n-harmonicity, minimality, conformality and cohomology

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Abstract. By studying cohomology classes that are related with \( n \)-harmonic morphisms and \( F \)-harmonic maps, we augment and extend several results on \( F \)-harmonic maps, harmonic maps in [1, 3, 15], \( p \)-harmonic morphisms in [23], and also revisit our previous results in [10, 11, 29] on Riemannian submersions and \( n \)-harmonic morphisms which are submersions. The results, for example Theorem 3.2 obtained by utilizing the \( n \)-conservation law (2.6), are sharp.

Keywords. \( p \)-harmonic maps, \( n \)-harmonic morphism, cohomology class, minimal submanifold, submersion

1 Introduction

Harmonicity and its variants are related with the topology and geometry of manifolds. It was shown in [27] that homotopy classes can be represented by \( p \)-harmonic maps (see, e.g. [29], for definition and examples of \( p \)-harmonic maps):

**Theorem A.** If \( N^n \) is a compact Riemannian \( n \)-manifold, then for any positive integer \( i \), each class in the \( i \)-th homotopy group \( \pi_i(N^n) \) can be represented by a \( C^{1,\alpha} \) \( p \)-harmonic map \( u_0 \) from an \( i \)-dimensional sphere \( S^i \) into \( N^n \) minimizing \( p \)-energy in its homotopy class for any \( p > i \).

On the other hand, B.-Y. Chen established in [7] the following result involving Riemannian submersion, minimal immersion, and cohomology class.

**Theorem B. ([7])** Let \( \pi: (M^m, g_M) \to (B^b, g_B) \) be a Riemannian submersion with minimal fibers and orientable base manifold \( B^b \). If \( M^m \) is a closed manifold with cohomology class \( H^b(M^m, \mathbb{R}) = 0 \), then the horizontal distribution \( \mathcal{H} \) of the Riemannian submersion is never integrable. Thus the submersion \( \pi \) is never non-trivial.

Whereas \( p \)-harmonic maps represent homotopy classes, B.-Y. Chen and S.W. Wei connected the two seemingly unrelated areas of \( p \)-harmonic morphisms and cohomology classes in the following.

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Theorem C. ([10, 11]) Let $u : (M^m, g_M) \to (N^n, g_N)$ be an $n$-harmonic morphism which is a submersion. If $N^n$ is an orientable $n$-manifold and $M^m$ is a closed $m$-manifold with $n$-th cohomology class $H^n(M^m, \mathbb{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of $u$ is never integrable. Hence the submersion $u$ is always non-trivial.

This recaptures Theorem B when $\pi : M^m \to B^b$ is a Riemannian submersion with minimal fibers and orientable base manifold $B^b$. While a horizontally weak conformal $p$-harmonic map is a $p$-harmonic morphism (cf. e.g., [11, Theorem 4]), $p$-harmonic morphism is also linked to cohomology class as follows.

Theorem D. ([10, 11]) Let $u : (M^m, g_M) \to (N^n, g_N)$ be an $n$-harmonic morphism which is a submersion. Then the pull back of the volume element of the base manifold $N^n$ is a harmonic $n$-form if and only if the horizontal distribution $\mathcal{H}$ of $u$ is completely integrable.

Following the proofs given in [10, 11], and by applying a characterization theorem of a $p$-harmonic morphism from [4, 6], and [29, Theorem 2.5], we seek a dual version of Theorem D. In particular, $p$-harmonic maps and cohomology classes are interrelated in [29] as follows.

Theorem E. Let $M^m$ be a closed $m$-manifold and $u : (M^m, g_M) \to (N^n, g_N)$ be an $n$-harmonic map which is a submersion. If $M^m$ is a closed $m$-manifold and the horizontal distribution $\mathcal{H}$ of $u$ is integrable and $u$ is an $n$-harmonic morphism, then we have $H^n(M, \mathbb{R}) \neq 0$.

Theorem F. ([29]) Let $u : (M^m, g_M) \to (N^n, g_N)$ be an $n$-harmonic map which is a submersion such that the horizontal distribution $\mathcal{H}$ of $u$ is integrable. If $M^m$ is a closed manifold with cohomology class $H^n(M^m, \mathbb{R}) = 0$. Then $u$ is not an $n$-harmonic morphism. Thus the submersion $u$ is always nontrivial.

The purpose of this paper is to point out the underlying essence of the foregoing Theorems C, D, E, and F is an application of stress-energy tensor and a conservation law. The results, for example Theorem 3.2 obtained by utilizing the $n$-conservation law (2.6), are sharp.

2 Preliminaries

2.1 Submersions

A differential map $u : (M^m, g_M) \to (N^n, g_N)$ between two Riemannian manifolds is called a submersion at a point $x \in M^m$ if its differential $du_x : T_x(M^m) \to T_{u(x)}(N^n)$ is a surjective linear map. A differentiable map $u$ that is a submersion at each point $x \in M^m$ is called a submersion. For each point $x \in N^n$, $u^{-1}(x)$ is called a fiber. For a submersion $u : M \to N$, let $\mathcal{H}_x$ denote the orthogonal complement of Kernel $(du_x : T_x(M^m) \to T_{u(x)}(N^n))$. Let $\mathcal{H} = \{\mathcal{H}_x : x \in M^m\}$ denote the horizontal distribution of $u$.

A submersion $u : (M^m, g_M) \to (N^n, g_N)$ is called horizontally weakly conformal if the restriction of $du_x$ to $\mathcal{H}_x$ is conformal, i.e., there exists a smooth function $\lambda$ on $M^m$ such that

$$u^*g_N = \lambda^2 g_M|_{\mathcal{H}_x} \quad \text{or} \quad g_N(du_x(X), du_x(Y)) = \lambda^2(x) g_M(X, Y) \quad (2.1)$$

for all $X, Y \in \mathcal{H}_x$ and $x \in M^m$. If the function $\lambda$ in (2.1) is positive, then $u$ is called horizontally conformal and $\lambda$ is called the dilation of $u$. For a horizontally conformal submersion $u$ with dilation $\lambda$, the energy density of $u$ is $\epsilon_u = \frac{1}{n}\lambda^2$ (cf. (2.2)).
A horizontally conformal submersion with dilation $\lambda \equiv 1$ is called a Riemannian submersion. Recall that a $k$-form $\omega$ on a compact Riemannian manifold is called harmonic if $\omega$ is both closed and co-closed, i.e., $d\omega = \delta \omega = 0$.

In [29, 4], generalizing the work of P. Baird and J. Eells for the case $n = 2$, and the necessary condition for the fibers being minimal, S. W. Wei linked $p$-harmonicity for every $p > 1$, and P. Baird and S. Gudmundsson linked $n$-harmonicity, $n = p = \dim N$ with minimal fibers as follows.

**Theorem 2.1** ([29], Theorem 2.5). Let $u : M \to N$ be a Riemannian submersion. Then $u$ is a $p$-harmonic map, for every $p > 1$, if and only if all fibers $u^{-1}(y)$, $y \in N$ are minimal submanifolds in $M$.

**Proposition 2.1** ([29], Proposition 2.4). Let $u : M \to N$ be a Riemannian submersion. Then $u$ is a $p$-harmonic morphism, for every $p > 1$, if and only if all fibers $u^{-1}(y)$, $y \in N$ are minimal submanifolds in $M$.

The case $p = 2$ in Theorem 2.1 and Proposition 2.1 are due to Eells-Sampson [16].

**Theorem 2.2** (P. Baird and S. Gudmundsson [4], Corollary 2.6). If $u : (M^m, g_M) \to (N^n, g_N)$ is a horizontally conformal submersion from a Riemannian manifold $M^m$ onto a Riemannian manifold $N^n$, then $u$ is $n$-harmonic if and only if the fibers of $u$ are minimal in $M^m$.

**Remark 1.** (i). The results of linking $p$-harmonicity for every $p > 1$, with minimal fibers in Theorem 2.1 can be extended to $p = 1 = n$ with minimal fibers. We refer to the celebrated work of E. Bombieri - E. De Gorgi - E. Giusti on minimal cones and the Bernstein problem ([5]), S.W. Wei on 1-harmonic functions ([28]), P. Baird - S. Gudmundsson on $p$-harmonic maps and minimal submanifolds ([4]), Y.I. Lee - S.W. Wei - A.N. Wang on a generalized 1-harmonic equation and the inverse mean curvature flow ([21]), etc. (ii). We also note that utilizing symmetry, Wu-Yi Hsiang pioneered the study of the inverse image of minimal submanifolds being minimal under appropriate conditions ([18]), which marked the birth of equivariant differential geometry (cf. e.g. W.Y. Hsiang - H.B. Lawson [19], S.W. Wei [26], etc.).

### 2.2 $F$- and $p$-harmonic morphisms

Let $u : (M^m, g_M) \to (N^n, g_N)$ be a differential map between two Riemannian manifolds $M$ and $N$. Denote $e_u$ the energy density of $u$, which is given by

$$e_u = \frac{1}{2} \sum_{i=1}^{m} g_N(du(e_i), du(e_i)) = \frac{1}{2} |du|^2,$$

where $\{e_1, \cdots, e_m\}$ is a local orthonormal frame field on $M^m$ and $|du|$ is the Hilbert-Schmidt norm of $du$, determined by the metric $g_M$ of $M$ and the metric $g_N$ of $N^n$. The energy of $u$, denoted by $E(u)$, is defined to be

$$E(u) = \int_M e_u \, dv_g,$$

A smooth map $u : M^m \to N^n$ is called harmonic if $u$ is a critical point of the energy functional $E$ with respect to any compactly supported variation.

Let $F : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $F(0) = 0$ and let $u : (M, g_M) \to (N, g_N)$ be a smooth map between two compact Riemannian manifolds. Then the map $u : M \to N$ is called $F$-harmonic if it is a critical point of the $F$-energy functional:

$$E_F(u) = \int_M F\left(\frac{|du|^2}{2}\right) \, dv_g.$$

(2.3)
In particular, if \( F(t) = \frac{1}{p}(2t)^{\frac{p}{2}} \), then the \( F \)-energy \( E_F(u) \) becomes \( p \)-energy, and its critical point \( u \) is called \( p \)-harmonic map. A map \( u : (M^m, g_M) \rightarrow (N^n, g_N) \) is a \( p \)-harmonic morphism if for any \( p \)-harmonic function \( f \) defined on an open set \( V \) of \( N^n \), the composition \( f \circ u \) is \( p \)-harmonic on \( u^{-1}(V) \).

### 2.3 Stress-Energy tensor

Let \( (M^m, g) \) be a smooth Riemannian \( m \)-manifold. Let \( \xi : E \rightarrow M^m \) be a smooth Riemannian vector bundle over \( (M^m, g) \), i.e. a vector bundle such that at each fiber is equipped with a positive inner product \( (\cdot, \cdot)_E \). Set \( A^p(\xi) = \Gamma(A^pT^*M \otimes E) \) the space of smooth \( p \)-forms on \( M^m \) with values in the vector bundle \( \xi : E \rightarrow M^m \).

For \( \omega \in A^p(\xi) \), set \(|\omega|^2 = (\omega, \omega) \) defined as in ([13, (2.3)]). The authors of [20] defined the following \( \mathcal{E}_{F,g} \)-energy functional given by

\[
\mathcal{E}_{F,g}(\omega) = \int_{M^m} F \left( \frac{|\omega|^2}{2} \right) dv_g
\]

where \( F : [0, +\infty) \rightarrow [0, +\infty) \) is as before.

The **stress-energy associated with the \( \mathcal{E}_{F,g} \)-energy functional** is defined as follows:

\[
S_{F,\omega}(X, Y) = F \left( \frac{|\omega|^2}{2} \right) g_M(X, Y) - F' \left( \frac{|\omega|^2}{2} \right) u^* g_N(i_X \omega, i_Y \omega)
\]

where \( i_X \omega \) is the interior multiplication by the vector field \( X \) given by

\[
(i_X \omega)(Y_1, \ldots, Y_{p-1}) = \omega(X, Y_1, \ldots, Y_{p-1})
\]

for \( \omega \in A^p(\xi) \) and any vector fields \( Y_i \) on \( M^m \), \( 1 \leq l \leq p - 1 \).

When \( F(t) = t \) and \( \omega = du \) for a map \( u : M^m \rightarrow N^n \), \( S_{F,\omega} \) is just the stress-energy tensor introduced in [3]. And when \( F(t) = \frac{1}{n}(2t)^{\frac{p}{2}} \) and \( \omega = du \) for a map \( u : M^m \rightarrow N^n \), \( S_{F,\omega} \) is the \( n \)-stress energy tensor \( S_n \) given by

\[
S_n = \frac{1}{n} |du|^n g_M - |du|^{n-2} u^* g_N.
\]

**Definition 1.** \( \omega \in A^p(\xi) (p \geq 1) \) is said to satisfy an **\( F \)-conservation law** if \( S_{F,\omega} \) is divergence free, i.e., the \((0,1)\)-type tensor field \( \text{div} S_{F,\omega} \) vanishes identically (i.e., \( \text{div} S_{F,\omega} \equiv 0 \)).

**Definition 2.1.** \( \omega \in A^p(\xi) (p \geq 1) \) is said to satisfy an **\( F \)-conservation law** if \( S_{F,\omega} \) is divergence free, i.e., the \((0,1)\)-type tensor field \( \text{div} S_{F,\omega} \) vanishes identically (i.e., \( \text{div} S_{F,\omega} \equiv 0 \)).

The \( n \)-conservation law is given by

\[
\text{div}(S_n) = 0
\]

(cf. [13, 22] for details), in which coarea formula was first employed by Y.X. Dong and S.W. Wei to derive monotonicity formulas, vanishing theorems, and Liouville theorems on complete noncompact manifolds from conservation laws.
3 Main Theorems and Their Proofs

Assume that \( \dim M^m = m \) and \( \dim N^n = n \).

**Theorem 3.1.** Let \( u : (M^m, g_M) \to (N^n, g_N) \) be a non-constant map. Then the \( n \)-stress tensor \( S_n = 0 \) if and only if \( m = n \) and \( u \) is conformal.

*Proof.* If \( S_n = 0 \), then \( u^*g_N = \frac{1}{n}|du|^2g_M = \lambda^2g_M \) (3.1) in the region \( du \neq 0 \), where \( \lambda \) is the dilation and thus

\[
0 = \text{trace } S_n = \frac{1}{n}|du|^n \text{trace } g_M - |du|^{n-2} \text{trace } u^*g_N
\]

\[
= e_n m - ne_n
\]

\[
= (m - n) e_n,
\]

where \( e_n \) is the \( n \)-energy density of \( u \) given by \( e_n = \frac{1}{n}|du|^n \). Hence, we get \( m = n \).

Conversely, if \( u^*g_N = \lambda^2g_M \) and \( m = n \), then we find

\[
|du|^2 = m\lambda^2, \quad \frac{1}{n}|du|^n = \frac{1}{n}(m\lambda^2)^{\frac{n}{2}}.
\]

Therefore, we obtain

\[
S_n = m^{\frac{n-2}{2}} (m-n) \frac{n}{n}\lambda^n g_M = 0,
\]

which shows that the \( n \)-stress tensor \( S_n \) vanishes identically. \( \square \)

**Theorem 3.2.** If \( m > n \) and \( u : (M^m, g_M) \to (N^n, g_N) \) is an \( n \)-harmonic and conformal map, then \( u \) is homothetic.

*Proof.* If \( u \) is \( n \)-harmonic, then it follows from [13, Corollary 2.2] that \( u \) satisfies \( n \)-conservation law, i.e., \( \text{div}(S_n) = 0 \).

In virtue of Theorem 3.1 and (3.3), with these hypotheses, we find

\[
0 = \text{div}(S_n) = \left( m^{\frac{n-2}{2}} \frac{m-n}{n} \right) \text{div}(\lambda^n g_M) = \left( m^{\frac{n-2}{2}} \frac{m-n}{n} \right) \langle d(\lambda^n), g_M \rangle.
\]

(3.4)

Thus, it follows from the assumption \( m > n \) that \( \lambda \) is a constant. Therefore, \( u \) is homothetic. \( \square \)

Theorems 3.2 is sharp in dimensions \( m > n \). That is, if \( m = n \), then the results no longer hold. Counterexamples can be provided and based on the fact that a conformal map between equal dimensional \( n \)-manifolds, such as stereographic projections \( u : \mathbb{E}^n \to S^n \) is \( n \)-harmonic, but \( u \) is not homothetic (cf. [30, 23]). In fact, Y. L. Ou and S. W. Wei proved the following:

**Theorem G.** ([23]) Let \( u : (M^m, g_M) \to (N^n, g_N) \) be a non-constant map between Riemannian manifolds with \( \dim M = \dim N = n \geq 2 \). Then \( u \) is an \( n \)-harmonic morphism if and only if \( u \) is weakly conformal.

While Theorems 3.2 on the one hand, augments Theorem G, on the other hand, Theorems 3.1 and 3.2 generalize the work of J. Eells and L. Lemaire ([15]) in which \( n = 2 \). Furthermore, Theorems 3.1 and 3.2 augment a theorem of M. Ara in [1] for the case the zeros of \( (n-2)F'(t) - 2tF''(t) \) are being isolated for horizontally conformal \( F \)-harmonic maps. Hence we obtain:
Theorem 3.3. Let \( u : (M^m, g_M) \to (N^n, g_N), m > n, \) be an \( F \)-harmonic map, which is horizontally conformal with dilation \( \lambda \).

Case 1. Assume that the zeros of \( (n-2)F'(t) - 2tF''(t) \) are isolated. Then the following three properties are equivalent:

1. The fibers of \( u \) are minimal submanifolds.
2. \( \text{grad}(\lambda^2) \) is vertical.
3. The horizontally distribution of \( u \) has mean curvature vector \( \frac{\text{grad}(\lambda^2)}{2N^2} \).

Case 2. Assume that the zeros of \( (n-2)F'(t) - 2tF''(t) \) are not isolated. Then

1. The fibers of \( u \) are minimal submanifolds.
2. \( u \) is homothetic, i.e. \( \lambda = C, \) a positive constant.
3. \( \text{grad}(\lambda^2) = 0 \), hence it is vertical.

Proof. Case 1 is exactly [1, Theorem 5.1] proved by M. Ara.

For Case 2, statement (1) follows from that fact that general solutions of
\[
(n-2)F'(t) - 2tF''(t) = 0
\]
are given by \( F(t) = at^\frac{n}{2} + b \) with constants \( a, b \). Hence, \( u \) is an \( n \)-harmonic map, and so we may apply Theorem 2.2 to conclude that fibers of \( u \) are minimal in \( M^m \). Statements (2) and (3) of Case 2 follow from Theorem 3.2 and the fact that \( u \) is an \( n \)-harmonic map.

In examining the converse of Theorem 3.3, Case 2, (1), we characterize the minimal fibers of a horizontally conformal map from the previously untreated case in \( F \)-harmonic maps:

Theorem 3.4. Let \( u : (M^m, g_M) \to (N^n, g_N), m > n, \) be a horizontally conformal map. Assume that the zeros of \( (n-2)F'(t) - 2tF''(t) \) are not isolated. Then the fibers of \( u \) are minimal submanifolds if and only if \( u \) is an \( F \)-harmonic map; if and only if \( u \) is an \( n \)-harmonic map.

Proof. This follows from the fact that when the zeros of \( (n-2)F'(t) - 2tF''(t) \) are not isolated, \( F \)-harmonic map is an \( n \)-harmonic map, and Theorem 2.2.

When the target manifold of \( u \) is a Riemann surface, i.e. \( n = 2 \), then we associate \( u \) with a harmonic map in the following way:

Theorem 3.5. Let \( (N^2, g_N) \) be a Riemann surface, and \( u : (M^m, g_M) \to (N^2, g_N), m > 2, \) be a horizontally conformal map. Assume that the zeros of \( -2tF''(t) \) are not isolated. Then the fibers of \( u \) are minimal submanifolds if and only if \( u \) is an \( F \)-harmonic map; if and only if \( u \) is a harmonic map.

Proof. This follows from the fact that when the zeros of \( -2tF''(t) \) are not isolated, \( F \)-harmonic map is a harmonic map, and Theorem 3.4.
4 Applications

As an application of Theorems 3.1 and 3.2, we revisit

**Theorem 4.1 (Theorem C. ([10, 11]))**. Let \( u : (M^m, g_M) \to (N^n, g_N) \) be an \( n \)-harmonic morphism which is a submersion. If \( N^n \) is an orientable manifold and \( M^m \) is a closed manifold with the \( n \)-th cohomology class \( H^n(M, \mathbb{R}) = 0 \), then the horizontal distribution \( \mathcal{H} \) of \( u \) is never integrable.

**Proof.** Under the hypothesis, in view of Theorem 2.2, \( u \) has minimal fibers and, according to Theorem 3.2, \( \lambda \) is constant. Let \( \{ \bar{e}_1, \ldots, \bar{e}_n \} \) be an oriented local orthonormal frame of the base manifold \( (N^n, g_N) \) and let \( \bar{\omega}^1, \ldots, \bar{\omega}^n \) denote the dual 1-forms of \( \{ \bar{e}_1, \ldots, \bar{e}_n \} \) on \( N^n \). Then \( \bar{\omega} = \bar{\omega}^1 \wedge \cdots \bar{\omega}^n \) is the volume form of \( (N^n, g_N) \), which is a closed \( n \)-form on \( N^n \).

Consider the pull back of the volume form \( \bar{\omega} \) of \( N^n \) via \( u \), which is denoted by \( u^*(\bar{\omega}) \). Then \( u^*(\bar{\omega}) \) is a simple \( n \)-form on \( M^m \) satisfying

\[
d(u^*(\bar{\omega})) = u^*(d\bar{\omega}) = 0, \tag{4.1}
\]
due to the fact that the exterior differentiation \( d \) and the pullback \( u^* \) commute.

Assume that \( m = \dim M^m = n + k \) and let \( e_1, \ldots, e_{n+k} \) be a local orthonormal frame field with \( \omega^1, \ldots, \omega^{n+k} \) being its dual coframe fields on \( M^m \) such that

(i) \( e_1, \ldots, e_n \) are basic horizontal vector fields satisfying \( du(e_i) = \lambda \bar{e}_i \), \( i = 1, \ldots, n \), and \( du(e_1), \ldots, du(e_n) \) give a positive orientation of \( N^n \); and

(ii) \( e_{n+1}, \ldots, e_{n+k} \) are vertical vector fields.

Then we have

\[
\omega^j(e_s) = 0, \quad \omega^j(e_j) = \delta_{ij}, \quad 1 \leq i, j \leq n; \quad n + 1 \leq s \leq n + k. \tag{4.2}
\]

Also, it follows from (i) that

\[
u^*\bar{\omega}^i = \frac{1}{\lambda} \omega^i, \quad i = 1, \ldots, n. \tag{4.3}\]

If we put

\[
\omega = \omega^1 \wedge \cdots \omega^n \quad \text{and} \quad \omega^\perp = \omega^{n+1} \wedge \cdots \wedge \omega^{n+k}, \tag{4.4}\]

then

\[
d\omega^\perp = \sum_{i=1}^k (-1)^i \omega^{n+1} \wedge \cdots \wedge d\omega^{n+i} \wedge \cdots \wedge \omega^{n+k}. \tag{4.5}\]

It follows from (4.2) and (4.5) that \( d\omega^\perp = 0 \) holds identically if and only if the following two conditions are satisfied:

\[
d\omega^\perp(e_i, e_{n+1}, \ldots, e_{n+k}) = 0, \quad i = 1, \ldots, n, \tag{4.6}\]

and

\[
d\omega^\perp(X, Y, V_1, \ldots, V_{k-1}) = 0. \tag{4.7}\]

for any horizontal vector fields \( X, Y \) and for vertical vector fields \( V_1, \ldots, V_{k-1} \).
Since the fibers of $u$ are minimal submanifolds of $M^m$, we find for each $1 \leq i \leq n$ that
\[
d\omega^\perp(e_i, e_{n+1}, \ldots, e_{n+k})
= \sum_{j=1}^{k} (-1)^{j+1} \omega^\perp([e_i, e_{n+j}], e_{n+1}, \ldots, \hat{e}_{n+j}, \ldots, e_{n+k})
= \sum_{j=1}^{k} (-1)^{j+1} \omega^{n+j}(\nabla e_i, e_{n+j}) - \omega^{n+j}(\nabla e_{n+j}, e_i)
= \sum_{j=1}^{k} -\langle \nabla e_{n+j} e_i, e_{n+j} \rangle
= \sum_{j=1}^{k} \langle h(e_{n+j}, e_{n+j}), e_i \rangle
= 0,
\]
(4.8)
where “\(\hat{\cdot}\)” denotes the missing term and $h$ denotes the second fundamental form of fibers in $M$, which prove that condition (4.2) holds.

Now, suppose that the horizontal distribution $\mathcal{H}$ is integrable. If $X, Y$ are horizontal vector fields, then $[X, Y]$ is also horizontal by Frobenius theorem. So, for vertical vector fields $V_1, \ldots, V_k$, we find (cf. [7, formula (6.7)] or [29, formula (3.5)])
\[
d\omega^\perp(X, Y, V_1, \ldots, V_{k-1}) = \omega^\perp([X, Y], V_1, \ldots, V_{k-1}) = 0.
\]
(4.9)
Consequently, from (4.8) and (4.9) we get
\[
d\omega^\perp = 0.
\]
(4.10)

Next, we show that if $\mathcal{H}$ is integrable, then we have $d((u^* \tilde{\omega})^\perp) = 0$. Since $u$ is a horizontally conformal submersion with constant dilation $\lambda$, it preserves orthogonality, which is crucial to horizontal and vertical distributions, and the pullback $u^*$ expands the length of 1-form constantly by $\frac{1}{\lambda}$ in every direction. This, via (4.3) and (4.10) leads to
\[
d((u^* \tilde{\omega})^\perp) = d((u^* \tilde{\omega}^1 \wedge \cdots \wedge u^* \tilde{\omega}^n)^\perp)
= d\left(\frac{1}{\lambda} \omega^1 \wedge \cdots \wedge \frac{1}{\lambda} \omega^n\right)^\perp
= \frac{1}{\lambda^n} d\omega^\perp
= 0.
\]
(4.11)
Since $d((u^* \omega)^\perp) = 0$ is equivalent to $u^* \omega$ being co-closed, it follows that, under the condition that $\mathcal{H}$ is integrable, the pullback of the volume form, $u^* \omega$ is a harmonic $n$-form on $M$. Thus, $u^* \omega$ gives rise to a non-trivial cohomology class in $H^n(M, \mathbb{R})$ by Hodge Theory [17]. Therefore, if $H^n(M, \mathbb{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of $u$ is never integrable.

From the proof of Theorem 4.1, we have the following.

**Theorem 4.2.** Let $u : (M, g_M) \rightarrow (N, g_N)$ be an $n$-harmonic morphism with $n = \dim N$ which is a submersion. Then the pull back of the volume element of $N$ is a harmonic $n$-form if and only if the horizontal distribution $\mathcal{H}$ of $u$ is completely integrable.
References


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