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APPROXIMATING COMMON FIXED POINTS OF TWO ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

ISA YILDIRIM AND MURAT OZDEMIR

Abstract. In this paper, we consider a composite iterative algorithm for approximating common fixed points of two nonself asymptotically quasi-nonexpansive mappings and we prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

1. Introduction

Let K be a nonempty subset of a real normed space E. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$ and asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0,\infty)$, $u_n \to 0$ as $n \to \infty$ such that $\|T^n x - T^n y\| \leqslant (1+u_n)\|x - y\|$ for all $x, y \in K$ and $n \geqslant 1$. T is said to be uniformly L-Lipschitzian if there exists a real number $L > 0$ such that $\|T^n x - T^n y\| \leqslant L\|x - y\|$ for all $x, y \in K$ and $n \geqslant 1$. The mapping T is said to be quasi-nonexpansive if $F(T) := \{x \in X : Tx = x\} \neq \emptyset$ and $\|Tx - p\| \leqslant \|x - p\|$ for all $x \in K$ and $p \in F(T)$. T is said to be asymptotically quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0,\infty)$ with $u_n \to 0$ as $n \to \infty$ such that $\|T^n x - p\| \leqslant (1+u_n)\|x - p\|$ for all $x \in K$, $p \in F(T)$ and $n \geqslant 1$.

From the above definition, it follows that; a nonexpansive mapping must be quasi-nonexpansive; an asymptotically nonexpansive mapping is uniformly L-Lipschitzian as well as asymptotically quasi-nonexpansive. However, the converse of these statements is not true, in general.

Iterative process for asymptotically nonexpansive self-mapping in Hilbert spaces and Banach spaces incluiding Mann-type and Ishikawa-type iteration process have been studied extensively by many authors, see for example [4, 6, 9, 11, 13]. The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [3] as an important generalization of the

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class of nonexpansive maps. They [3] proved that if K is a nonempty closed convex bounded subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self mapping on K, then T has a fixed point.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that, if a map P is a retraction, then $Pz = z$ for all z in the range of P.

Recently, Chidume [1] introduce the concept of nonself asymptotically nonexpansive mappings, which is the generalization of asymptotically nonexpansive mappings. Similarly, the concept of nonself asymptotically quasi-nonexpansive mappings can also be defined as the generalization of asymptotically quasi-nonexpansive mappings and nonself asymptotically nonexpansive mappings. These mappings are defined as follows:

Definition 1.1. Let K be a nonempty subset of a real normed space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K and $T : K \to E$ be a nonself mapping.

(1) T is said to be a nonself asymptotically nonexpansive mapping, if there exists a sequence $\{u_n\} \subset [0,\infty)$, $u_n \to 0$ as $n \to \infty$ such that

$$\left\| T(PT)^{n-1}x - T(PT)^{n-1}y \right\| \leqslant (1+u_n)\left\| x-y \right\|$$

for all $x, y \in K$ and $n \geqslant 1$;

(2) T is said to be a nonself uniformly L-Lipschitzian mapping, if there exists a constant $L > 0$ such that

$$\left\| T(PT)^{n-1}x - T(PT)^{n-1}y \right\| \leqslant L\left\| x-y \right\|$$

for all $x, y \in K$ and $n \geqslant 1$;

(3) T is said to be a nonself asymptotically quasi-nonexpansive mapping, if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0,\infty)$, $u_n \to 0$ as $n \to \infty$ such that

$$\left\| T(PT)^{n-1}x - p \right\| \leqslant (1+u_n)\left\| x-p \right\|$$

for all $x, y \in K$, $p \in F(T)$ and $n \geqslant 1$.

By studying the following iteration process (Mann-type iteration)

$$x_1 \in K, \; x_{n+1} = P\left((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\right) \tag{1.1}$$

Chidume, Ofoedu and Zeyege [1] studied the strong and weak convergence theorems for nonself asymptotically nonexpansive mapping T. Wang [15] generalizes the iteration process (1.1) as follows (Ishikawa-type iteration):

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P\left((1-\alpha_n)\, x_n + \alpha_n T_1 (PT_1)^{n-1}\, y_n\right), \\ y_n = P\left(\left(1-\beta_n\right) x_n + \beta_n T_2 (PT_2)^{n-1}\, x_n\right),\ n \geqslant 1, \end{cases} \tag{1.2}$$

where $T_1, T_2 : K \to E$ are nonself asymptotically nonexpansive mappings. And he also got several convergence theorems of the iterative scheme (1.2) under proper conditions.

In addition, Petryshn and Williamson [7] proved a sufficient and necessary condition for the Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings. Ghosh and Debnath [2] extended the result of [7] and gave a sufficient and necessary conditions for strong convergence of Ishikawa-type iteration process to a fixed point of a quasi-nonexpansive mapping in a real Banach space. Qihou [8] extended the resul of Ghosh and Debnath to asymptotically quasi-nonexpansive mappings. Shahzad and Udomene [12] established necessary and sufficient conditions for the convergence of the Ishikawa-type iterative sequences involving two asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty closed convex subset of a Banach space.

Recently, Thianwan [14] studied the convergence of an projection type Ishikawa iteration process to a common fixed point of two nonself asymptotically nonexpansive mapppings. This scheme defined as follows:

Let K be a nonempty convex subset of a real normed space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K and $T_1, T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings.

$$\begin{cases} x_1 \in K \\ x_{n+1} = P\left((1-\alpha_n)\, y_n + \alpha_n T_1 (PT_1)^{n-1}\, y_n\right), \\ y_n = P\left(\left(1-\beta_n\right) x_n + \beta_n T_2 (PT_2)^{n-1}\, x_n\right),\ n \geqslant 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0,1)$. In the iterative schemes (1.3) and (1.2), if $T_1 = T_2 = T$ and $\beta_n = 0$ for all $n \geqslant 1$, then this schemes reduce to (1.1). The iterative schemes (1.2) and (1.3) are not general each others. That is, one of them are not obtain from the other. Thianwan [14] gave the following strong and weak convergence theorems.

Theorem 1.2. ([14]) *Let E be a uniformly convex Banach space and K a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself mappings of K satisfying condition (A') with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty, \sum_{n=1}^{\infty} (l_n - 1) < \infty$ and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Then the sequences*

$\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.3) converge strongly to a common fixed point of T_1 and T_2.

Theorem 1.3. ([14]) *Let E be a uniformly convex Banach space which satisfies Opial's condition and K a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself mappings of K with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by (1.3). Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a common fixed point of T_1 and T_2.*

The purpose of this paper is to establish:

(1) necessary and sufficient conditions for the convergence of the projection type Ishikawa iteration process involving two nonself asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty closed convex subset of a Banach space, and

(2) a sufficient condition for the convergence of the projection type Ishikawa iteration process involving two uniformly L-Lipschitzian, nonself asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty closed convex subset of a uniformly convex Banach space.

Our results are significant generalizations of the corresponding results of Petryshyn and Williamson [7], Chidume, Ofoedu and Zeyege [1], Thianwan [14].

2. Preliminaries

Let E be a real normed linear space. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \varepsilon = \|x-y\|\right\}.$$

E is called uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

A mapping $T : K \to K$ is said to be semicompact if, for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence say $\{x_{nj}\}$ of $\{x_n\}$ such that $\{x_{nj}\}$ converges strongly to some x^* in K. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \to x^* \in D(T)$ and $Tx_n \to p$ then $Tx^* = p$.

A Banach space E said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \to x$ converges weakly implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for all $y \in E$ and $y \neq x$.

Two mappings $T_1, T_2 : K \to E$ with $F := F(T_1) \cap F(T_2) = \{x \in K : T_1 x = T_2 x = x\} \neq \emptyset$ is said to satisfy condition (A') [5] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t > 0$ such that

$$\|x - T_1 x\| \geqslant f(d(x,F)) \text{ or } \|x - T_2 x\| \geqslant f(d(x,F))$$

for all $x \in K$, where $d(x,F) = \inf\{\|x - q\| : q \in F\}$.

In what follows, we will state the following useful lemmas:

Lemma 2.4.([13]) *Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers such that $\lambda_{n+1} \leqslant \lambda_n + \sigma_n$ for all $n \geqslant 1$, and $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ as $n \to \infty$.*

Lemma 2.5. ([11]) *Let E be a real uniformly convex Banach space and $0 \leqslant p \leqslant t_n \leqslant q < 1$ for all positive integers $n \geqslant 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \leqslant s, \quad \limsup_{n\to\infty} \|y_n\| \leqslant s \text{ and } \lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = s$$

hold for some $s \geqslant 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. Main results

In this section, we shall prove convergence of the iteration scheme (1.3) to a common fixed point of two nonself asymptotically quasi-nonexpansive mappings. Also, we always assume $F = F(T_1) \cap F(T_2) \neq \emptyset$. In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *Let E be a real Banach space, K be a closed convex nonempty subset of E which is also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two nonself asymptotically quasi-nonexpansive mappings with sequences (respectively), $\{u_n\}, \{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1)$, $x^* \in F$ and $\{x_n\}$ is defined by (1.3). Then, $\{x_n\}$ is bounded and the limits $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} d(x_n, F)$ exists, where $\lim_{n\to\infty} d(x_n, F) = \inf_{x^* \in F} \|x_n - x^*\|$.*

Proof. Let $x^* \in F$.We know that $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$. Using (1.3), we have

$$\begin{aligned}
\left\| y_n - x^* \right\| &= \left\| P\left(\left(1-\beta_n\right) x_n + \beta_n T_2 \left(PT_2\right)^{n-1} x_n\right) - P\left(x^*\right) \right\| \\
&\leqslant \left(1-\beta_n\right) \left\| x_n - x^* \right\| + \beta_n \left\| T_2 \left(PT_2\right)^{n-1} x_n - x^* \right\| \\
&\leqslant \left(1-\beta_n\right) \left\| x_n - x^* \right\| + \beta_n \left(1+v_n\right) \left\| x_n - x^* \right\| \\
&\leqslant \left(1+v_n\right) \left\| x_n - x^* \right\|. \qquad (3.1)
\end{aligned}$$

From this and by (1.3), we have also

$$\begin{aligned}
\left\| x_{n+1} - x^* \right\| &= \left\| P\left(\left(1-\alpha_n\right) y_n + \alpha_n T_1 \left(PT_1\right)^{n-1} y_n\right) - P\left(x^*\right) \right\| \\
&\leqslant \left(1-\alpha_n\right) \left\| y_n - x^* \right\| + \alpha_n \left\| T_1 \left(PT_1\right)^{n-1} y_n - x^* \right\| \\
&\leqslant \left(1-\alpha_n\right) \left\| y_n - x^* \right\| + \alpha_n \left(1+u_n\right) \left\| y_n - x^* \right\| \\
&= \left(1-\alpha_n+\alpha_n+\alpha_n u_n\right) \left\| y_n - x^* \right\| \\
&\leqslant \left(1+u_n\right) \left\| y_n - x^* \right\| \\
&\leqslant \left(1+u_n\right)\left(1+v_n\right) \left\| x_n - x^* \right\| \\
&= \left(1+u_n+v_n+u_n v_n\right) \left\| x_n - x^* \right\| \\
&\leqslant e^{\left(u_n+v_n+u_n v_n\right)} \left\| x_n - x^* \right\| \\
&\leqslant e^{\sum_{n=1}^{\infty}\left(u_n+v_n+u_n v_n\right)} \left\| x_1 - x^* \right\|. \qquad (3.2)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\left(u_n+v_n+u_n v_n\right) < \infty$, it follows that $\{x_n\}$ is bounded. Thus, there exists constant $M > 0$ such that $\|x_n - x^*\| \leqslant M$ for all $n \geqslant 1$. Then, we obtain

$$\left\| x_{n+1} - x^* \right\| \leqslant \left\| x_n - x^* \right\| + M\left(u_n+v_n+u_n v_n\right). \qquad (3.3)$$

Since (3.3) is true for each x^* in F. This implies that

$$d\left(x_{n+1}, F\right) \leqslant d\left(x_n, F\right) + M\left(u_n+v_n+u_n v_n\right).$$

From Lemma 2.1, we obtain that $\lim_{n\to\infty}\|x_n - x^*\|$ and $\lim_{n\to\infty} d\left(x_n, F\right)$ exists. This completes the proof. □

The above lemma generalizes Theorem 3.5 of Chidume, Ofoedu, Zeyege [1] and Lemma 2.1 of Thianwan [14] for nonself asymptotically nonexpansive mappings to nonself asymptotically quasi-nonexpansive mappings.

Theorem 3.2. *Let K be a nonempty closed convex subset of a real Banach space$E$. Let $T_1, T_2 : K \to E$ be two nonself asymptotically quasi-nonexpansive mappings with sequences $\{u_n\}, \{v_n\}, \{\alpha_n\}$ and $\left\{\beta_n\right\}$ as in Lemma 3.1. Suppose that $F \neq \emptyset$ and $\{x_n\}$ is defined by (1.3). Then $\{x_n\}$ converges to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d\left(x_n, F\right) = 0$.*

Proof. It suffices that we only prove the sufficiency. That is, let $\liminf_{n\to\infty} d(x_n, F) = 0$. Then, the proof follows as in the proof of Theorem 2.4 of [10].

Now, we establish some weak and strong convergence results for the iterative scheme (1.3) by removing the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ from the results obtained in above; for this we have to consider the class of uniformly L-Lipschitzian and nonself asymptotically quasi-nonexpansive mappings on a uniformly convex Banach space.

For our next theorems, we start by proving the following lemma which will be needed in the sequel.

Lemma 3.3. *Let E be a uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two uniformly L-Lipschitzian, nonself asymptotically quasi-nonexpansive mappings with sequences $\{u_n\}, \{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0,1)$. Suppose $\{x_n\}$ is generated iteratively by (1.3). Then,*

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2 x_n\|.$$

Proof. For any $x^* \in F$, by Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Assume $\lim_{n\to\infty} \|x_n - x^*\| = b$ for some $b \geqslant 0$. From (3.1) and (3.2), we get

$$\left\| y_n - x^* \right\| \leqslant (1+v_n) \left\| x_n - x^* \right\| \tag{3.4}$$

and

$$\left\| T_1 (PT_1)^{n-1} y_n - x^* \right\| \leqslant (1+u_n) \left\| y_n - x^* \right\|. \tag{3.5}$$

Taking lim sup on both sides in the inequalities (3.4) and (3.5), we obtain

$$\limsup_{n\to\infty} \left\| y_n - x^* \right\| \leqslant b \tag{3.6}$$

and

$$\limsup_{n\to\infty} \left\| T_1 (PT_1)^{n-1} y_n - x^* \right\| \leqslant \limsup_{n\to\infty} \left\| y_n - x^* \right\| \leqslant b. \tag{3.7}$$

And also, by using (1.3) we get that

$$\begin{aligned} b &= \lim_{n\to\infty} \left\| x_n - x^* \right\| = \lim_{n\to\infty} \left\| x_{n+1} - x^* \right\| \\ &= \lim_{n\to\infty} \left\| P(1-\alpha_n) y_n + \alpha_n T_1 (PT_1)^{n-1} y_n - P\left(x^*\right) \right\| \\ &= \lim_{n\to\infty} \left\| (1-\alpha_n)\left(y_n - x^*\right) + \alpha_n \left(T_1 (PT_1)^{n-1} y_n - x^*\right) \right\|. \end{aligned} \tag{3.8}$$

This together with (3.6), (3.7) and Lemma 2.5 imply that

$$\lim_{n\to\infty} \left\| y_n - T_1 (PT_1)^{n-1} y_n \right\| = 0. \tag{3.9}$$

On the other hand, $\left\| T_2 (PT_2)^{n-1} x_n - x^* \right\| \leqslant (1+v_n) \|x_n - x^*\|$, taking lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \left\| T_2 (PT_2)^{n-1} x_n - x^* \right\| \leqslant \limsup_{n\to\infty} \left\| x_n - x^* \right\| \leqslant b. \tag{3.10}$$

From (1.3) we get that

$$\begin{aligned}
\left\| x_{n+1} - x^* \right\| &\leqslant \left\| P\left((1-\alpha_n) y_n + \alpha_n T_1 (PT_1)^{n-1} y_n \right) - x^* \right\| \\
&\leqslant (1-\alpha_n) \left\| y_n - x^* \right\| + \alpha_n \left\| T_1 (PT_1)^{n-1} y_n - x^* \right\| \\
&\leqslant (1-\alpha_n) \left\| y_n - x^* \right\| + \alpha_n \left\| T_1 (PT_1)^{n-1} y_n - y_n \right\| + \alpha_n \left\| y_n - x^* \right\| \\
&\leqslant \left\| y_n - x^* \right\| + \left\| T_1 (PT_1)^{n-1} y_n - y_n \right\|.
\end{aligned}$$

Putting $\lim_{n\to\infty} \|x_{n+1} - x^*\| = b$ in (3.8), we have

$$b \leqslant \liminf_{n\to\infty} \left\| y_n - x^* \right\|. \tag{3.11}$$

Combining (3.6) and (3.11), we obtain

$$\lim_{n\to\infty} \left\| y_n - x^* \right\| = b. \tag{3.12}$$

It follows from (3.12) that

$$\begin{aligned}
b = \lim_{n\to\infty} \left\| y_n - x^* \right\| &\leqslant \lim_{n\to\infty} \left\| (1-\beta_n)(x_n - x^*) + \beta_n \left(T_2 (PT_2)^{n-1} x_n - x^* \right) \right\| \\
&\leqslant \lim_{n\to\infty} \left\| x_n - x^* \right\| = b,
\end{aligned}$$

and so

$$\lim_{n\to\infty} \left\| (1-\beta_n)(x_n - x^*) + \beta_n \left(T_2 (PT_2)^{n-1} x_n - x^* \right) \right\| = b.$$

Using (3.10) and Lemma 2.5, we obtain

$$\lim_{n\to\infty} \left\| x_n - T_2 (PT_2)^{n-1} x_n \right\| = 0. \tag{3.13}$$

From $y_n = P\left((1-\beta_n) x_n + \beta_n T_2 (PT_2)^{n-1} x_n \right)$ and (3.13), we have

$$\begin{aligned}
\left\| y_n - x_n \right\| &= \left\| P\left((1-\beta_n) x_n + \beta_n T_2 (PT_2)^{n-1} x_n \right) - x_n \right\| \\
&\leqslant \left\| (1-\beta_n)(x_n - x_n) + \beta_n \left(T_2 (PT_2)^{n-1} x_n - x_n \right) \right\| \\
&\leqslant (1-\beta_n) \|x_n - x_n\| + \beta_n \left\| T_2 (PT_2)^{n-1} x_n - x_n \right\| \\
&\leqslant \left\| T_2 (PT_2)^{n-1} x_n - x_n \right\|
\end{aligned}$$

and it implies that

$$\lim_{n\to\infty} \left\| y_n - x_n \right\| = 0. \tag{3.14}$$

Since T_1 uniformly L-Lipschitzian mapping, it follows that

$$\begin{aligned}\left\|x_n - T_1 (PT_1)^{n-1} x_n\right\| &\leqslant \left\|x_n - y_n + y_n - T_1 (PT_1)^{n-1} x_n\right\| \\ &\leqslant \left\|x_n - y_n\right\| + \left\|y_n - T_1 (PT_1)^{n-1} x_n\right\| \\ &\leqslant \left\|x_n - y_n\right\| + \left\|y_n - T_1 (PT_1)^{n-1} y_n\right\| + \left\|T_1 (PT_1)^{n-1} y_n - T_1 (PT_1)^{n-1} x_n\right\| \\ &\leqslant \left\|x_n - y_n\right\| + \left\|y_n - T_1 (PT_1)^{n-1} y_n\right\| + L\left\|x_n - y_n\right\| \\ &\leqslant (1+L)\left\|x_n - y_n\right\| + \left\|y_n - T_1 (PT_1)^{n-1} y_n\right\|.\end{aligned}$$

Thus, from (3.9) and (3.14) we obtain

$$\lim_{n\to\infty} \left\|T_1 (PT_1)^{n-1} x_n - x_n\right\| = 0. \tag{3.15}$$

Next, we shall prove that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Since

$$\begin{aligned}\|x_n - T_1 x_n\| &\leqslant \left\|x_n - T_1 (PT_1)^{n-1} x_n\right\| + \left\|T_1 (PT_1)^{n-1} x_n - T_1 (PT_1)^{n-1} y_n\right\| \\ &\quad + \left\|T_1 (PT_1)^{n-1} y_n - T_1 x_n\right\| \\ &\leqslant \left\|x_n - T_1 (PT_1)^{n-1} x_n\right\| + L\left\|x_n - y_n\right\| + L\left(\left\|T_1 (PT_1)^{n-2} y_n - y_n\right\| + \left\|y_n - x_n\right\|\right) \\ &= \left\|x_n - T_1 (PT_1)^{n-1} x_n\right\| + L\left\|x_n - y_n\right\| + L\left\|x_n - y_n\right\| + L\left\|T_1 (PT_1)^{n-2} y_n - y_n\right\|\end{aligned}$$

this together with (3.9), (3.14) and (3.15) imply that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0.$$

Moreover,

$$\begin{aligned}\|x_n - T_2 x_n\| &\leqslant \left\|x_n - T_2 (PT_2)^{n-1} x_n\right\| + \left\|T_2 (PT_2)^{n-1} x_n - T_2 x_n\right\| \\ &\leqslant \left\|x_n - T_2 (PT_2)^{n-1} x_n\right\| + L\left\|T_2 (PT_2)^{n-2} x_n - x_n\right\|.\end{aligned}$$

Thus, since T_2 uniformly$L$-Lipschitzian mapping and by the equality (3.13) we have

$$\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0.$$

This completes the proof. □

Now we give some strongly convergence theorems and proofs. Note that, the condition (A') is weaker than the compactness of the domain of the mappings.

Theorem 3.4. *Let E be a uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two uniformly L-Lipschitzian, nonself asymptotically quasi-nonexpansive mappings of K satisfying condition (A') with sequences $\{u_n\}, \{v_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ as in Lemma 3.3. Then the sequence $\{x_n\}$ defined by the iterative scheme (1.3) converge strongly to a common fixed point of T_1 and T_2.*

Proof. From Lemma 3.3, we know that

$$\lim_{n\to\infty}\|x_n - T_1x_n\| = 0 = \lim_{n\to\infty}\|x_n - T_2x_n\|.$$

Since T_1 and T_2 satisfies condition (A'), we get that

$$\lim_{n\to\infty} f(d(x_n,F)) \leqslant \lim_{n\to\infty}\|x_n - T_1x_n\| \text{ or } \lim_{n\to\infty} f(d(x_n,F)) \leqslant \lim_{n\to\infty}\|x_n - T_2x_n\|.$$

Therefore, $\lim_{n\to\infty} d(x_n,F) = 0$. For any given $\varepsilon > 0$, there exists $x^* \in F$ and $n_0 > 0$ such that for all $n \geqslant n_0$

$$\left\|x_n - x^*\right\| < \frac{\varepsilon}{2}$$

from Lemma 3.1. It implies that

$$\|x_{n+m} - x_n\| \leqslant \left\|x_{n+m} - x^*\right\| + \left\|x_n - x^*\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geqslant n_0$ and $m \geqslant 0$. Thus, $\{x_n\}$ is a Cauchy sequence. Since E is complete, we can obtain $\{x_n\}$ is convergent. That is, $\lim_{n\to\infty} x_n = x^*$. Since K is closed, so we get $x^* \in K$. Now, we prove that $x^* \in F$. We can write the following inequality

$$\left|d\left(x^*,F\right) - d(x_n,F)\right| \leqslant \left\|x^* - x_n\right\|$$

for all $n \geqslant 1$. Since $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} d(x_n,F) = 0$, we get that $d(x^*,F) = 0$. So, $x^* \in F$. Therefore $\{x_n\}$ converges strongly to a common fixed point $x^* \in F$. This completes the proof. □

Theorem 3.5. *Let E be a uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two uniformly L-Lipschitzian, nonself asymptotically quasi-nonexpansive mappings of K with sequences $\{u_n\}, \{v_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ as in Lemma 3.3. Let $\{x_n\}$ be the sequences defined by (1.3). If one of T_1 and T_2 is semicompact, then $\{x_n\}$ converge strongly to a common fixed point of T_1 and T_2.*

Proof. We may assume that one of T_1 and T_2 is semicompact. Since $\lim_{n\to\infty}\|x_n - T_1x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T_2x_n\| = 0$ from Lemma 3.3, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that x_{n_j} converges strongly to x^*. Hence from Lemma 3.3,

$$\left\|x^* - T_ix^*\right\| = \lim_{n\to\infty}\left\|x_{n_j} - T_ix_{n_j}\right\| = 0, \;\; i = 1,2.$$

This implies that $x^* \in F$. Thus $\lim_{n\to\infty}\|x_n - x^*\|$ exists by Lemma 3.1. Since the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to x^*. Then $\{x_n\}$ converges strongly to a common fixed point $x^* \in F$. This completes the proof.

Finally, we prove weak convergence theorem given as follows:

Theorem 3.6. *Let E be a uniformly convex Banach space and let K be a closed convex nonempty subset of E which is also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two uniformly L-Lipschitzian, nonself asymptotically quasi-nonexpansive mappings of K with sequences $\{u_n\}, \{v_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ as in Lemma 3.3. Let $\{x_n\}$ be the sequences defined by (1.3). If E satisfies Opial's condition and each $I - T_i, i = 1,2,$ is demiclosed at 0, then the sequence $\{x_n\}$ converge weakly to a common fixed point of T_1 and T_2.*

Proof. From Lemma 3.1, $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Since a uniformly convex Banach space is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some $y^* \in K$. From Lemma 3.3, $\lim_{n\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ and $I - T_i$ is demiclosed at 0 for $i = 1,2$. Hence, we have $T_i y^* = y^*$. That is, $y^* \in F$. Now, we show that $\{x_n\}$ converges weakly to y^*. So, we suppose that another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $z^* \in K$. Again, we can prove that $z^* \in F$, as above. Next, we show that $y^* = z^*$. Assume $y^* \neq z^*$. Then, by using the Opial's condition, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - y^*\| &= \lim_{n_j\to\infty} \|x_{n_j} - y^*\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z^*\| = \lim_{n\to\infty} \|x_n - z^*\| \\
&= \lim_{n_k\to\infty} \|x_{n_k} - z^*\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - y^*\| = \lim_{n\to\infty} \|x_n - y^*\|.
\end{aligned}$$

which is a contradiction, hence $y^* = z^*$. Then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2. This completes the proof. □

Remark 3.7.

(1) Since an asymptotically nonexpansive mapping is uniformly L-Lipschitzian and asymptotically quasi-nonexpansive, the uniformly L-Lipschitz and nonself asymptotically quasi-nonexpansive mappings in Lemma 3.3 can be replaced by a asymptotically nonexpansive mappings. Thus, Lemma 3.3 extends Lemma 2.5 of Thianwan [14] for two asymptotically nonexpansive mappings to two uniformly L-Lipschitz and nonself asymptotically quasi-nonexpansive mappings.

(2) Theorem 3.4 and Theorem 3.6 contain as special cases, Theorem 2.5 and Theorem 2.6 of Thianwan [14], respectively.

(3) In the iterative scheme (1.3), if $T_1 = T_2 = T$ and $\beta_n = 0$ for all $n \geq 1$, then (1.3) reduces to (1.1). Thus Theorem 3.5 contains as special cases, Theorem 3.7 of Chidume, Ofoedu, Zegeye [1].

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Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey.

E-mail: isayildirim@atauni.edu.tr

Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey.

E-mail: mozdemir@atauni.edu.tr