\right)\|u\|^2 + \frac{1}{4}\int_\Omega \frac{g(u)u - 4G(u)}{|x|^\beta}dx.$$

Hence, using (1.2) and Lemma 4.3, we obtain

$$I(u) \geq 0.$$

Now we separate the proof into three cases.

Case 1. $c_* = 0$. If this is the case, we use (5.6) and (5.7)

$$0 \leq I(u) \leq \liminf_{n\to+\infty} I(u_n) = \int_\Omega \frac{G(u)}{|x|^\beta}dx - \int_\Omega \frac{G(u)}{|x|^\beta}dx = 0.$$

So $M\left(\|u_n\|^2\right) \to M\left(\|u\|^2\right)$ and then $\|u_n\| \to \|u\|$ which implies that $u_n \to u$ in $H_0^1(\Omega)$.

Case 2. $c_* \neq 0$, $u = 0$. We show that this cannot hapen for a Palais-Smale sequence. First we claim that

$$\int_\Omega \frac{\left|u_n h(u_n)e^{\alpha u_n^2}\right|}{|x|^\beta}dx \to 0 \text{ as } n \to \infty.$$

Since $u = 0$, we have $\int_\Omega \frac{G(u_n)}{|x|^\beta}dx \to 0$ and so

$$\frac{1}{2}M\left(\|u_n\|^2\right) \to c_* < \frac{1}{2}M\left(\frac{4\pi}{\alpha_0}\left(1 - \frac{\beta}{2}\right)\right).$$

Let $M^{-1}(2\rho_0) < \eta < \frac{4\pi}{\alpha_0}\left(1 - \frac{\beta}{2}\right)$. Then, $\|u_n\| < \sqrt{\eta}$ for all $n \geq n_0$ and for some $n_0 \in \mathbb{N}$. Now, choose $q = \frac{4\pi}{\eta\alpha}\left(1 - \frac{\beta}{2}\right) > 1$ and $\frac{1}{1-\frac{1}{q}} < r < \frac{2}{\beta\left(1-\frac{1}{q}\right)}$. By the Hölder inequality,

$$\int_\Omega \frac{\left|u_n h(u_n)e^{\alpha u_n^2}\right|}{|x|^\beta}dx \leq \left(\int_\Omega |u_n h(u_n)|^p\, dx\right)^{\frac{1}{p}}\left(\int_\Omega \frac{e^{q\alpha u_n^2}}{|x|^\beta}dx\right)^{\frac{1}{q}}\left(\int_\Omega \frac{1}{|x|^{\beta r\left(1-\frac{1}{q}\right)}}dx\right)^{\frac{1}{r}},$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Since the function $th(t)$ is bounded and $u = 0$, the first integral on the right-hand side converges to zero , the second integral is bounded for $n \geq n_0$ by lemma 2.1 since $q\alpha u_n^2 = 4\pi\left(1 - \frac{\beta}{2}\right)U_n^2$, where $U_n = \frac{u_n}{\sqrt{\eta}}$ satisfies $\|U_n\| \leq 1$, and the last integral is finite because $\beta r\left(1 - \frac{1}{q}\right) < 2$. So

$$\int_\Omega \frac{\left|u_n h(u_n)e^{\alpha u_n^2}\right|}{|x|^\beta}dx \to 0.$$

Then $m\left(\|u_n\|^2\right)\|u_n\|^2 \to 0$ by(5.2) and consequently by (M_1), $\|u_n\| \to 0$. This contradicts (5.7), which says in this case that $\|u_n\|^2 \to 2c_* \neq 0$.

Case 3. $c_* \neq 0$, $u \neq 0$. In this case we claim that

$$I(u) = c_*. \tag{5.8}$$

As u_n is bounded, up to a subsequence, $\|u_n\| \to r > 0$. By using (5.6) and semicontinuity of norm, we have $I(u) \le c_*$. So it remains to prove (5.8), we assume by contradiction that $I(u) < c_*$. Then,

$$\|u\| < r.$$

Next, defining $w_n = \frac{u_n}{\|u_n\|}$ and $w = \frac{u}{r}$, we have

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega) \quad \text{and} \quad \|w\| < 1.$$

Thus, by lemma 2.2

$$\sup_{n\in\mathbb{N}} \int_\Omega \frac{e^{pw_n^2}}{|x|^\beta} dx < \infty, \quad \forall p < \frac{4\pi\left(1-\frac{\beta}{2}\right)}{1-\|w\|^2}. \tag{5.9}$$

On the other hand,

$$2c_* - 2I(u) = M\left(r^2\right) - M\left(\|u\|^2\right). \tag{5.10}$$

Using this equality, lemma 4.2 and the fact that $I(u) \ge 0$, we get

$$M\left(r^2\right) < M\left(\frac{4\pi}{\alpha}\left(1-\frac{\beta}{2}\right)\right) + M\left(\|u\|^2\right).$$

From (M_1) and (M_2), it follows that

$$r^2 < M^{-1}\left(M\left(\frac{4\pi}{\alpha}\left(1-\frac{\beta}{2}\right)\right) + M\left(\|u\|^2\right)\right) \le \frac{4\pi}{\alpha}\left(1-\frac{\beta}{2}\right) + \|u\|^2. \tag{5.11}$$

Now, we observe that

$$r^2 = \frac{r^2 - \|u\|^2}{1-\|w\|^2},$$

and from (5.11), it follows that

$$r^2 < \frac{\frac{4\pi}{\alpha}\left(1-\frac{\beta}{2}\right)}{1-\|w\|^2}.$$

Then, there exists $\rho > 0$ such that $\alpha\|(u_n\|^2 < \rho < \frac{4\pi\left(1-\frac{\beta}{2}\right)}{1-\|w\|^2}$ for n sufficiently large. Now, taking $q > 1$ close to 1 such that

$$q\alpha\|u_n\|^2 < \rho < \frac{4\pi\left(1-\frac{\beta}{2}\right)}{1-\|w\|^2}, \quad \text{for } n \text{ large enough}$$

and invoking (5.9), for some $C > 0$, we conclude that

$$\int_\Omega \frac{e^{q\alpha u_n^2}}{|x|^\beta} dx \le \int_\Omega \frac{e^{\rho w_n^2}}{|x|^\beta} dx \le C.$$

Hence, using $(H2)$ and Hölder inequality, for some $p > 1$, we reach

$$\left| \int_\Omega \frac{h(u_n)e^{\alpha u_n^2}(u_n - u)}{|x|^\beta} dx\, dx \right| \le C_1 \int_\Omega \frac{e^{\alpha u_n^2}|u_n - u|}{|x|^\beta} dx$$

$$\leq C_2 \|u_n - u\|_{L^p(\Omega)} \to 0 \text{ as } n \to \infty,$$

Since $\langle I(u_n), (u_n - u)\rangle = o(1)$, we get

$$m\left(\|u_n\|^2\right) \int_\Omega \nabla u_n (\nabla u_n - \nabla u)\ dx \to 0.$$

On the other hand,

$$m\left(\|u_n\|^2\right) \int_\Omega \nabla u_n (\nabla u_n - \nabla u_0)\ dx = m\left(\|u_n\|^2\right) \|u_n\|^2 - m\left(\|u_n\|^2\right) \int_\Omega \nabla u_n \nabla u\, dx$$
$$\to m\left(r^2\right) r^2 - m\left(r^2\right) \|u\|^2 .$$

which implies that $\|u\| = r$ and so $u_n \to u$ in $H_0^1(\Omega)$. In view of the continuity of I, we must have $I(u) = c_*$ that is an absurde. Thus, the proof of Proposition 1 is complete. □

Proof of Theorem 1.1. It follows the assumptions that the functiona I satisfies the Plais-Smale condition at any level $c_* < \frac{1}{2} M\left(\frac{4\pi}{\alpha_0}\left(1 - \frac{\beta}{2}\right)\right)$, see Proposition 5.1. To finish the proof of theorem 1.1, we use Lemma 3.1 and 3.2 and apply the mountain Pass Theorem. □

References

[1] Adimurthi, Existence of Positive Solutions of the Semilinear Dirichlet Problem with Critical Growth for the n-Laplacian, Ann. Della. Scuola. Norm. Sup. di Pisa, Serie IV, Vol.XVII, Fasc. **3** (1990), 393-413 .

[2] C. O. Alves, F. Correa and G. M. Figueiredo, On a class of nonlocal elliptic problmes with critical growth, Differential equations and applications, **2** (2010), 409-417.

[3] Adimurthi and K. Sandeep, A singular Moser-Trudinger embedding and its applications, Nonlinear Differential Equations and Applications, **13** (2007), 585-603.

[4] A. Benaissa and B. Khaldi, On a Singular Class of Hamiltonian Systems in Dimension Two, J. of Analysis and its Applications, **33** (2014), 199-215.

[5] N. T. Chung, Multiplicity results for a class of p(x)-Kirchhoff type equations with combined nonlinearities, E. J. Qualit. Theory Diff. Equ., **42** (2012), 1-13.

[6] N. T. Chung, An existence result for a class of Kirchhoff type systems via sub and super-solutions method, Appl. Math. Lett., **35** (2014), 95-101.

[7] E. Colorado and I Peral, Eigenvalues and bifurcation for elliptic equations with mixed Dirichlet-Neumann boundary conditions related to Caffarelli-Kohn-Nirenberg inequalities, Topol. Methods Nonlinear Anal., **23** (2004) 239-273.

[8] G. M. Figueiredo, U. B. Severo, Ground state solution for a Kirchhoff problem with exponential critical growth, Milan J. Math, **84** (2015), 23-39.

[9] X. Han and G. Dai, On the sub-supersolution method for p(x)-Kirchhoff type equations, J. Inequal. Appl., **283** (2012), 1-11.

[10] G. Kirchhoff; Mechanik, Teubner, Leipzig, Germany, 1883.

[11] P. L. Lions, The concentration compactness principle in the calculus of variations part-I, Revista Matem atica Iberoamerican, **1** (1985), 145–201.

[12] C. S. Lin and J. V. Prajapat, Vortex Condensates for Relativistic Abelian Chern-Simons Model with Two Higgs Scalar Fields and Two Gauge Fields on a Torus, Commun. Math. Phys. **288** (2009), 311-347.

[13] C. S. Lin and S. Yan, Bubbling Solutions for Relativistic Abelian Chern-Simons Model on a Torus, Commun. Math. Phys. **297** (2010), 733-758.

[14] J. Moser, A sharp form of an inequality by N.Trudinger, Indiana Univ. Math. Jour., **20** (1971), 1077-1092.

[15] N. Megrez, K. Sreenadh, B. Khaldi, Multiplicity of positive solutions for a gradient system with an exponential nonlinearity, EJDE, **236** (2012), 1-16.

[16] O. H. Miyagaki and R. S. Rodrigues, On positive solutions for a class of singular quasilinear elliptic systems, J. Math. Anal. Appl., **334** (2007), 818-833.

[17] M. de Souza, J. M. do Ó, On a singular and nonhomogeneous N-Laplacian equation involving critical growth, J. Math. Anal. Appl. **280** (2011), 241-263.

[18] N.S. Trudinger, On embedding into Orlicz spaces and some applications, Journal of Math. Mech., **17** (1967), 473-484.

[19] Y.T. Tang, J.F. Liao, C.L. Tang, Two positive solutions for Kirchhoff type problems with Hardy-Sobolev critical exponent and singular nonlinearities, Taiwanese Journal of Mathematics **23** (2019), 231-253.

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