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n-harmonicity, minimality, conformality and cohomology

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Abstract. By studying cohomology classes that are related with n-harmonic morphisms and F-harmonic maps, we augment and extend several results on F-harmonic maps, harmonic maps in [1, 3, 15], p-harmonic morphisms in [23], and also revisit our previous results in [10, 11, 29] on Riemannian submersions and n-harmonic morphisms which are submersions. The results, for example Theorem 3.2 obtained by utilizing the n-conservation law (2.6), are sharp.

Keywords. p-harmonic maps, n-harmonic morphism, cohomology class, minimal submanifold, submersion

1 Introduction

Harmonicity and its variants are related with the topology and geometry of manifolds. It was shown in [27] that homotopy classes can be represented by p-harmonic maps (see, e.g. [29], for definition and examples of p-harmonic maps):

Theorem A. *If N^n is a compact Riemannian n-manifold, then for any positive integer i, each class in the i-th homotopy group $\pi_i(N^n)$ can be represented by a $C^{1,\alpha}$ p-harmonic map u_0 from an i-dimensional sphere S^i into N^n minimizing p-energy in its homotopy class for any $p > i$.*

On the other hand, B.-Y. Chen established in [7] the following result involving Riemannian submersion, minimal immersion, and cohomology class.

Theorem B. ([7]) *Let $\pi : (M^m, g_M) \to (B^b, g_B)$ be a Riemannian submersion with minimal fibers and orientable base manifold B^b. If M^m is a closed manifold with cohomology class $H^b(M^m, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of the Riemannian submersion is never integrable. Thus the submersion π is never non-trivial.*

Whereas p-harmonic maps represent homotopy classes, B.-Y. Chen and S. W. Wei connected the two seemingly unrelated areas of p-harmonic morphisms and cohomology classes in the following.

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Theorem C. ([10, 11]) *Let $u : (M^m, g_M) \to (N^n, g_N)$ be an n-harmonic morphism which is a submersion. If N^n is an orientable n-manifold and M^m is a closed m-manifold with n-th cohomology class $H^n(M^m, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of u is never integrable. Hence the submersion u is always non-trivial.*

This recaptures Theorem B when $\pi : M^m \to B^b$ is a Riemannian submersion with minimal fibers and orientable base manifold B^b. While a horizontally weak conformal p-harmonic map is a p-harmonic morphism (cf. e.g., [11, Theorem 4]), p-harmonic morphism is also linked to cohomology class as follows.

Theorem D. ([10, 11]) *Let $u : (M^m, g_M) \to (N^n, g_N)$ be an n-harmonic morphism which is a submersion. Then the pull back of the volume element of the base manifold N^n is a harmonic n-form if and only if the horizontal distribution $\mathcal{H}$ of u is completely integrable.*

Following the proofs given in [10, 11], and by applying a characterization theorem of a p-harmonic morphism from [4, 6], and [29, Theorem 2.5], we seek a dual version of Theorem D. In particular, p-harmonic maps and cohomology classes are interrelated in [29] as follows.

Theorem E. *Let M^m be a closed m-manifold and $u : (M^m, g_M) \to (N^n, g_N)$ be an n-harmonic map which is a submersion. If M^m is a closed m-manifold and the horizontal distribution $\mathcal{H}$ of u is integrable and u is an n-harmonic morphism, then we have $H^n(M, \mathbf{R}) \neq 0$.*

Theorem F. ([29]) *Let $u : (M^m, g_M) \to (N^n, g_N)$ be an n-harmonic map which is a submersion such that the horizontal distribution $\mathcal{H}$ of u is integrable. If M^m is a closed manifold with cohomology class $H^n(M^m, \mathbf{R}) = 0$. Then u is not an n-harmonic morphism. Thus the submersion u is always nontrivial.*

The purpose of this paper is to point out the underlying essence of the foregoing Theorems C, D, E, and F is an application of stress-energy tensor and a conservation law. The results, for example Theorem 3.2 obtained by utilizing the n-conservation law (2.6), are sharp.

2 Preliminaries

2.1 Submersions

A differential map $u : (M^m, g_M) \to (N^n, g_N)$ between two Riemannian manifolds is called a submersion at a point $x \in M^m$ if its differential $du_x : T_x(M^m) \to T_{u(x)}(N^n)$ is a surjective linear map. A differentiable map u that is a submersion at each point $x \in M^m$ is called a *submersion*. For each point $x \in N^n$, $u^{-1}(x)$ is called a *fiber*. For a submersion $u : M \to N$, let $\mathcal{H}_x$ denote the orthogonal complement of Kernel $\big(du_x : T_x(M^m) \to T_{u(x)}(N^n)\big)$. Let $\mathcal{H} = \{\mathcal{H}_x : x \in M^m\}$ denote the horizontal distribution of u.

A submersion $u : (M^m, g_M) \to (N^n, g_N)$ is called *horizontally weakly conformal* if the restriction of du_x to $\mathcal{H}_x$ is conformal, i.e., there exists a smooth function λ on M^m such that

$$u^* g_N = \lambda^2 g_{M_{|\mathcal{H}}} \quad \text{or} \quad g_N\big(du_x(X), du_x(Y)\big) = \lambda^2(x) g_M(X, Y) \tag{2.1}$$

for all $X, Y \in \mathcal{H}_x$ and $x \in M^n$. If the function λ in (2.1) is positive, then u is called *horizontally conformal* and λ is called the *dilation* of u. For a horizontally conformal submersion u with dilation λ, the *energy density* of u is $e_u = \frac{1}{2} n \lambda^2$ (cf. (2.2)).

A horizontally conformal submersion with dilation $\lambda \equiv 1$ is called a *Riemannian submersion*. Recall that a k-form ω on a compact Riemannian manifold is called *harmonic* if ω is both closed and co-closed, i.e., $d\omega = \delta\omega = 0$.

In [29, 4], generalizing the work of P. Baird and J. Eells for the case $n = 2$, and the necessary condition for the fibers being minimal, S. W. Wei linked p-harmonicity for every $p > 1$, and P. Baird and S. Gudmundsson linked n-harmonicity, $n = p = \dim N$ with minimal fibers as follows.

Theorem 2.1 ([29], Theorem 2.5)**.** *Let $u : M \to N$ be a Riemannian submersion. Then u is a p-harmonic map, for every $p > 1$, if and only if all fibers $u^{-1}(y)$, $y \in N$ are* minimal *submanifolds in M .*

Proposition 2.1 ([29], Propostion 2.4)**.** *Let $u : M \to N$ be a Riemannian submersion. Then u is a p-harmonic morphism, for every $p > 1$, if and only if all fibers $u^{-1}(y)$, $y \in N$ are* minimal *submanifolds in M .*

The case $p = 2$ in Theorem 2.1 and Proposition 2.1 are due to Eells-Sampson [16].

Theorem 2.2 (P. Baird and S. Gudmundsson [4], Corollary 2.6)**.** *If $u : (M^m, g_M) \to (N^n, g_N)$ is a horizontally conformal submersion from a Riemannian manifold M^m onto a Riemannian manifold N^n, then u is n-harmonic if and only if the fibers of u are minimal in M^m.*

Remark 1. (i). The results of linking p-harmonicity for every $p > 1$, with minimal fibers in Theorem 2.1 can be extended to $p = 1 = n$ with minimal fibers. We refer to the celebrated work of E. Bombieri - E. De Gorgi - E. Jiusti on minimal cones and the Bernstein problem ([5]), S.W. Wei on 1-harmonic functions ([28]), P. Baird - S. Gudmundsson on p-harmonic maps and minimal submanifods ([4]), Y.I. Lee - S.W. Wei - A.N. Wang on a generalized 1-harmonic equation and the inverse mean curvature flow ([21]), etc. (ii). We also note that utilizing symmetry, Wu-Yi Hsiang pioneered the study of the inverse image of minimal submanifolds being minimal under appropriate conditions ([18]), which marked the birth of *equivariant differential geometry* (cf. e.g. W.Y. Hsiang - H.B. Lawson [19], S.W. Wei [26], etc.).

2.2 F- and p-harmonic morphisms

Let $u : (M^m, g_M) \to (N^n, g_N)$ be a differential map between two Riemannian manifolds M and N. Denote e_u the *energy density* of u, which is given by

$$e_u = \frac{1}{2}\sum_{i=1}^{m} g_N\big(du(e_i), du(e_i)\big) = \frac{1}{2}|du|^2 , \tag{2.2}$$

where $\{e_1, \cdots, e_m\}$ is a local orthonormal frame field on M^m and $|du|$ is the Hilbert-Schmidt norm of du, determined by the metric g_M of M and the metric g_N of N^n. *The energy of* u, denoted by $E(u)$, is defined to be

$$E(u) = \int_M e_u \, dv_g.$$

A smooth map $u : M^m \to N^n$ is called *harmonic* if u is a critical point of the energy functional E with respect to any compactly supported variation.

Let $F : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $F(0) = 0$ and let $u : (M, g_M) \to (N, g_N)$ be a smooth map between two compact Riemannian manifolds. Then the map $u : M \to N$ is called F*-harmonic* if it is a critical point of the F-energy functional:

$$E_F(u) = \int_M F\left(\frac{|du|^2}{2}\right) dv_g. \tag{2.3}$$

In particular, if $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, then the F-energy $E_F(u)$ becomes p-energy, and its critical point u is called *p-harmonic map*. A map $u : (M^m, g_M) \to (N^n, g_N)$ is a *p-harmonic morphism* if for any p-harmonic function f defined on an open set V of N^n, the composition $f \circ u$ is p-harmonic on $u^{-1}(V)$.

2.3 Stress-Energy tensor

Let (M^m, g) be a smooth Riemannian m-manifold. Let $\xi : E \to M^m$ be a smooth Riemannian vector bundle over (M^m, g), i.e. a vector bundle such that at each fiber is equipped with a positive inner product $\langle \quad , \quad \rangle_E$. Set $A^p(\xi) = \Gamma(\Lambda^p T^*M \otimes E)$ the space of smooth p-forms on M^m with values in the vector bundle $\xi : E \to M^m$.

For $\omega \in A^p(\xi)$, set $|\omega|^2 = \langle \omega, \omega \rangle$ defined as in ([13, (2.3)],). The authors of [20] defined the following $\mathcal{E}_{F,g}$-energy functional given by

$$\mathcal{E}_{F,g}(\omega) = \int_{M^m} F\left(\frac{|\omega|^2}{2}\right) dv_g$$

where $F : [0, +\infty) \to [0, +\infty)$ is as before.

The *stress-energy associated with the $\mathcal{E}_{F,g}$-energy functional* is defined as follows:

$$S_{F,\omega}(X, Y) = F\left(\frac{|\omega|^2}{2}\right) g_M(X, Y) - F'\left(\frac{|\omega|^2}{2}\right) u^* g_N(i_X\omega, i_Y\omega) \tag{2.4}$$

where $i_X\omega$ is the interior multiplication by the vector field X given by

$$(i_X\omega)(Y_1, \dots, Y_{p-1}) = \omega(X, Y_1, \dots, Y_{p-1})$$

for $\omega \in A^p(\xi)$ and any vector fields Y_l on M^m, $1 \le l \le p - 1$.

When $F(t) = t$ and $\omega = du$ for a map $u : M^m \to N^n$, $S_{F,\omega}$ is just the *stress-energy tensor* introduced in [3]. And when $F(t) = \frac{1}{n}(2t)^{\frac{n}{2}}$ and $\omega = du$ for a map $u : M^m \to N^n$, $S_{F,\omega}$ is the *n-stress energy tensor* S_n given by

$$S_n = \frac{1}{n}|du|^n g_M - |du|^{n-2} u^* g_N\,. \tag{2.5}$$

Definition 1. $\omega \in A^p(\xi)\,(p \ge 1)$ is said to satisfy an *F-conservation law* if $S_{F,\omega}$ is divergence free, i.e., the $(0, 1)$-type tensor field $\operatorname{div} S_{F,\omega}$ vanishes identically (i.e., $\operatorname{div} S_{F,\omega} \equiv 0$).

Definition 2.1. $\omega \in A^p(\xi)\,(p \ge 1)$ *is said to satisfy an* F-conservation law *if* $S_{F,\omega}$ *is divergence free, i.e., the* $(0, 1)$*-type tensor field* $\operatorname{div} S_{F,\omega}$ *vanishes identically* (*i.e.,* $\operatorname{div} S_{F,\omega} \equiv 0$).

The n-conservation law is given by

$$\operatorname{div}(S_n) = 0 \tag{2.6}$$

(cf. [13, 22] for details), in which coarea formula was *first* employed by Y. X. Dong and S. W. Wei to derive monotonicity formulas, vanishing theorems, and Liouville theorems on complete noncompact manifolds from conservation laws.

3 Main Theorems and Their Proofs

Assume that $\dim M^m = m$ and $\dim N^n = n$.

Theorem 3.1. *Let $u : (M^m, g_M) \to (N^n, g_N)$ be a non-constant map. Then the n-stress tensor $S_n = 0$ if and only if $m = n$ and u is conformal.*

Proof. If $S_n = 0$, then

$$u^* g_N = \frac{1}{n}|du|^2 g_M = \lambda^2 g_M \tag{3.1}$$

in the region $du \neq 0$, where λ is the dilation and thus

$$\begin{aligned} 0 = \text{trace } S_n &= \frac{1}{n}|du|^n \text{ trace } g_M - |du|^{n-2} \text{ trace } u^* g_N \\ &= e_n\, m - n e_n \\ &= (m-n)\, e_n, \end{aligned} \tag{3.2}$$

where e_n is the n-energy density of u given by $e_n = \frac{1}{n}|du|^n$. Hence, we get $m = n$.

Conversely, if $u^* g_N = \lambda^2 g_M$ and $m = n$, then we find

$$|du|^2 = m\lambda^2,\ \frac{1}{n}|du|^n = \frac{1}{n}(m\lambda^2)^{\frac{n}{2}}.$$

Therefore, we obtain

$$S_n = m^{\frac{n-2}{2}} \frac{(m-n)}{n} \lambda^n g_M = 0, \tag{3.3}$$

which shows that the n-stress tensor S_n vanishes identically. □

Theorem 3.2. *If $m > n$ and $u : (M^m, g_M) \to (N^n, g_N)$ is an n-harmonic and conformal map, then u is homothetic.*

Proof. If u is n-harmonic, then it follows from [13, Corollary 2.2] that u satisfies n-conservation law, i.e., $\text{div}(S_n) = 0$.

In virtue of Theorem 3.1 and (3.3), with these hypotheses, we find

$$0 = \text{div}(S_n) = \left(m^{\frac{n-2}{2}} \frac{m-n}{n}\right) \text{div}(\lambda^n g_M) = \left(m^{\frac{n-2}{2}} \frac{m-n}{n}\right) \langle d(\lambda^n), g_M \rangle. \tag{3.4}$$

Thus, it follows from the assumption $m > n$ that λ is a constant. Therefore, u is homothetic. □

Theorems 3.2 is sharp in dimensions $m > n$. That is, if $m = n$, then the results no longer hold. Counterexamples can be provided and based on the fact that a conformal map between equal dimensional n-manifolds, such as stereographic projections $u : \mathbb{E}^n \to S^n$ is n-harmonic, but u is not homothetic (cf. [30, 23]). In fact, Y. L. Ou and S. W. Wei proved the following:

Theorem G. ([23]) *Let $u : (M^m, g_M) \to (N^n, g_N)$ be a non-constant map between Riemannian manifolds with $\dim M = \dim N = n \geq 2$. Then u is an n-harmonic morphism if and only if u is weakly conformal.*

While Theorems 3.2 on the one hand, augments Theorem G, on the other hand, Theorems 3.1 and 3.2 generalize the work of J. Eells and L. Lemaire ([15]) in which $n = 2$. Furthermore, Theorems 3.1 and 3.2 augment a theorem of M. Ara in [1] for the case the zeros of $(n-2)F'(t) - 2tF''(t)$ are being isolated for horizontally conformal F-harmonic maps. Hence we obtain:

Theorem 3.3. *Let $u : (M^m, g_M) \to (N^n, g_N)$, $m > n$, be an F-harmonic map, which is horizontally conformal with dilation λ.*

Case 1. Assume that the zeros of $(n-2)F'(t) - 2tF''(t)$ are isolated. Then the following three properties are equivalent:

(1) *The fibers of u are minimal submanifolds.*

(2) $\mathrm{grad}(\lambda^2)$ *is vertical.*

(3) *The horizontally distribution of u has mean curvature vector $\frac{\mathrm{grad}(\lambda^2)}{2\lambda^2}$.*

Case 2. Assume that the zeros of $(n-2)F'(t) - 2tF''(t)$ are not isolated. Then

(1) *The fibers of u are minimal submanifolds.*

(2) *u is homothetic, i.e. $\lambda = C$, a positive constant.*

(3) $\mathrm{grad}(\lambda^2) = 0$*, hence it is vertical.*

Proof. Case 1 is exactly [1, Theorem 5.1] proved by M. Ara.

For Case 2, statement (1) follows from that fact that general solutions of

$$(n-2)F'(t) - 2tF''(t) = 0$$

are given by $F(t) = at^{\frac{n}{2}} + b$ with constants a, b. Hence, u is an n-harmonic map, and so we may apply Theorem 2.2 to conclude that fibers of u are minimal in M^m. Statements (2) and (3) of Case 2 follow from Theorem 3.2 and the fact that u is n-harmonic. □

In examining the converse of Theorem 3.3, Case 2, (1), we characterize the minimal fibers of a horizontally conformal map from the previously untreated case in F-harmonic maps:

Theorem 3.4. *Let $u : (M^m, g_M) \to (N^n, g_N)$, $m > n$, be a horizontally conformal map. Assume that the zeros of $(n-2)F'(t) - 2tF''(t)$ are not isolated. Then the fibers of u are minimal submanifolds if and only if u is an F-harmonic map; if and only if u is an n-harmonic map.*

Proof. This follows from the fact that when the zeros of $(n-2)F'(t) - 2tF''(t)$ are not isolated, F-harmonic map is an n-harmonic map, and Theorem 2.2. □

When the target manifold of u is a Riemann surface, i.e. $n = 2$, then we associate u with a harmonic map in the following way:

Theorem 3.5. *Let (N^2, g_N) be a Riemann surface, and $u : (M^m, g_M) \to (N^2, g_N)$, $m > 2$, be a horizontally conformal map. Assume that the zeros of $-2tF''(t)$ are not isolated. Then the fibers of u are minimal submanifolds if and only if u is an F-harmonic map; if and only if u is a harmonic map.*

Proof. This follows from the fact that when the zeros of $-2tF''(t)$ are not isolated, F-harmonic map is a harmonic map, and Theorem 3.4. □

4 Applications

As an application of Theorems 3.1 and 3.2, we revisit

Theorem 4.1 (**Theorem C.** ([10, 11]). *Let $u : (M^m, g_M) \to (N^n, g_N)$ be an n-harmonic morphism which is a submersion. If N^n is an orientable manifold and M^m is a closed manifold with the n-th cohomology class $H^n(M, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of u is never integrable.*

Proof. Under the hypothesis, in view of Theorem 2.2, u has minimal fibers and, according to Theorem 3.2, λ is constant. Let $\{\bar{e}_1, \ldots, \bar{e}_n\}$ be an oriented local orthonormal frame of the base manifold (N^n, g_N) and let $\bar{\omega}^1, \ldots, \bar{\omega}^n$ denote the dual 1-forms of $\{\bar{e}_1, \ldots, \bar{e}_n\}$ on N^n. Then $\bar{\omega} = \bar{\omega}^1 \wedge \cdots \bar{\omega}^n$ is the volume form of (N^n, g_N), which is a closed n-form on N^n.

Consider the pull back of the volume form $\bar{\omega}$ of N^n via u, which is denoted by $u^*(\bar{\omega})$. Then $u^*(\bar{\omega})$ is a simple n-form on M^m satisfying

$$d\big(u^*(\bar{\omega})\big) = u^*(d\bar{\omega}) = 0, \tag{4.1}$$

due to the fact that the exterior differentiation d and the pullback u^* commute.

Assume that $m = \dim M^m = n + k$ and let $e_1, \ldots, e_{n+k}$ be a local orthonormal frame field with $\omega^1, \ldots, \omega^{n+k}$ being its dual coframe fields on M^m such that

(i) $e_1, \ldots, e_n$ are basic horizontal vector fields satisfying $du(e_i) = \lambda \bar{e}_i$, $i = 1, \ldots, n$, and $du(e_1), \ldots, du(e_n)$ give a positive orientation of N^n; and

(ii) $e_{n+1}, \ldots, e_{n+k}$ are vertical vector fields.

Then we have

$$\omega^j(e_s) = 0, \quad \omega^i(e_j) = \delta_{ij}, \quad 1 \le i, j \le n;\ n+1 \le s \le n+k\,. \tag{4.2}$$

Also, it follows from (i) that

$$u^*\bar{\omega}^i = \frac{1}{\lambda}\omega^i, \quad i = 1, \ldots, n. \tag{4.3}$$

If we put

$$\omega = \omega^1 \wedge \cdots \wedge \omega^n \quad \text{and} \quad \omega^\perp = \omega^{n+1} \wedge \cdots \wedge \omega^{n+k}\,, \tag{4.4}$$

then

$$d\omega^\perp = \sum_{i=1}^{k} (-1)^i \omega^{n+1} \wedge \cdots \wedge d\omega^{n+i} \wedge \cdots \wedge \omega^{n+k}. \tag{4.5}$$

It follows from (4.2) and (4.5) that $d\omega^\perp = 0$ holds identically if and only if the following two conditions are satisfied:

$$d\omega^\perp(e_i, e_{n+1}, \ldots, e_{n+k}) = 0, \quad i = 1, \ldots, n, \tag{4.6}$$

and

$$d\omega^\perp(X, Y, V_1, \ldots, V_{k-1}) = 0. \tag{4.7}$$

for any horizontal vector fields X, Y and for vertical vector fields $V_1, \ldots, V_{k-1}$.

Since the fibers of u are minimal submanifolds of M^m, we find for each $1 \le i \le n$ that

$$
\begin{aligned}
&d\omega^{\perp}(e_i, e_{n+1}, \dots, e_{n+k}) \\
&\qquad = \sum_{j=1}^{k} (-1)^{j+1} \omega^{\perp}([e_i, e_{n+j}], e_{n+1}, \dots, \hat{e}_{n+j}, \dots, e_{n+k}) \\
&\qquad = \sum_{j=1}^{k} (-1)^{j+1} \big(\omega^{n+j}(\nabla_{e_i} e_{n+j}) - \omega^{n+j}(\nabla_{e_{n+j}} e_i)\big) \\
&\qquad = \sum_{j=1}^{k} -\langle \nabla_{e_{n+j}} e_i, e_{n+j} \rangle \\
&\qquad = \sum_{j=1}^{k} \langle h(e_{n+j}, e_{n+j}), e_i \rangle \\
&\qquad = 0,
\end{aligned}
\tag{4.8}
$$

where "$\hat{\ }$" denotes the missing term and h denotes the second fundamental form of fibers in M, which prove that condition (4.2) holds.

Now, suppose that the horizontal distribution $\mathcal{H}$ is integrable. If X, Y are horizontal vector fields, then $[X, Y]$ is also horizontal by Frobenius theorem. So, for vertical vector fields $V_1, \dots, V_k$, we find (cf. [7, formula (6.7)] or [29, formula (3.5)])

$$
d\omega^{\perp}(X, Y, V_1, \dots, V_{k-1}) = \omega^{\perp}([X, Y], V_1, \dots, V_{k-1}) = 0. \tag{4.9}
$$

Consequently, from (4.8) and (4.9) we get

$$
d\omega^{\perp} = 0. \tag{4.10}
$$

Next, we show that if $\mathcal{H}$ is integrable, then we have $d\big((u^*\bar{\omega})^{\perp}\big) = 0$. Since u is a horizontally conformal submersion with constant dilation λ, it preserves orthogonality, which is crucial to horizontal and vertical distributions, and the pullback u^* expands the length of 1-form constantly by $\frac{1}{\lambda}$ in every direction. This, via (4.3) and (4.10) leads to

$$
\begin{aligned}
d\big((u^*\bar{\omega})^{\perp}\big) &= d\big((u^*\bar{\omega}^1 \wedge \cdots \wedge u^*\bar{\omega}^n)^{\perp}\big) \\
&= d\left(\frac{1}{\lambda}\omega^1 \wedge \cdots \wedge \frac{1}{\lambda}\omega^n\right)^{\perp} \\
&= \frac{1}{\lambda^n} d\omega^{\perp} \\
&= 0.
\end{aligned}
\tag{4.11}
$$

Since $d\big((u^*\omega)^{\perp}\big) = 0$ is equivalent to $u^*\omega$ being co-closed, it follows that, under the condition that $\mathcal{H}$ is integrable, the pullback of the volume form, $u^*\omega$ is a harmonic n-form on M. Thus, $u^*\omega$ gives rise to a non-trivial cohomology class in $H^n(M, \mathbf{R})$ by Hodge Theory [17]. Therefore, if $H^n(M, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of u is never integrable. □

From the proof of Theorem 4.1, we have the following.

Theorem 4.2. *Let $u : (M, g_M) \to (N, g_N)$ be an n-harmonic morphism with $n = \dim N$ which is a submersion. Then the pull back of the volume element of N is a harmonic n-form if and only if the horizontal distribution $\mathcal{H}$ of u is completely integrable.*

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