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Hypergeometric type extended bivariate zeta function

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Abstract. Based on the generalized extended beta function, we shall introduce and study a new hypergeometric-type extended zeta function together with related integral representations, differential relations, finite sums, and series expansions. Also, we present a relationship between the extended zeta function and the Laguerre polynomials. Our hypergeometric type extended zeta function involves several known zeta functions including the Riemann, Hurwitz, Hurwitz-Lerch, and Barnes zeta functions as particular cases.

Keywords. Extended beta function, extended bivariate zeta function, Hurwitz-Lerch zeta function, Mellin-Barnes integrals, hypergeometric functions

1 Introduction

The Hurwitz zeta function [9] is defined by

$$\zeta(z,a)=\sum_{n=0}^{\infty}(a+n)^{-z} \tag{1.1}$$

$$(a\neq\mathbb{Z}_0^-;\Re(z)>1),$$

which is a generalization of the Riemann zeta function

$$\zeta(z)=\sum_{n=0}^{\infty}n^{-z}. \tag{1.2}$$

Here and elsewhere, let $\mathbb{C},\mathbb{R},\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}_0^-$ denote the complex numbers, real numbers, positive integers, and non-negative integers and the non-positive integers.
As a generalization of both Riemann and Hurwitz zeta functions the so-called Hurwitz-Lerch zeta function is defined by [8,p.27(1)]:

$$\Phi(y,z,a)=\sum_{n=0}^{\infty}\frac{y^n}{(a+n)^z}, \tag{1.3}$$

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$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; |y| < 1).$$

Φ is an analytic function in both variables y and z in a suitable region and it reduces to the ordinary Lerch zeta function when $y = e^{2\pi i\lambda}$:

$$\Phi(e^{2\pi i\lambda}, z, a) = \phi(\lambda, z, a) = \sum_{n=0}^{\infty} \frac{e^{2\pi in\lambda}}{(a+n)^z}. \tag{1.4}$$

Next, we recall here a further generalization of the Hurwitz-Lerch zeta function $\Phi(y, z, a)$ in the form (see [13,p.100,Equation (1.5)]):

$$\Phi_\mu^*(x, z, a) = \sum_{n=0}^{\infty} \frac{(\mu)_n x^n}{(a+n)^z}, \tag{1.5}$$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)...(\alpha+n-1)$ denotes the Pochhammer's symbol, $\mu \in \mathbb{C}, a \notin \mathbb{Z}_0^-$ and $|x| < 1$. Obviously, when $\mu = 1$, (1.5) reduces to (1.3).
In [3] Bin-Saad introduced the hypergeometric type generating function of the generalized zeta function defined by (1.3) and (1.5) in the form:

$$\zeta_\lambda^\mu(x, y; z, a) = \sum_{m=0}^{\infty} (\mu)_m \Phi(y, z, a+\lambda m) \frac{x^m}{m!}, \tag{1.6}$$

where $|x| < 1, |y| < 1; \mu \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda \in \mathbb{C} \setminus \{0\}; a \in \mathbb{C} \setminus \{-(n+\lambda m)\}, n, m \in \mathbb{N}_0$ and Φ is the Hurwitz-Lerch zeta function defined by (1.3).
The alternative representation

$$\zeta_\lambda^\mu(x, y; z, a) = \sum_{n=0}^{\infty} \Phi_\mu^*\left(x, z, \frac{a+n}{\lambda}\right) \frac{y^n}{\lambda^z}, \tag{1.7}$$

where Φ_μ^* is the generalized zeta function defined by (1.5), follows by changing the order of summations and considering equation (1.5) . The case when $y = 0$ of the definition (1.6) gives us the following further generalization of the zeta function defined by (1.5) [3]:

$$\zeta_\lambda^\mu(x, 0; z, a) = \Phi_{\mu,\lambda}^*(x, z, a) = \sum_{m=0}^{\infty} \frac{(\mu)_m x^m}{m!(a+\lambda m)^z}, \tag{1.8}$$

where $|x| < 1; \mu \in \mathbb{C} \setminus \mathbb{Z}_0^-; a \in \mathbb{C} \setminus \{-(\lambda m)\}, m \in \mathbb{N}_0$.

A further generalization of the Hurwitz-Lerch zeta function Φ_μ^* (see (1.5)):

$$\Phi_{\lambda,\mu;\nu}(x, z, a) = \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n \ x^n}{(\nu)_n (a+n)^z}, \tag{1.9}$$

where $\lambda, \mu \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; x \in \mathbb{C}$ when $|x| < 1; \Re(z+\nu-\lambda-\mu) > 1$ when $|x| = 1$,
was investigated earlier by Garg et al. [12,p.313, Eq(1.7)]. In recent years, several extensions of well known special functions have been considered by several authors [4-7]. In [5], the following extension of Euler's beta function

$$B(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt; \Re(p) \geq 0, \Re(x) > 0, \Re(y) > 0, \tag{1.10}$$

has been defined in [5] (see also [6]) and it has been proved that this extension has connection with Macdonald, error and Whittaker's functions. It is obvious that $p=0$ gives the original beta function:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt; \Re(x)>0, \Re(y)>0.$$

Recently, in [6] Chaudhry et al. generalized the Gaussian hypergeometric function ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$ as follows

$$F_p\left(a,b;c;x\right)=\sum_{n=0}^{\infty}\frac{B(b+n,c-b;p)(a)_n}{B(b,c-b)}\frac{x^n}{n!}; p\geq 0, |x|<1, \Re(c)>\Re(b)>0, \tag{1.11}$$

and

$$\Phi_p\left(a,b;c;x\right)=\sum_{n=0}^{\infty}\frac{B(b+n,c-b;p)}{B(b,c-b)}\frac{x^n}{n!}; p\geq 0, |x|<1, \Re(c)>\Re(b)>0. \tag{1.12}$$

The generalization of Euler's beta function is represented by the various following integral representations [7]:

$$B(x,y;p)=2\int_0^{\frac{\pi}{2}}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\exp(-p\ \sec^2\theta\csc^2\theta), \tag{1.13}$$

$$B(x,y;p)=e^{-2p}\int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}\exp\left[-p(u+u^{-1})\right]du, \tag{1.14}$$

$$B(a,b;p)=2^{1-a-b}\int_{-\infty}^{\infty}\exp\left[(a-b)x-4\,p\cosh^2 x)\right]\frac{dx}{(\cosh x)^{a+b}}dx. \tag{1.15}$$

Formulas (1.13), (1.14) and (1.15) can be obtained by using $t-\cos^2\theta, t=\frac{u}{(1+u)}$ and $t=\tanh x$ in formula (1.10) respectively. The present work aims at introducing and investigating a new kind of hypergeometric type generating functions $\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)$ given by (2.1) or (2.2) below. The results we will obtain and discuss are a further contribution along the line developed in [2] and [3].

The layout of the paper is as follows. In Section 2 we introduce and describe some properties and relationships for the function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$. Relevant connections of the function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ with those considered in [3] are also indicated. In Section 3, we establish several integral representations for the function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ involving integral representations of contour and Mellin-Barnes type of integrals. Section 4 is devoted to the differentiation of the function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ with respect to arguments x,y,z,λ,δ and a. In the final section, we present some series expansions for the function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ involving Saran's function of three variables F_K and the generalized hypergeometric function ${}_3F_2$. Also, we present a connection between our new extended zeta function and the Laguerre polynomials.

2 The Extended Bivariate Zeta Function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)$

By virtue of the extension of Euler's beta function (1.10), we aim in the present work to introduce and study an extended bivariate zeta function of the form

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m y^n}{m!(a+n+\lambda m)^z}, \tag{2.1}$$

$\left(\{|x|, |y|\} < 1; \{\delta, \mu, \nu\} \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda \in \mathbb{C} \setminus \{0\}; a \in \mathbb{C} \setminus \{-(n + \lambda m)\}, \Re(p) \geq 0, n, m \in \mathbb{N}_0\right).$

The alternative representation

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x, y; z, a; p) = \sum_{m=0}^{\infty} \frac{B(\mu + m, \nu - \mu; p)(\delta)_m}{B(\mu, \nu - \mu)} \Phi(y, z, a + \lambda m) \frac{x^m}{m!}, \tag{2.2}$$

where Φ is the generalized zeta function defined by (1.3), follows by changing the order of summations and using the definition (1.3). Indeed, formula (2.2) shows that the function $\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x, y; z, a; p)$ is a hypergeometric type generating function of the function Φ defined by (1.3). We have the following relationships

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(0, 1; z, a; 0) = \zeta(z, a), \tag{2.3}$$

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(0, y; z, a; 0) = \Phi(y, z, a), \tag{2.4}$$

$$\zeta_{\nu,1}^{*(\delta,\mu)}(x, 0; z, a; 0) = \Phi_{\delta,\mu;\nu}(x, z, a), \tag{2.5}$$

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x, 0; z, a; 0) = \Phi_{\delta,\lambda}^{*}(x, z, a), \tag{2.6}$$

$$\zeta_{\nu,1}^{*(\delta,\mu)}(x, 0; z, a; 0) = \Phi_{\delta}^{*}(x, z, a), \tag{2.7}$$

$$\zeta_{\mu,\lambda}^{*(\delta,\mu)}(x, y; z, a; 0) = \zeta_{\lambda}^{\delta}(x, y; z, a). \tag{2.8}$$

The case when $y = 0$ of the definition (2.1) suggests us to define the generalization of the zeta function $\Phi_{\lambda,\mu;\nu}$ defined by (1.9) in the following interesting form:

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x; z, a; p) = \sum_{m=0}^{\infty} \frac{B(\mu + m, \nu - \mu; p)}{B(\mu, \nu - \mu)} \frac{(\delta)_m x^m}{m!(a + \lambda m)^z}. \tag{2.9}$$

Whereas the case when $p = 0$ of the definition (2.1) gives us another generalization of the zeta function $\Phi_{\lambda,\mu;\nu}$ in the form:

$$\Phi_{\nu,\lambda}^{\delta,\mu}(x, y; z, a) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta)_m (\mu)_m x^m y^n}{(\nu)_m m!(a + n + \lambda m)^z}, \tag{2.10}$$

$\left(\max\{|x|, |y|\} < 1; \{\delta, \mu, \nu\} \in \mathbb{C}^2 \times \left(\mathbb{C} \setminus \mathbb{Z}_0^-\right), \lambda \in \mathbb{C} \setminus \{0\}; a \in \mathbb{C} \setminus \{-(n + \lambda m)\},\right.$
$\left.\{n, m\} \in \mathbb{N} \cup \{0\}\right),$
which for $y = 0$, reduces to the interesting special case:

$$\Phi_{\mu,\nu;\lambda}^{\delta}(x; z, a) = \sum_{m=0}^{\infty} \frac{(\delta)_m (\mu)_m x^m}{(\nu)_m m!(a + \lambda m)^z}. \tag{2.11}$$

Based on the definition (2.11), we can present another alternative representation for the extended zeta function $\zeta_{\nu,\lambda}^{*(\delta,\mu)}$ in the form:

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x, y; z, a; p) = \sum_{n=0}^{\infty} \Phi_{\mu,\nu;\lambda}^{\delta}\left(x; z, \frac{a + n}{\lambda}; p\right) \frac{y^n}{\lambda^z}. \tag{2.12}$$

In the case when $\lambda = p = 0$, we have simply

$$\zeta_{\nu,0}^{*(\delta,\mu)}(x,y;z,a;0) = a^{-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_n(\delta)_m(\mu)_m(1)_n x^m y^n}{(\nu)_m(a+1)_n m!n!},$$

which implies the next result .

Corollary 2.1. *Let $max\{|x|,|y|\} < 1, \Re(a) > 0$. Then*

$$\zeta_{\nu,0}^{*(\delta,\mu)}(x,y;z,a;0) = a^{-1}{}_2F_1[\delta,\mu;\nu;x] \times {}_2F_1[a,1;a+1;y], \tag{2.13}$$

where ${}_2F_1$ is the Gaussian hypergeometric function of one variables (cf. [24]).
According to the relationship (2.4), equation (2.13) yields the following known result [8,p.30(10)]:

$$\Phi(y,1,a) = a^{-1}{}_2F_1[a,1;a+1;y].$$

Corollary 2.2. *Let $\lambda = 1, max\{|x|,|y|\} < 1$. Then*

$$\zeta_{\nu,1}^{\delta,\mu}(xy,y;z,a;p) = \Phi(y,z,a) \times F_p(\delta,\mu;\nu;x), \tag{2.14}$$

and

$$\zeta_{\nu,1}^{\delta,\mu}(x,yx;z,a;0) = \sum_{m=0}^{\infty}\frac{(\delta)_m(\mu)_m}{(\nu)_m(a+m)^z} \times {}_3F_2\left[\begin{array}{c}-m,1-\nu-m,1;\\ \\ 1-\delta,1-\mu-m;\end{array} y\right]\frac{x^m}{m!}. \tag{2.15}$$

Proof. We have

$$\zeta_{\nu,1}^{\delta,\mu}(xy,y;z,a;p) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m y^{n+m}}{m!(a+n+m)^z}.$$

Then by letting $n \to n-m$ and considering the Hurwitz-Lerch zeta function $\Phi(y,z,a)$ and the extended Gaussian hypergeometric function F_p (see (1.3) and (1.11)), we get (2.14) . Similarly, one can prove the result (2.15). □

Further, we recall the definition of the derivative operator D_x^m (see [20] and [18]):

$$D_x^m x^{\delta+m-1} = \frac{\Gamma(\delta+m)}{\Gamma(\delta)}x^{\delta-1} = (\delta)_m x^{\delta-1}, m \in \mathbb{N}_0, \tag{2.16}$$

Now, from (2.14) it is not difficult to infer the following interesting special case

Corollary 2.3. *Let $\Re(\delta) > 0, \lambda = 1, max\{|x|,|y|\} < 1$. Then*

$$\zeta_{\nu,1}^{*(\delta,\mu)}(xy,y;z,a;p) = x^{1-\delta}\Phi(y,z,a) \times \Phi_p(\mu;\nu;D_x x)\, x^{\delta-1}, \tag{2.17}$$

where Φ_p is the extended confluent hypergeometric function defined by (1.12).
Proof. We refer to the proof of (2.14).
Next, we present a series representation for the function $\zeta_{\nu,\lambda}^{*(\delta,\mu)}$. First, we recall the following well-known expansion formula of the Hurwitz-Lerch zeta function [8,p.29(8)]:

$$\Phi(y,z,a) = \frac{\Gamma(1-z)}{y^a}\left[\log\frac{1}{y}\right]^{z-1} + \frac{1}{y^a}\sum_{k=0}^{\infty}\zeta(z-k,a)\frac{(\log y)^k}{k!}, \tag{2.18}$$

valid for $|\log(y)| < 2\pi, z \neq \mathbb{N}; a \neq \mathbb{Z}_0^-$.

Theorem 2.1. *Let* $\lambda > 0, |\log(y)| < 2\pi$ *and* $|\frac{x}{y^\lambda}| < 1$. *Then*

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p) = \frac{1}{y^a}\left[\Gamma(1-z)\left[\log\frac{1}{y}\right]^{z-1} F_p\left(\delta,\mu;\nu;\frac{x}{y^\lambda}\right)\right.$$

$$\left. + \sum_{k=0}^{\infty} \zeta^{\delta,\mu}_{\nu,\lambda}\left(\frac{x}{y^\lambda}; z-k, a; p\right)\frac{(\log y)^k}{k!}\right], \tag{2.19}$$

valid for $z \neq \mathbb{N}; a \neq -(n+\lambda m), \{m,n\} \in \mathbb{N}_0$.
Proof. Use the series representation (2.18) in the definition (2.1). □

The relationships (2.3) to (2.7) give us the intention to derive new series representations for the other zeta functions involved in those relationships. For instance, if $p = 0$ and $\lambda = 1$, in formula (2.19) , we get an expansion for the zeta function of Garg et al. (1.9):

$$\sum_{k=0}^{\infty} \Phi_{\delta,\mu;\nu}\left(\frac{x}{y}, z-k, a\right)\frac{(\log y)^k}{k!} = y^a \zeta^{\delta,1}_{\mu,\nu}(x,y;z,a) - \Gamma(1-z)\left[\log\frac{1}{y}\right]^{z-1} {}_2F_1\left[\delta,\mu;\nu'\frac{x}{y}\right], \tag{2.20}$$

valid for $|\frac{1}{y}| > 1, |\frac{x}{y}| < 1, z \notin \mathbb{N}; \Re(a) > 0$.
Finally, putting $\delta = \alpha+\beta$ in (2.1) and using the classical formula of Nörlund for the Pochhammer symbol (cf. [1,Section 1, Chapter 3]):

$$(a+b)_k = \sum_{m=0}^{k} \binom{k}{m} (a)_{k-m}\, (b)_m, \tag{2.21}$$

we find form (2.1) that

$$\zeta^{*(\alpha+\beta,\mu)}_{\nu,\lambda}(x,y;z,a;p) = \sum_{m=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}(\alpha)_m \Phi(y,z,a+\lambda m)$$

$$\times\ {}_2F_1\left[\begin{matrix} -m,\beta; \\ 1-\alpha-m; \end{matrix}\ 1\right]\frac{x^m}{m!}. \tag{2.22}$$

3 Integral Images for $\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)$

First, by exploiting the integral representation of the generalized extended beta function $B(x,y;p)$(see (1.10)) and the results

$$(a+n+\lambda m)^{-z} x^m y^n = \frac{1}{\Gamma(z)}\int_0^\infty \left(xe^{-\lambda}\right)^m \left(ye^{-t}\right)^n t^{z-1} e^{-at} dt, \tag{3.1}$$

and

$$(\delta)_m = \frac{1}{\Gamma(\delta)}\int_0^\infty u^{\delta+m-1} e^u du, \tag{3.2}$$

which follows from the Eulerian integral[2]:

$$a^{-z}\Gamma(z)=\int_0^\infty e^{-at}t^{z-1}dt, \tag{3.3}$$

we can derive the following triple integral representation:

Theorem 3.1. *Let* $\{\Re(a),\Re(z),\Re(\delta),\Re(\mu)\}>0$. *Then*

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{e^{-2p}}{B(\mu,\nu-\mu)\Gamma(\delta)\Gamma(z)}\int_0^\infty\int_0^\infty\int_0^\infty\frac{t^{z-1}s^{\mu-1}u^{\delta-1}}{(1-u)^{\nu}}e^{-(at+s)}$$
$$\times e^{-p(u+u^{-1})}\times e^{\left(\frac{uxse^{-\lambda}}{1+u}\right)}(1-ye^{-t})^{-1}dudsdt, \tag{3.4}$$

Proof. In view of the definitions (2.1) and (1.14), it is easily seen that

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{e^{-2p}}{B(\mu,\nu-\mu)}\sum_{m,n=0}^{\infty}\int_0^\infty\frac{u^{\mu+m-1}}{(1+u)^{\nu+m}}\exp\left[-p(u+u^{-1})\right]$$
$$\times\frac{(\delta)_m x^m y^n}{m!(a+n+\lambda m)^z}du. \tag{3.5}$$

Now, with the aid of the results (3.1) and (3.2) and by interchanging the order of summation and integration, equation (3.5) gives us the left-hand side of assertion (3.4). □

Theorem 3.2. *Let* $\{\Re(a),\Re(\delta),\Re(\mu)\}>0$. *Then*

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{1}{\Gamma(z)}\int_0^\infty e^{-at}t^{z-1}(1-ye^{-t})^{-1}F_p\left(\delta,\mu;\nu;xe^{-\lambda t}\right)dt, \tag{3.6}$$

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{\Gamma(\nu)e^{-2p}}{\Gamma(\mu)\Gamma(\nu-\mu)}\int_0^\infty\frac{s^{\mu-1}}{(1+s)^{\nu}}\exp\left[-p(s+s^{-1})\right]$$
$$\times\zeta_\lambda^\delta\left(\frac{xs}{1+s},y;z,a\right)ds. \tag{3.7}$$

Proof. To prove the formulas (3.6) and (3.7), we employ the integral relations (3.3) and (1.14) respectively, and exploit the same procedure leading to (3.4). □

Next, utilizing the integral image of the generalized extended Euler's beta function, we can derive the following results.

Theorem 3.3. *Let* $\Re(a)>0$ *and* $\Re(z)>0$. *Then*

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{1}{B(\mu,\nu-mu)}\int_0^1 t^{\mu-1}(1-t)^{\nu-\mu-1}\exp\left[\frac{-p}{t(1-t)}\right]\zeta_\lambda^\delta(xt,y;z,a)dt, \tag{3.8}$$

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{1}{B(\mu,\nu-mu)}\times\int_0^{\frac{\pi}{2}}(\cos\theta)^{2\mu-1}(\sin\theta)^{2(nu-\mu)-1}\exp\left(-p\sec^2\theta\csc^2\theta\right)$$
$$\times\zeta_\lambda^\delta(x\cos^2\theta,y;z,a)dt, \tag{3.9}$$

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{2^{1-\nu}}{B(\mu,\nu-\mu)}\int_{-\infty}^{\infty}(\cosh x)^{\nu}\exp\left[-(x\nu+4p\cosh^2 x)\right]$$

$$\times\zeta_{\lambda}^{\delta}(x\cosh x\ e^{-\nu},y;z,a)dx. \tag{3.10}$$

Proof. The results follow directly from the formulas (1.10), (1.13) and (1.15) respectively. □
We shall now treat $\zeta_{\nu,\lambda}^{*(\delta,\mu)}$ by applying the Mellin- Barnes type integrals. Our starting point is the formula [24,section 14.51,p.289, Corollary]; also see [15]:

$$(1-\omega)^{-z}=\frac{1}{2\pi i}\int_{c}\frac{\Gamma(z+\nu)\Gamma(-\nu)(-\omega)^{\nu}}{\Gamma(z)}d\nu, \tag{3.11}$$

where z and ω are complex with $\Re(z)>0,|arg(\omega)|<\pi,\omega\neq 0$, and the path is the vertical line from $c-i\infty$ to $c+i\infty$. In [26] this formula is stated with $c=0$, (with suitabe modification of the path near the point $z=0$), but it is clear that the formula is also valid for $-\Re(z)<c<0$.
Theorem 3.4. *Let* $\Re(z)>0,\Re(a-b)>0,\Re(b)>0,|x|<1$ *and* $|y|<1$. *Then*

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{1}{2\pi i}\int_{c}\frac{\Gamma(\nu+z)\Gamma(-\nu)}{\Gamma(z)}\zeta_{\mu,\nu}^{\delta,\lambda}(x;-\nu,a-b;p)\times\Phi(y,z+\nu,b)d\nu, \tag{3.12}$$

Proof. Let $\omega=(b-a-\lambda m)/(b+n)$ in (3.11) and multiply both the sides by

$$\frac{B(\mu+m,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m y^n}{m!},m,n\in\mathbb{N}_0,$$

to obtain

$$\left(\frac{B(\mu+m,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m y^n}{m!}\right)(a+n+\lambda m)^{-z}$$
$$=\int_{c}\frac{(\Gamma(\nu+z)\Gamma(-\nu)}{\Gamma(z)}\times\left(\frac{B(\mu+m,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m}{m!(a-b+\lambda m)^{-\nu}}\right)\times\frac{y^n}{(b+n)^{z+\nu}}d\nu,$$
$$(m\geq 0,n>0).$$

Therefore, if we assume $(1-\Re(\nu)<c<-1$, then from (1.3) and (2.11) we get (3.12). □

Further, by using the Mellin transform representation of the generalized beta function in terms of Mellin-Barnes type contour integral[7]

$$B(x,y;z,a;p)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(\Gamma(s)\Gamma(x+s)\Gamma(y+s)}{\Gamma(x+y+2s)}p^{-s}ds, \tag{3.13}$$

we have the following complex integral representation for $\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)$.

Theorem 3.5. *Let* $\Re(p)>0,m>0,$ $max\{|x|,|y|\}<1$ *and* $\gamma>0$. *Then*

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(\Gamma(s)\Gamma(\nu-\mu+s)\Gamma(\mu+s)\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)\Gamma(\nu+2s)}p^{-s}\Phi_{\nu+2s,\lambda}^{\delta,\mu+s}(x,y;z,a)ds. \tag{3.14}$$

where $\Phi_{\nu,\lambda}^{\delta,\mu}$ is zeta function defined by (2.10).
Proof. Using the formula (3.13) in the definition (2.1), interchanging the order of summation and integration and considering the definition (2.10), we led to the desired result (3.14). □

4 Differential Relations

The extended zeta function $\zeta_{\nu,\lambda}^{(\delta,\mu)}$, as a function satisfies some differential recurrence relations. Fortunately these properties of $\zeta_{\nu,\lambda}^{(\delta,\mu)}$ can be developed directly from the definition (2.1). Firstly, we recall the following result [20]

$$D_x^m x^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m}, n-m \geq 0, D_x = \frac{d}{dx}, \tag{4.1}$$

Theorem 4.1. *Let $k \in \mathbb{N}$. Then*

$$D_x^k \left\{ \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p) \right\}$$
$$= \frac{(\delta)_k(\mu)_k}{(\nu)_k} \zeta_{\nu+k,\lambda}^{*(\delta+k,\mu+k)}(x,y;z,a+\lambda k;p), \tag{4.2}$$

$$D_y^k \left\{ \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p) \right\}$$
$$= k! \sum_{m=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu+m,\nu-\mu)} (\mu)_m \Phi_{k+1}^*(y,z,a+k+\lambda m) \frac{x^m}{m!}, \tag{4.3}$$

$$D_a^k \left\{ \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p) \right\} = (-1)^k (z)_k \; \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z+k,a;p). \tag{4.4}$$

Proof. Using (4.1), we get

$$D_x^k \left\{ \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p) \right\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^{m-k} y^n}{(m-k)!(a+n+\lambda m)^z}. \tag{4.5}$$

Now, replacing m by $m+k$ in (4.5) and considering the definition (2.1), we get the right-hand side of formula (4.2). Similarly, one can proof the formulas (4.3) and (4.4). □

According to the relation (2.3) formula (4.4)reduces to the result

$$D_a^k \zeta(z,a) = (-1)^k (z)_k \zeta(z+k,a), \tag{4.6}$$

which is a known result (see e.g. [10,p.2(1.8)]). In view of the relationship (2.5), we find from equation (4.4) that

$$D_a^k \left(\Phi_{\delta,\mu;\nu}(x,z,a) \right) = (-1)^k (z)_k \Phi_{\delta,\mu;\nu}(x,z+k,a). \tag{4.6}$$

Secondly, we show that the zeta function $\zeta_{\mu,\nu}^{\delta,\lambda}$ in (2.9) is related to the extended function $\zeta_{\mu,\nu}^{*(\delta,\lambda)}$ for $\delta,\mu,\nu \neq \mathbb{N}$ by the following differential relation.

Theorem 4.2. *Let $\delta-n, \lambda-n, \nu-n \neq \mathbb{Z}_0^-$. Then*

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a;p) = \sum_{n=0}^{\infty} \frac{(1-\nu)_n (-1)^n y^n}{(1-\delta)_n (1-\mu)_n} D_x^n \left[\zeta_{\nu-n,\lambda}^{\delta-n,\mu-n}(x,y;z,a+(1-\lambda)n;p) \right]. \tag{4.8}$$

Proof. Let I denote the right-hand side of assertion (4.8). Then in view of (2.11) and (4.1) we have

$$I=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m-n,\nu-\mu;p)(\delta-n)_m(1-\nu)_n(-1)^n y^n x^{m-n}}{B(\mu,\nu-\mu)(1-\mu)_n(1-\delta)^n(a+(1-\lambda)n+\lambda m)^z(m-n)!}. \tag{4.9}$$

Now, by using the identities

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+k), \tag{4.10}$$

$$B(\mu-n,\nu-\mu)=\frac{(1-\nu)_n}{(1-\mu)_n}B(\mu,\nu-\mu), \tag{4.11}$$

and

$$(a)_{-n}=\frac{(-1)^n}{(1-a)_n}, \tag{4.12}$$

we led to the left-hand side of (4.8). □

Next, we establish the derivative of the function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ with respect to the argument λ.

Theorem 4.3. *Let $b\in\mathbb{R}$. Then*

$$\frac{\partial}{\partial\lambda}\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z-1,a+\lambda b;p)$$
$$=(1-z)\left[x\delta\zeta^{*(\delta+1,\mu)}_{\nu,\lambda}(x,y;z,a+\lambda(b+1);p)+b\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a+\lambda b;p)\right]. \tag{4.13}$$

Proof. We have
$\frac{\partial}{\partial\lambda}\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z-1,a+\lambda b;p)$

$$=(1-z)\left[\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m+1,\nu-\mu;p)(\delta)_m x^m y^n}{B(\mu,\nu-\mu)(m-1)!(a+n+\lambda(m+b))^z}\right.$$
$$\left.+b\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)(\delta)_m x^m y^n}{B(\mu,\nu-\mu)m!(a+n+\lambda(m+b))^z}\right]. \tag{4.14}$$

Now , let $m\to m+1$ in the first summation of (4.14) and then use the identity

$$(\mu)_{m+n}=(\mu)_n(\mu+n)_m,$$

to obtain (4.13). □

The same type of differentiation gives the next result.

Theorem 4.4. *Let $q\in\mathbb{R}$. Then*

$$\frac{\partial}{\partial q}\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z-1,a+bq;p)=b(1-z)\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a+bq;p). \tag{4.15}$$

Proof. We refer to the proof of Theorem 4.3. □

It is easily observed that the relations (4.13) and (4.15) are generalization of the known results (see e.g. [9,p.451]):

$$\frac{\partial}{\partial\lambda}\zeta(z,\lambda)=-z\zeta(z+1,\lambda), \tag{4.16}$$

and

$$\frac{\partial}{\partial q}\zeta(z-1,a+\lambda b)=b(1-z)\zeta(z,a+qb). \tag{4.17}$$

Closely associated with the derivative of the gamma function is the digamma function defined by (see e.g.[17])

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma^{'}(x)}{\Gamma(x)}, x\neq 0,-1,-2,\cdots. \tag{4.18}$$

Now, we establish the derivative of the function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ with respect to the parameter δ.

Theorem 4.5. *Let $\delta\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then*

$$\frac{\partial}{\partial\delta}\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)$$

$$=\sum_{m=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\Phi(y,z,a+\lambda m)\left[\psi(\delta+m)-\psi(\delta)\right]\frac{(\delta)_m x^m}{m!}. \tag{4.19}$$

Proof. By noting that

$$\frac{\partial}{\partial\delta}\left[(\delta)_m\right]=\frac{\partial}{\partial\delta}\left[\frac{\Gamma(\delta+m)}{\Gamma(\delta)}\right]=(\delta)_m\left[\psi(\delta+m)-\psi(\delta)\right], \tag{4.20}$$

we obtain the result (4.19). □

According to the algebraic identity (cf. [22,p.295(6.7)]):

$$\psi(x+1)-\psi(x+m+1)=\sum_{k=1}^{m}\frac{(-1)^k m!\Gamma(x+1)}{k(m-k)!\Gamma(x+k+1)}, \tag{4.21}$$

the formula (4.19) can be rewritten in the following more compact form

$$\frac{\partial}{\partial\delta}\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)=\sum_{m=0}^{\infty}\sum_{k=1}^{m}\frac{(-1)^{k+1}\Gamma(\delta+m)}{k\ (m-k)!\ \Gamma(\delta+k)}\Phi(y,z,a+\lambda m)x^m. \tag{4.22}$$

Further, let us recall the definition of the Weyl fractional derivative of the exponential function $e^{-at}, a>0$ of order ν in the form (see [18,p.248(7.4)]):

$$D^{\nu}\left\{e^{-at}\right\}=a^{\nu}e^{-at}, \tag{4.23}$$

(ν not restricted to be postive integer).

We now proceed to find the fractional derivative of the function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ with respect to z.

Theorem 4.6. *Let $\nu>0$. Then*

$$D_z^{\nu}\left[\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m y^n}{m!(a+n+\lambda m)^z}\times\left[\log(a+n+\lambda m)\right]^{\nu}. \quad (4.24)$$

Proof. Since

$$(a+n+\lambda m)^{-z} = e^{-z log(a+n+\lambda m)},$$

we have

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{(\mu)_m x^m y^n}{m!}e^{-z\log(a+n+\lambda m)}.$$

The desired result now follows by applying the formula (4.23) to the above identity. □
Finally, it is interesting to note that the $n-th$ derivative of the extended zeta function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ concerning the parameter p is given by the following theorem.

Theorem 4.7. *Let $k\in\mathbb{N}$. Then*

$$\hat{D}^k_p\left(\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)\right) = (-1)^k\frac{B(\mu-k,\nu-\mu-k)}{B(\mu,\nu-\mu)}\zeta^{*(\delta,\mu-k)}_{\nu-2k,\lambda}(x,y;z,a;p). \quad (4.25)$$

Proof. From definitions (2.1) and (1.10) and the formula (4.1), we can state that

$$\hat{D}^k_p\left(\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a;p)\right) = (-1)^k\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m-k,\nu-\mu-k;p)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m y^n}{m!(a+n+\lambda m)^z}. \quad (3.26)$$

Now, by interpreting the above series in the form of the definition (2.1), we obtain the desired result (4.25). □

5 Series Expansions

This section aims at establishing some series relations for the extende bivariate series zeta function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$. First, based on the two forms of Taylor's theorem for the deduction of addition and multiplication theorems for the confluent hypergeometric function (cf.[11,p.63, Equations(2.8.8) and (2.8.9)] or [22,p.21-22]):

$$f(x+y) = \sum_{m=0}^{\infty} f^{(m)}(x)\frac{y^m}{m!}, \quad (5.1)$$

and

$$f(xy) = \sum_{m=0}^{\infty} f^{(m)}(x)\frac{[(y-1)x]^m}{m!}, \quad (5.2)$$

where $|y|<\rho,\ \rho$ being the radius of convergence of the analytic function $f(x)$, we aim to discuss certain addition and multiplication theorems of the extended bivariate zeta function $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$.

Theorem 5.1. *Let $|\omega|<1$. Then*

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x+\omega,y;z,a;p) = \sum_{k=0}^{\infty}\frac{(\delta)_k(\mu)_k}{(\nu)_k}\zeta^{*(\delta+k,\mu+k)}_{\nu+k,\lambda}(x+\omega,y;z,a+\lambda k;p)\frac{\omega^k}{k!}, \quad (5.3)$$

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y+\omega;z,a;p)=\sum_{m,n=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}(\delta)_m\Phi^{*}_{n+1}(\omega,z,a+n+\lambda m)\frac{x^m y^n}{m!},\quad(5.4)$$

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x\omega,y;z,a;p)=\sum_{k=0}^{\infty}\frac{(\delta)_k(\mu)_k}{(\nu)_k}\zeta^{*(\delta+k,\mu+k)}_{\nu+k,\lambda}(x(\omega-1),y;z,a+\lambda k;p)\frac{x^k}{k!},\quad(5.5)$$

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y\omega;z,a;p)=\sum_{m,n=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}(\delta)_m\Phi^{*}_{n+1}(y(\omega-1),z,a+n+\lambda m)\frac{x^m y^n}{m!}.\quad(5.6)$$

Proof. The proof is a direct application of the formulas (5.1),(5.2) and the first two results of Theorem 4.1. □

Next , we derive the Taylor expansion of $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ in the fourth variable a.

Theorem 5.2. *Let* $|\omega|<\Re(a)$*. Then*

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a+\omega;p)=\sum_{k=0}^{\infty}(-1)^k\Phi(y,z+k,a)\times\zeta^{*\delta,\lambda}_{\mu,\nu}(x,-k,\omega)\frac{(z)_k}{k!}.\quad(5.7)$$

Proof. We have

$$\zeta^{*(\delta,\mu)}_{\nu,\lambda}(x,y;z,a+\omega;p)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m y^n}{m!}(a+n)^{-z}\left(1+\frac{\omega+\lambda m}{a+n}\right)^{-z}.\quad(5.8)$$

The result now follows from the binomial expansion and the definitions (1.3) and (2.9). □
In fact, equation (5.7) gives a number of known and new series expansions as particular cases. For instance, in view of the relation (2.4) we find from (5.7) that

$$\Phi(y,z,a+\omega)=\sum_{k=0}^{\infty}(z)_k\Phi(y,z+k,a)\frac{\omega^k}{k!},(z\neq1,|\omega|<|a|),\quad(5.9)$$

which is a known result due to Raina and Chhajed [21,p.93(3.3)]. Moreover according to the relationship (2.3), equation (5.7) yields

$$\zeta(z,a+\omega)=\sum_{k=0}^{\infty}(-1)^k(z)_k\zeta(z+k,a)\frac{\omega^k}{k!},|\omega|<|a|.\quad(5.10)$$

Note that, formula (5.10) is a known result due to Kanemitsu et al. [15,p.5(2.6)].
Furthermore, if in (5.9) we let $y=e^{2\pi i\alpha}$(in conjunction with (1.4)), formula (5.9) reduces to a known power series expansion due to Klusch [16]:

$$\phi(\alpha,a+\omega,z)=\sum_{k=0}^{\infty}(-1)^k(z)_k\phi(\alpha,a,z+k)\frac{\omega^k}{k!},|\omega|<a.\quad(5.11)$$

Another expansion function for $\zeta^{*(\delta,\mu)}_{\nu,\lambda}$ can be derived by using the result [17,p.374, Exercise 9.4(7)]:

$${}_2F_1\left[a,a+\frac{1}{2};\frac{1}{2};x\right]=\frac{1}{2}\left(1+\sqrt{x}\right)^{-2a}+\frac{1}{2}\left(1-\sqrt{x}\right)^{-2a}.\quad(5.12)$$

Theorem 5.3. *Let $\mu \geq 1, Re(a) > 0, |x| < 1, |y| < 1$ and $|\omega| < |a|$. Then*

$$\sum_{k=0}^{\infty} \binom{z+2k-1}{2k} \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z+2k,a;p)\omega^{2k}$$
$$= \frac{1}{2}\left[\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a+\omega;p) + \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z,a-\omega;p)\right]. \tag{5.13}$$

Proof. We have

$$\sum_{k=0}^{\infty} \binom{z+2k-1}{2k} \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z+2k,a;p)\omega^{2k}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m y^n}{m!(a+n+\lambda m)^z} \sum_{k=0}^{\infty} \frac{(z)_{2k}\omega^{2k}}{(2k)!(a+n+\lambda m)^{2k}}. \tag{5.14}$$

By applying the formula (5.12) to the last summation in the right-hand side of equation (5.14), we led to the result (5.13). □

Next, we derive a series expansion for the extended zeta function $\zeta_{\nu,\lambda}^{*(\delta,\mu)}$ involving Saran hypergeometric function F_K of three variables defined by the series (see e.g. [23]):

$$F_K\left[a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z\right] = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_1)_{n+p} (b_1)_{m+p} (b_2)_n x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!}, \tag{5.15}$$

$$|x| < 1 \wedge |y| < 1 \wedge |z| < (1-|x|)(1-|y|).$$

Theorem 5.4. *Let $max\{|x/b|, |y/b|\} < 1, |b| < \Re(a)$ and $\lambda \neq 0$. Then*

$$\sum_{k=0}^{\infty} (z)_k \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z+k,a+b;p)\frac{\omega^k}{k!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m y^n}{m!}$$
$$\times F_K\left[1,z,z,1;1,1,z;\frac{-b}{a+n+\lambda m},\frac{-\lambda m}{a+n},\frac{-\omega}{a+n}\right](a+n)^{-z}. \tag{5.16}$$

Proof. Since

$$(a+n+\lambda m+b)^{-(z+k)} = (a+n)^{-(z+k)}\left(1+\frac{\lambda m}{a+n}\right)^{-(z+k)}\left(1+\frac{b}{a+n+\lambda m}\right)^{-(z+k)},$$

it follows that

$$\sum_{k=0}^{\infty} (z)_k \zeta_{\nu,\lambda}^{*(\delta,\mu)}(x,y;z+k,a+b;p)\frac{\omega^k}{k!}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m y^n}{m!(a+n)^z}$$
$$\times \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(z)_{k+s}(z)_{k+r}(1)_s(1)_r}{k!s!r!(z)_k(1)_s(1)_r} \times \left(\frac{-b}{a+n+\lambda m}\right)^s \left(\frac{-\lambda m}{a+n}\right)^r \left(\frac{-\omega}{a+n}\right)^k. \tag{5.17}$$

The result (5.16) now follows from the definition (5.15). □

Indeed, for $x = p = b = 0$ equation (5.16) reduces to the well-known result of Ramanujan [20,p.396(6)]:

$$\zeta(z, a-\omega) = \sum_{k=0}^{\infty} \frac{(z)_k}{k!}\zeta(z+n, a)\omega^n.$$

Finally, we give a representation of the extended zeta function $\zeta_{\nu,\lambda}^{*(\delta,\mu)}$ in terms of Laguerre polynomials. We start by recalling the useful identity used in [19]

$$e^{\left(\frac{-p}{t(1-t)}\right)} = e^{-2p} \sum_{m,n=0}^{\infty} L(p)_m L(p)_n t^{m+1}(1-t)^{n+1}; |t| < 1. \tag{5.18}$$

Theorem 5.5. *Let* $\{\delta, \mu, \nu\} \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda \in \mathbb{C} \setminus \{0\}; a \in \mathbb{C} \setminus \{-(n+\lambda m)\}, \Re(p) \geq 0$. *Then*

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x, y; z, a; p) = \frac{e^{-2p}}{B(\mu, \nu-\mu)} \sum_{m,n=0}^{\infty} \sum_{s,r=0}^{\infty} \frac{B(\mu+\delta+m+1, \nu-\mu+r+1)}{m!(a+n+\lambda m)^z} \times (\delta)_s L_s(p) L_r(p) x^m y^n. \tag{5.19}$$

Proof. Using (5.18) in (3.8), employing the series expansion of the zeta function ζ_λ^δ and interchange the order of integration and summation, we obtain the result (5.19). □

According to the definition of beta function[24]

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

the assertion (5.19) can be rewritten in the following alternative representation:

$$\zeta_{\nu,\lambda}^{*(\delta,\mu)}(x, y; z, a; p) = \frac{e^{-2p}\mu(\nu-\mu)}{\nu} \sum_{s,r=0}^{\infty} \frac{(\mu+1)_s(\nu-\mu+1)_r L_s(p) L_r(p)}{(\nu+1)_{s+r}} \zeta_{\mu+s+1,\nu+s+r+1}^{\delta,\lambda}(x, y; z, a), \tag{5.20}$$

where $\zeta_{\nu,\lambda}^{\delta,\mu}(x, y; z, a)$ is the generalized zeta function defined by (2.9).

Data Availability Statement

The results data used to support the findings of this study are included within the article.

Disclosure statement

No potential conflict of interest was reported by the authors.

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