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A unique continuation result for a system of nonlinear differential equations

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Abstract. Using an appropriate Carleman-type estimate, we establish a result of unique continuation for a special class of one-dimensional systems that model the evolution of long water waves with small amplitude in the presence of surface tension.

Keywords. Nonlinear equations, long waves, Carleman estimate, unique continuation.

1 Introduction

The focus of the present work is the following one-dimensional system

$$\begin{cases} \left(I - a\partial_x^2\right)\eta_t + \partial_x^2\Phi - b\partial_x^4\Phi + \partial_x(\eta\partial_x\Phi) &= 0, \\ \left(I - c\partial_x^2\right)\Phi_t + \eta - d\partial_x^2\eta + \frac{1}{2}\left(\partial_x\Phi\right)^2 &= 0, \end{cases} \tag{1.1}$$

that describes the propagation of long water waves with small amplitude in the presence of surface tension, where $\Phi = \Phi(x,t)$ represents the rescale nondimensional velocity potential on the bottom $z = 0$, the variable $\eta = \eta(x,t)$ corresponds the rescaled free surface elevation, and the constants $a, b, c, d > 0$ are such that

$$a + c - (b + d) = \frac{1}{3} - \sigma,$$

where σ^{-1} is known as the Bond number (associated with the surface tension). This model is the 1D version of systems derived in [5] and [6].

As happens in water wave models, there is a Hamiltonian type structure which is clever to find the appropriate space for special solutions (solitary waves for example) and also provide relevant information for the study of the Cauchy problem. For the particular system (1.1), the Hamiltonian functional $\mathcal{H} = \mathcal{H}(t)$ is defined as

$$\mathcal{H}\begin{pmatrix}\eta\\ \Phi\end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}}\left(\eta^2 + d(\partial_x\eta)^2 + (\partial_x\Phi)^2 + b(\partial_x^2\Phi)^2 + \eta\left(\partial_x\Phi\right)^2\right)dx,$$

and the Hamiltonian type structure is given by

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix} = \mathcal{J}\mathcal{H}'\begin{pmatrix}\eta\\ \Phi\end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & \left(I - c\partial_x^2\right)^{-1} \\ -\left(I - a\partial_x^2\right)^{-1} & 0 \end{pmatrix}.$$

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We see directly that the functional $\mathcal{H}$ is well defined when $\eta(\cdot,t), \partial_x\Phi(\cdot,t) \in H^1(\mathbb{R})$, for t in some interval. These conditions already characterize the natural space for the study of solutions of the system (1.1). Certainly, J. Quintero and A. Montes in [8] showed for the model (1.1) the existence of solitary wave solutions which propagate with speed of wave $\theta > 0$, i. e. solutions of the form

$$\eta(x,t) = u(x-\theta t),\ \ \Phi(x,t) = v(x-\theta t),$$

in the energy space $H^1(\mathbb{R}) \times \mathcal{V}^2(\mathbb{R})$, where $H^1(\mathbb{R})$ is the usual Sobolev space of order 1 and the space $\mathcal{V}^2(\mathbb{R})$ is defined with respect to the norm given by

$$\|w\|^2_{\mathcal{V}^2(\mathbb{R})} = \|w'\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \left((w')^2 + (w'')^2\right) dx.$$

They also showed the local well-posedness for the Cauchy problem associated to the system (1.1) in the Sobolev type space $H^s(\mathbb{R}) \times \mathcal{V}^{s+1}(\mathbb{R})$, where $H^s(\mathbb{R})$ is the usual Sobolev space of order s defined as the completion of the Schwartz class with respect to the norm

$$\|w\|_{H^s(\mathbb{R})} = \|\,(1+|\xi|)^s\,\widehat{w}(\xi)\|_{L^2_\xi}$$

and $\mathcal{V}^{s+1}(\mathbb{R})$ denotes the completion of the Schwartz class with respect to the norm

$$\|w\|_{\mathcal{V}^{s+1}(\mathbb{R})} = \|\,(1+|\xi|)^s\,|\xi|\widehat{w}(\xi)\|_{L^2_\xi},$$

where $\widehat{w}$ is the Fourier transform of w in the space variable x and ξ is the variable in the frequency space related to the variable x. For $a, b, c, d > 0$, using a bilinear estimate obtained by J. Bona and N. Tzvetkov in [1], Quintero and Montes showed that for $(\eta_0, \Phi_0) \in H^s(\mathbb{R}) \times \mathcal{V}^{s+1}(\mathbb{R})$ with $s \geq 0$, there exists a time $T > 0$ and unique solution

$$(\eta,\Phi) \in C\left([-T,T], H^s(\mathbb{R}) \times \mathcal{V}^{s+1}(\mathbb{R})\right) \cap C^1\left([-T,T], H^{s-1}(\mathbb{R}) \times \mathcal{V}^{s}(\mathbb{R})\right)$$

of the Cauchy problem associated to the model (1.1) with the initial condition

$$\eta(x,t) = \eta_0(x), \quad \Phi(x,t) = \Phi_0(x).$$

On the case of the periodic domain $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (the one-dimensional torus), it was proved in [7] the local well-posedness of the Cauchy problem associated to system (1.1) in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, for $s \geq 0$, where the periodic Sobolev space $H^s(\mathbb{T})$ is defined by

$$H^s(\mathbb{T}) = \left\{ w = \sum_{k\in\mathbb{Z}} w_k e^{ikx} \;\;:\;\; \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |w_k|^2 \;<\; +\infty \right\}$$

and the space $\mathcal{V}^{s+1}(\mathbb{T})$ is defined by the norm

$$\|w\|_{\mathcal{V}^{s+1}(\mathbb{T})} = \left[\sum_{k\in\mathbb{Z}} (1+|k|^2)^s |k|^2 |w_k|^2\right]^{1/2}$$

where $w_k = \widehat{w}(k)$ denotes the k-Fourier coefficient with respect to the spatial variable x.

In the present work, for $a, b, c, d > 0$ we will prove a unique continuation result for the system (1.1). More precisely, we show that if $(\eta, \Phi) = (\eta(x,t), \Phi(x,t))$ is a solution of the system (1.1) in a suitable function space, for example

$$\eta, \eta_t \in L^2\left(-T, T; H^2_{loc}(\mathbb{R})\right),\ \ \Phi \in L^2\left(-T, T; H^4_{loc}(\mathbb{R})\right),\ \ \Phi_t \in L^2\left(-T, T; H^2_{loc}(\mathbb{R})\right),$$

and (η, Φ) vanishes on an open subset Ω of $\mathbb{R} \times [-T, T]$, then $(\eta, \Phi) \equiv 0$ in the horizontal component of Ω. We recall that the horizontal component Ω_1 of an open subset $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ is defined as the union of all segments $t = constant$ in $\mathbb{R} \times \mathbb{R}$ which contain a point of Ω, this is,

$$\Omega_1 = \big\{(x,t) \in \mathbb{R} \times [-T,T] \ : \ \exists x_1 \in \mathbb{R}, \ (x_1,t) \in \Omega\big\}.$$

The unique continuation property has been intensively studied for a long time. An important work on the subject was done by J.C. Saut and B. Scheurer in [9]. They showed a unique continuation result for a general class of dispersive equations including the well known KdV equation,

$$u_t + uu_x + u_{xxx} = 0,$$

and various generalizations. In the work [3], M. Davila and G. Menzala proved a similar result for the Benjamin-Bona-Mahony equation,

$$u_t + u_x - u_{xxt} + uu_x = 0,$$

and for the Boussinesq equation,

$$u_{tt} - u_{xx} + \big(u^2 + u_{xx}\big)_{xxx} = 0.$$

In a similar way, Y. Shang showed in [10] a unique continuation result for the symmetric regularized long wave equation,

$$u_{tt} - u_{xx} + \frac{1}{2}\big(u^2\big)_{xt} - u_{xxtt} = 0.$$

In the previous equations, a Carleman estimate is established to prove that if a solution u vanishes on an open subset Ω, then $u \equiv 0$ in the horizontal component of Ω. By using the inverse scattering transform and some results from the Hardy function theory, B. Zhang in [11] established that if u is a solution of the KdV equation, then it cannot have compact support at two different moments unless it vanishes identically. In the paper [2], J. Bourgain introduced a different approach and prove that if a solution u to the KdV equation has compact support in a nontrivial time interval $I = [t_1, t_2]$, then $u \equiv 0$. His argument is based on an analytic continuation of the Fourier transform via the Paley-Wiener Theorem and the dispersion relation of the linear part of the equation. It also applies to higher order dispersive nonlinear models, and to higher spatial dimensions. More recently, C. Kenig, G. Ponce and L. Vega in [4] proposed a new method and proved that if a sufficiently smooth solution u to a generalized KdV equation is supported in a half line at two different instants of time, then $u \equiv 0$.

Following from close the works of Saut-Scheurer [9], we base our analysis in finding an appropriate Carleman-type estimate for the linear operator $\mathcal{L}$ associated to the system (1.1). In order to do this we use a particular version of the well known Treves' inequality. For the operator $\mathcal{L}$ we also prove that if a solution vanishes in a ball in the xt plane, which passes through the origin, then it also vanishes in a neighborhood of the origin.

The paper is organized as follows. In Section 2, using a particular version of the Treves inequality, we establish a Carleman estimate for a differential operator $\mathcal{L}$ closely related to our problem. In Section 3, first we give some useful technical results. Later, we show the unique continuation result for the system (1.1).

2 Carleman estimate

In this section, using a particular version of the Treves' inequality, we establish a Carleman estimate for the differential operator $\mathcal{L}$ defined as

$$\mathcal{L} = \begin{pmatrix} \partial_t - a\partial_x^2\partial_t + c_1\partial_x^3 + f_1(x,t)\partial_x & f_2(x,t)\partial_x^2 - b\partial_x^4 \\ I - d\partial_x^2 & \partial_t - c\partial_x^2\partial_t + c_2\partial_x^3 + f_3(x,t)\partial_x \end{pmatrix}. \tag{2.1}$$

In what follows $a, b, c, d > 0$. In addition, we are going to use the notation $D = (\partial_x, \partial_t)$. Moreover, if $P = P(\xi_1, \xi_2)$ is a polynomial in two variables, has constant coefficients and degree m, then we consider the differential operator of order m associated to P,

$$P(D) = P(\partial_x, \partial_t) = \sum_{|\alpha| \le m} a_\alpha D^\alpha,$$

where $D^\alpha = \partial_x^{\alpha_1}\partial_t^{\alpha_2}$ and $|\alpha| = \alpha_1 + \alpha_2$. By definition $P^{(\beta)}(\xi_1, \xi_2) = \partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}P(\xi_1, \xi_2)$ where β is given by $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$.

Theorem 2.1. *(Treves' Inequality). Let $P(D) = P(\partial_x, \partial_t)$ be a differential operator of order m with constant coefficients. Then for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, $\delta > 0$, $\tau > 0$, $\Psi \in C_0^\infty(\mathbb{R}^2)$ and $\psi(x,t) = (x-\delta)^2 + \delta^2 t^2$ we have that*

$$\frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_2}}{\alpha!}\int_{\mathbb{R}^2} |P^{(\alpha)}(D)\Psi|^2 e^{2\tau\psi}dxdt \le C(m,\alpha)\int_{\mathbb{R}^2} |P(D)\Psi|^2 e^{2\tau\psi}dxdt \tag{2.2}$$

with

$$|\alpha| = |\alpha_1| + |\alpha_2|, \quad \alpha! = \alpha_1!\alpha_2! \quad \text{and} \quad C(m,\alpha) = \begin{cases} \sup_{|r+\alpha|\le m} \binom{r+\alpha}{\alpha}, & \text{if} \quad |\alpha| \le m, \\ 0, & \text{if} \quad |\alpha| > m. \end{cases}$$

Proof. See Corollary 5.1 in [3] (see also Corollary 1 in [10]). □

We present the Carleman estimate for the differential operator $\mathcal{L}$.

Theorem 2.2. *Let $\mathcal{L}$ the differential operator defined in (2.1), where c_1, c_2 are real constants and $f_1, f_2, f_3 \in L^\infty_{loc}(\mathbb{R}^2)$. Let $\delta > 0$ and*

$$B_\delta := \{(x,t) \in \mathbb{R}^2 \ : \ x^2 + t^2 < \delta^2\}, \quad \psi(x,t) = (x-\delta)^2 + \delta^2 t^2.$$

Then, there exists $C > 0$ such that for all $\Psi = (\Psi_1, \Psi_2) \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$ and $\tau > 0$ with

$$\frac{\|f_1\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2a^2} \le \frac{1}{4}, \quad \frac{\|f_2\|^2_{L^\infty(B_\delta)}}{\tau^2b^2} \le \frac{1}{4}, \quad \frac{\|f_3\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2c^2} \le \frac{1}{4},$$

we have that

$$\begin{aligned} \tau^3c_1^2\int_{B_\delta} |\Psi_1|^2e^{2\tau\psi}dxdt + \tau^2\delta^2a^2\int_{B_\delta} |\partial_x\Psi_1|^2e^{2\tau\psi}dxdt + (\tau^3c_2^2 + \tau^4b^2)\int_{B_\delta} |\Psi_2|^2e^{2\tau\psi}dxdt \\ + (\tau^2\delta^2c^2 + \tau^3b^2)\int_{B_\delta} |\partial_x\Psi_2|^2e^{2\tau\psi}dxdt + \tau^2b^2\int_{B_\delta} |\partial_x^2\Psi_2|^2e^{2\tau\psi}dxdt \\ \le C\int_{B_\delta} |\mathcal{L}\Psi|^2e^{2\tau\psi}dxdt. \end{aligned} \tag{2.3}$$

Proof. Let $\Psi = (\Psi_1, \Psi_2) \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$. Consider the polynomial

$$P_1(\xi_1, \xi_2) = \xi_2 - a\xi_1^2\xi_2 + c_1\xi_1^3$$

and

$$P_1(D) = P_1(\partial_x, \partial_t) = \partial_t - a\partial_x^2\partial_t + c_1\partial_x^3$$

the differential operator associated to P_1. Then, simple calculations show that if $\alpha = (1,1)$ we have that

$$P_1^{(\alpha)}(\xi_1, \xi_2) = P_1^{(1,1)}(\xi_1, \xi_2) = -2a\xi_1, \quad P_1^{(\alpha)}(D)\Psi_1 = -2a\partial_x\Psi_1$$

and also

$$C(3, \alpha) = \sup_{|r+\alpha| \le 3} \binom{r+\alpha}{\alpha} = 2.$$

Thus, using Theorem 2.1 we see that

$$\begin{aligned} \tau^2\delta^2a^2 \int_{B_\delta} |\partial_x\Psi_1|^2 e^{2\tau\psi} dxdt &\le 32\tau^2\delta^2a^2 \int_{B_\delta} |\partial_x\Psi_1|^2 e^{2\tau\psi} dxdt \\ &= \frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_2}}{\alpha!} \int_{B_\delta} |P_1^{(\alpha)}(D)\Psi_1|^2 e^{2\tau\psi} dxdt \\ &\le \int_{B_\delta} |P_1(D)\Psi_1|^2 e^{2\tau\psi} dxdt. \end{aligned} \tag{2.4}$$

Moreover,

$$P_1^{(3,0)}(\xi_1, \xi_2) = 6c_1, \quad P_1^{(3,0)}(D)\Psi_1 = 6c_1\Psi_1, \quad C(3, (3,0)) = 1.$$

Then, using again the Theorem 2.1 we obtain that

$$\begin{aligned} \tau^3c_1^2 \int_{B_\delta} |\Psi_1|^2 e^{2\tau\psi} dxdt &\le \frac{2^6\tau^3}{6} \int_{B_\delta} |P_1^{(3,0)}(D)\Psi_1|^2 e^{2\tau\psi} dxdt \\ &\le \int_{B_\delta} |P_1(D)\Psi_1|^2 e^{2\tau\psi} dxdt. \end{aligned} \tag{2.5}$$

Now, by defining

$$P_2(\xi_1, \xi_2) = -b\xi_1^4, \quad P_2(D) = -b\partial_x^4,$$

we have that

$$P_2^{(4,0)}(\xi_1, \xi_2) = -24b, \quad P_2^{(4,0)}(D)\,\Psi_2 = -24b\Psi_2, \quad C(4, (4,0)) = 1$$

and

$$\begin{aligned} \tau^4b^2 \int_{B_\delta} |\Psi_2|^2 e^{2\tau\psi} dxdt &\le \frac{2^8\tau^4}{24} \int_{B_\delta} |P_2^{(4,0)}(D)\Psi_2|^2 e^{2\tau\psi} dxdt \\ &\le \int_{B_\delta} |P_2(D)\Psi_2|^2 e^{2\tau\psi} dxdt. \end{aligned} \tag{2.6}$$

In a similar fashion

$$P_2^{(3,0)}(D)\Psi_2 = -24b\partial_x\Psi_2, \quad P_2^{(2,0)}(D)\,\Psi_2 = -12b\partial_x^2\Psi_2, \quad C(4, (3,0)) = 4, \quad C(4, (2,0)) = 6.$$

Hence, we see that

$$\tau^3 b^2 \int_{B_\delta} |\partial_x \Psi_2|^2 e^{2\tau\psi} dxdt \leq \frac{2^6\tau^3}{24} \int_{B_\delta} |P_2^{(3,0)}(D)\Psi_2|^2 e^{2\tau\psi} dxdt \leq \int_{B_\delta} |P_2\,(D)\,\Psi_2|^2 e^{2\tau\psi} dxdt \tag{2.7}$$

and also that

$$\tau^2 b^2 \int_{B_\delta} |\partial_x^2 \Psi_2|^2 e^{2\tau\psi} dxdt \leq \frac{2^4\tau^2}{12} \int_{B_\delta} |P_2^{(2,0)}\,(D)\,\Psi_2|^2 e^{2\tau\psi} dxdt \leq \int_{B_\delta} |P_2\,(D)\,\Psi_2|^2 e^{2\tau\psi} dxdt. \tag{2.8}$$

By considering

$$P_4(\xi_1, \xi_2) = \xi_2 - c\xi_1^2\xi_2 + c_2\xi_2^3, \quad P_4(D) = P(\partial_x, \partial_t) = \partial_t - c\partial_x^2\partial_t + c_2\partial_x^3$$

we have that

$$P_4^{(3,0)}\,(D)\,\Psi_2 = 6c_2\Psi_2, \quad P_4^{(1,1)}\,(D)\,\Psi_2 = -2c\partial_x\Psi_2, \quad C(3,(3,0)) = 1, C(3,(1,1)) = 2.$$

Then, using Theorem 2.1 we obtain that

$$\tau^3 c_2^2 \int_{B_\delta} |\Psi_2|^2 e^{2\tau\psi} dxdt \leq \frac{2^6\tau^3}{6} \int_{B_\delta} |P_4^{(3,0)}\,(D)\,\Psi_2|^2 e^{2\tau\psi} dxdt \leq \int_{B_\delta} |P_4(D)\Psi_2|^2 e^{2\tau\psi} dxdt, \tag{2.9}$$

and

$$\tau^2\delta^2 c^2 \int_{B_\delta} |\partial_x \Psi_2|^2 e^{2\tau\psi} dxdt \leq 2^4\tau^2\delta^2 \int_{B_\delta} |P_4^{(1,1)}\,(D)\,\Psi_2|^2 e^{2\tau\psi} dxdt \leq \int_{B_\delta} |P_4(D)\Psi_2|^2 e^{2\tau\psi} dxdt. \tag{2.10}$$

From (2.4)-(2.10), there is $C > 0$ such that

$$\begin{aligned}
\tau^3 c_1^2 \int_{B_\delta} |\Psi_1|^2 e^{2\tau\psi} dxdt + \tau^2\delta^2 a^2 \int_{B_\delta} |\partial_x \Psi_1|^2 e^{2\tau\psi} dxdt + (\tau^3 c_2^2 + \tau^4 b^2) \int_{B_\delta} |\Psi_2|^2 e^{2\tau\psi} dxdt \\
+ (\tau^2\delta^2 c^2 + \tau^3 b^2) \int_{B_\delta} |\partial_x \Psi_2|^2 e^{2\tau\psi} dxdt + \tau^2 b^2 \int_{B_\delta} |\partial_x^2 \Psi_2|^2 e^{2\tau\psi} dxdt \\
\leq C \int_{B_\delta} \left(|P_1(D)\Psi_1|^2 + |P_2(D)\Psi_2|^2 + |P_4(D)\Psi_2|^2\right) e^{2\tau\psi} dxdt.
\end{aligned} \tag{2.11}$$

Now, we note that

$$\mathcal{L}_1 = \partial_t - a\partial_x^2\partial_t + c_1\partial_x^3 + f_1(x,t)\partial_x$$

implies $P_1(D)\Psi_1 = \mathcal{L}_1\Psi_1 - f_1(x,t)\partial_x\Psi_1$. Then, using inequality (2.4), we have that

$$\int_{B_\delta} |f_1(x,t)\partial_x\Psi_1|^2 e^{2\tau\psi} dxdt \leq \|f_1\|^2_{L^\infty(B_\delta)} \int_{B_\delta} |\partial_x \Psi_1|^2 e^{2\tau\psi} dxdt$$

$$\leq \frac{\|f_1\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 a^2}\int_{B_\delta}|P_1(D)\Psi_1|^2 e^{2\tau\psi}dxdt$$
$$\leq \frac{2\|f_1\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 a^2}\int_{B_\delta}\left(|\mathcal{L}_1\Psi_1|^2+|f_1(x,t)\partial_x\Psi_1|^2\right)e^{2\tau\psi}dxdt. \tag{2.12}$$

In a similar way, for

$$\mathcal{L}_2\Psi_2 = P_2(D)\Psi_2 + f_2(x,t)\partial_x^2\Psi_2, \quad \mathcal{L}_4\Psi_2 = P_4(D)\Psi_2 + f_3(x,t)\partial_x\Psi_2$$

we obtain, using (2.8) and (2.10), that

$$\int_{B_\delta}|f_2(x,t)\partial_x^2\Psi_2|^2 e^{2\tau\psi}dxdt \leq \|f_2\|^2_{L^\infty(B_\delta)}\int_{B_\delta}|\partial_x^2\Psi_2|^2 e^{2\tau\psi}dxdt$$
$$\leq \frac{\|f_2\|^2_{L^\infty(B_\delta)}}{\tau^2 b^2}\int_{B_\delta}|P_2(D)\Psi_2|^2 e^{2\tau\psi}dxdt$$
$$\leq \frac{2\|f_2\|^2_{L^\infty(B_\delta)}}{\tau^2 b^2}\int_{B_\delta}\left(|\mathcal{L}_2\Psi_2|^2+|f_2(x,t)\partial_x^2\Psi_2|^2\right)e^{2\tau\psi}dxdt \tag{2.13}$$

and also that

$$\int_{B_\delta}|f_3(x,t)\partial_x\Psi_2|^2 e^{2\tau\psi} \leq \|f_3\|^2_{L^\infty(B_\delta)}\int_{B_\delta}|\partial_x\Psi_2|^2 e^{2\tau\psi}dxdt$$
$$\leq \frac{\|f_2\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 c^2}\int_{B_\delta}|P_4(D)\Psi_2|^2 e^{2\tau\psi}dxdt$$
$$\leq \frac{2\|f_3\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 c^2}\int_{B_\delta}\left(|\mathcal{L}_4\Psi_2|^2+|f_3(x,t)\partial_x\Psi_2|^2\right)e^{2\tau\psi}dxdt. \tag{2.14}$$

Next, if we choose $\tau > 0$ large enough such that

$$\frac{\|f_1\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 a^2} \leq \frac{1}{4}, \quad \frac{\|f_2\|^2_{L^\infty(B_\delta)}}{\tau^2 b^2} \leq \frac{1}{4}, \quad \frac{\|f_3\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 c^2} \leq \frac{1}{4},$$

then from inequalities (2.12)-(2.14) we have that

$$\int_{B_\delta}\left(|f_1(x,t)\partial_x\Psi_1|^2+|f_2(x,t)\partial_x^2\Psi_2|^2+|f_3(x,t)\partial_x\Psi_2|^2\right)e^{2\tau\psi}dxdt$$
$$\leq \frac{1}{2}\int_{B_\delta}\left(|\mathcal{L}_1\Psi_1|^2+|\mathcal{L}_2\Psi_2|^2+|\mathcal{L}_4\Psi_2|^2\right)e^{2\tau\psi}dxdt$$
$$+\frac{1}{2}\int_{B_\delta}\left(|f_1(x,t)\partial_x\Psi_1|^2+|f_2(x,t)\partial_x^2\Psi_2|^2+|f_3(x,t)\partial_x\Psi_2|^2\right)e^{2\tau\psi}dxdt,$$

what implies

$$\int_{B_\delta}\left(|f_1(x,t)\partial_x\Psi_1|^2+|f_2(x,t)\partial_x^2\Psi_2|^2+|f_3(x,t)\partial_x\Psi_2|^2\right)e^{2\tau\psi}dxdt$$
$$\leq \int_{B_\delta}\left(|\mathcal{L}_1\Psi_1|^2+|\mathcal{L}_2\Psi_2|^2+|\mathcal{L}_3\Psi_1|^2+|\mathcal{L}_4\Psi_2|^2\right)e^{2\tau\psi}dxdt$$

$$= \int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt,$$

where

$$\mathcal{L}_3 = I - d\partial_x^2 \quad \text{and} \quad |\mathcal{L}\Psi| = \left(|\mathcal{L}_1\Psi_1|^2 + |\mathcal{L}_2\Psi_2|^2 + |\mathcal{L}_3\Psi_1|^2 + |\mathcal{L}_4\Psi_2|^2\right)^{1/2}.$$

Therefore

$$\begin{aligned}
\int_{B_\delta} & \left(|P_1(D)\Psi_1|^2 + |P_2(D)\Psi_2|^2 + |P_4(D)\Psi_2|^2\right) e^{2\tau\psi} dxdt \\
&\leq 2 \int_{B_\delta} \left(|\mathcal{L}_1\Psi_1|^2 + |f_1(x,t)\partial_x\Psi_1|^2\right) e^{2\tau\psi} dxdt \\
&+ 2 \int_{B_\delta} \left(|\mathcal{L}_2\Psi_2|^2 + |f_2(x,t)\partial_x^2\Psi_2|^2\right) e^{2\tau\psi} dxdt \\
&+ 2 \int_{B_\delta} \left(|\mathcal{L}_4\Psi_2|^2 + |f_3(x,t)\partial_x\Psi_2|^2\right) e^{2\tau\psi} dxdt \\
&\leq 4 \int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt.
\end{aligned}$$

Hence, from previous inequality and (2.11) we obtain the estimate (2.3).

□

Remark 1. The estimate (2.3) is invariant under changes of signs on the components of $\mathcal{L}$.

Corollary 2.3. *Let $T > 0$. Assume that in addition to the hypotheses of the Theorem 2.2 we have that*

$$\eta, \eta_t \in L^2\left(-T, T; H^2_{loc}(\mathbb{R})\right), \quad \Phi \in L^2\left(-T, T; H^4_{loc}(\mathbb{R})\right), \quad \Phi_t \in L^2\left(-T, T; H^2_{loc}(\mathbb{R})\right)$$

and the support of η and support of Φ are compact contained in B_δ. Then, the inequality (2.3) holds if we replace $\Psi = (\Psi_1, \Psi_2)$ by $U = (\eta, \Phi)$. Indeed,

$$\begin{aligned}
\tau^3 c_1^2 \int_{B_\delta} |\eta|^2 e^{2\tau\psi} dxdt &+ \tau^2\delta^2 a^2 \int_{B_\delta} |\partial_x \eta|^2 e^{2\tau\psi} dxdt + (\tau^3 c_2^2 + \tau^4 b^2) \int_{B_\delta} |\Phi|^2 e^{2\tau\psi} dxdt \\
&+ (\tau^2\delta^2 c^2 + \tau^3 b^2) \int_{B_\delta} |\partial_x \Phi|^2 e^{2\tau\psi} dxdt + \tau^2 b^2 \int_{B_\delta} |\partial_x^2 \Phi|^2 e^{2\tau\psi} dxdt \\
&\leq C \int_{B_\delta} |\mathcal{L}U|^2 e^{2\tau\psi} dxdt. \qquad (2.15)
\end{aligned}$$

Proof. Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a regularizing sequence (in two variables) and consider

$$U_\epsilon = (\rho_\epsilon * \eta, \rho_\epsilon * \Phi),$$

where $*$ denotes the usual convolution. Then we have that $U_\epsilon \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$ and the inequality (2.3) holds for U_ϵ, that is

$$\tau^3 c_1^2 \int_{B_\delta} |\rho_\epsilon * \eta|^2 e^{2\tau\psi} dxdt + \tau^2\delta^2 a^2 \int_{B_\delta} |\partial_x(\rho_\epsilon * \eta)|^2 e^{2\tau\psi} dxdt$$

$$+\left(\tau^3c_2^2+\tau^4b^2\right)\int_{B_\delta}|\rho_\epsilon*\Phi|^2e^{2\tau\psi}dxdt+\left(\tau^2\delta^2c^2+\tau^3b^2\right)\int_{B_\delta}|\partial_x(\rho_\epsilon*\Phi)|^2e^{2\tau\psi}dxdt$$
$$+\tau^2b^2\int_{B_\delta}|\partial_x^2(\rho_\epsilon*\Phi)|^2e^{2\tau\psi}dxdt\le C\int_{B_\delta}|\mathcal{L}U_\epsilon|^2e^{2\tau\psi}dxdt. \tag{2.16}$$

Now, for $n=0,1$ and $m=0,1,2$ we have that

$$\begin{aligned}\|\partial_x^n(\rho_\epsilon*\eta)e^{\tau\psi}-\partial_x^n\eta\, e^{\tau\psi}\|_{L^2(B_\delta)}&=\|(\rho_\epsilon*\partial_x^n\eta)e^{\tau\psi}-\partial_x^n\eta\, e^{\tau\psi}\|_{L^2(B_\delta)}\\&\le C\|\partial_x^n(\rho_\epsilon*\eta)-\partial_x^n\eta\|_{L^2(B_\delta)}\to 0\end{aligned}$$

and

$$\|\partial_x^m(\rho_\epsilon*\Phi)e^{\tau\psi}-\partial_x^m\Phi\, e^{\tau\psi}\|_{L^2(B_\delta)}\le C\|\partial_x^m(\rho_\epsilon*\Phi)-\partial_x^m\Phi\|_{L^2(B_\delta)}\to 0,\quad \text{as}\quad \epsilon\to 0^+,$$

where C is a positive constant depending only on τ and δ. Similarly we have that

$$\int_{B_\delta)}\left(|\mathcal{L}U_\epsilon|^2\,e^{2\tau\psi}-|\mathcal{L}U|^2\,e^{2\tau\psi}\right)dxdt\to 0,\quad \text{as}\quad \epsilon\to 0^+,$$

which allows us to pass to the limit in (2.16) to conclude the proof of Corollary 2.3.

□

3 Unique continuation

In this section we prove the unique continuation result for the system (1.1). Before to do the proof, we establish the following results.

Lemma 3.1. *Let $T>0$ and $f_1,f_2,f_3\in L^\infty_{loc}(\mathbb{R}\times(-T,T))$. Let $U=(\eta,\Phi)$ with*

$$\eta,\eta_t\in L^2(-T,T;H^2_{loc}(\mathbb{R})),\quad \Phi\in L^2(-T,T;H^4_{loc}(\mathbb{R})),\quad \Phi_t\in L^2(-T,T;H^2_{loc}(\mathbb{R}))$$

be a solution of $\mathcal{L}U=0$ in $\mathbb{R}\times(-T,T)$ where $\mathcal{L}$ is the differential operator defined in (2.1). Let

$$\widetilde{U}=\begin{cases}U & \text{if}\quad t\ge 0\\ 0 & \text{if}\quad t<0.\end{cases}$$

Suppose that $\widetilde{U}\equiv 0$ in the region $\{(x,t)\,:\,x<t\}$ intercepted with a neighborhood of $(0,0)$. Then there exists a neighborhood $\mathcal{O}_1$ of $(0,0)$ (in the plane xt) such that $\widetilde{U}\equiv 0$ in $\mathcal{O}_1$.

Proof. By hypotheses there is $0<\delta<1$ such that $\widetilde{U}\equiv 0$ in $R_\delta=R_1\cup R_2$, where

$$R_1=\{(x,t)\,:\,x<t\}\cap B_\delta,\quad R_2=\{(x,t)\,:\,t<0\}\cap B_\delta,\quad B_\delta=\{(x,t)\,:\,x^2+t^2<\delta^2\}.$$

Next, consider $\chi\in C_0^\infty(B_\delta)$ such that $\chi=1$ in a neighborhood $\mathcal{O}$ of $(0,0)$ and define

$$\Psi=(\Psi_1,\Psi_2)=\chi\widetilde{U}.$$

Then we have that

$$\Psi_1,\Psi_{1t}\in L^2(-T,T;H^2_{loc}(\mathbb{R})),\quad \Psi_2\in L^2(-T,T;H^4_{loc}(\mathbb{R})),\quad \Psi_{2t}\in L^2(-T,T;H^2_{loc}(\mathbb{R}))$$

and

$$supp\,\Psi \subset B_\delta.$$

By using the definition of χ, we note that $\mathcal{L}\Psi = 0$ in $\mathcal{O}$. Thus, using the Corollary 2.3, we have for $\psi(x,t) = (x-\delta)^2 + \delta^2 t^2$ and $\tau > 0$ large enough that

$$\begin{aligned}\tau^3 c_1^2 \int_{B_\delta} |\Psi_1|^2 e^{2\tau\psi} dxdt + \tau^2\delta^2 a^2 \int_{B_\delta} |\partial_x \Psi_1|^2 e^{2\tau\psi} dxdt + (\tau^3 c_2^2 + \tau^4 b^2) \int_{B_\delta} |\Psi_2|^2 e^{2\tau\psi} dxdt \\ + (\tau^2\delta^2 c^2 + \tau^3 b^2) \int_{B_\delta} |\partial_x \Psi_2|^2 e^{2\tau\psi} dxdt + \tau^2 b^2 \int_{B_\delta} |\partial_x^2 \Psi_2|^2 e^{2\tau\psi} dxdt \\ \le C \int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt = C \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt. \qquad (3.1)\end{aligned}$$

Now, using again the definition of χ and the fact that $\widetilde{U} \equiv 0$ in R_δ, we see that

$$supp\,\Psi \subset D, \quad supp\,\mathcal{L}\Psi \subset D \cap (B_\delta \setminus \mathcal{O}), \quad D = \{(x,t) : 0 \le t \le x < \delta < 1\}.$$

It follows that if $(x,t) \neq (0,0)$ and $(x,t) \in D$ then

$$\psi(x,t) = (x-\delta)^2 + \delta^2 t^2 \le (t-\delta)^2 + \delta^2 t^2 = t^2(1+\delta^2) - 2t\delta + \delta^2 < \delta^2.$$

Thus, there exists $0 < \epsilon < \delta^2$ such that

$$\psi(x,t) \le \delta^2 - \epsilon, \quad (x,t) \in D \cap (B_\delta \setminus \mathcal{O}).$$

Moreover, since $\psi(0,0) = \delta^2$, we can choose $\mathcal{O}_1 \subset \mathcal{O}$ a neighborhood of $(0,0)$ such that

$$\psi(x,t) > \delta^2 - \epsilon, \quad (x,t) \in \mathcal{O}_1.$$

From the above construction and inequality (3.1), we have that there exists $C_1 > 0$ such that

$$\begin{aligned}\tau^3 e^{2\tau(\delta^2-\epsilon)} \int_{\mathcal{O}_1} \left(|\Psi_1|^2 + |\Psi_2|^2\right) dxdt &\le \tau^3 \int_{\mathcal{O}_1} \left(|\Psi_1|^2 + |\Psi_2|^2\right) e^{2\tau\psi} dxdt \\ &\le \tau^3 \int_{B_\delta} \left(|\Psi_1|^2 + |\Psi_2|^2\right) e^{2\tau\psi} dxdt \\ &\le C_1 \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt \\ &\le C_1 e^{2\tau(\delta^2-\epsilon)} \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 dxdt.\end{aligned}$$

Therefore

$$\int_{\mathcal{O}_1} \left(|\Psi_1|^2 + |\Psi_2|^2\right) dxdt \le \frac{C_1}{\tau^3} \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 dxdt.$$

Then, passing to the limit as $\tau \to +\infty$, we have that $\Psi \equiv 0$ in $\mathcal{O}_1$. Since $\widetilde{U} = \Psi$ in $\mathcal{O}$ and $\mathcal{O}_1 \subset \mathcal{O}$, we see that $\widetilde{U} = 0$ in $\mathcal{O}_1$.

□

Similarly, we also have the following result.

Lemma 3.2. *Let $T > 0$ and $f_1, f_2, f_3 \in L^\infty_{loc}(\mathbb{R} \times (-T,T))$. Let $U = (\eta, \Phi)$ with*

$$\eta, \eta_t \in L^2(-T,T; H^2_{loc}(\mathbb{R})), \quad \Phi \in L^2(-T,T; H^4_{loc}(\mathbb{R})), \quad \Phi_t \in L^2(-T,T; H^2_{loc}(\mathbb{R}))$$

be a solution of $\mathcal{L}U = 0$ in $\mathbb{R} \times (-T,T)$ where $\mathcal{L}$ is the differential operator defined in (2.1). Let

$$\widetilde{U} = \begin{cases} 0 & \text{if} \quad t \geq 0 \\ U & \text{if} \quad t < 0. \end{cases}$$

Suppose that $\widetilde{U} \equiv 0$ in the region $\{ (x,t) \ : \ x < -t \}$ intercepted with a neighborhood of $(0,0)$. Then there exists a neighborhood $\mathcal{O}_2$ of $(0,0)$ (in the plane xt) such that $\widetilde{U} \equiv 0$ in $\mathcal{O}_2$.

Corollary 3.1. *Let $T > 0$ and $F_1, F_2, F_3 \in L^\infty_{loc}(\mathbb{R} \times (-T,T))$. Let $U = (\eta, \Phi)$ with*

$$\eta, \eta_t \in L^2(-T,T; H^2_{loc}(\mathbb{R})), \quad \Phi \in L^2(-T,T; H^4_{loc}(\mathbb{R})), \quad \Phi_t \in L^2(-T,T; H^2_{loc}(\mathbb{R}))$$

be a solution in $\mathbb{R} \times (-T,T)$ of the system

$$\begin{cases} (I - a\partial_x^2)\eta_t - b\partial_x^4\Phi + F_1(x,t)\partial_x\eta + F_2(x,t)\partial_x^2\Phi = 0, \\ \\ (I - c\partial_x^2)\Phi_t + \eta - d\partial_x^2\eta + F_3(x,t)\partial_x\Phi = 0. \end{cases}$$

Let γ be a circumference passing through the origin $(0,0)$. Suppose that $U \equiv 0$ in the interior of the circle (with boundary γ) in a neighborhood of $(0,0)$. Then, there exists a neighborhood of $(0,0)$ where $U \equiv 0$.

Proof. Let us assume that the circumference (a piece of it) γ is given by $x = g(t)$. By using the hypotheses, we have that $U \equiv 0$ in the region $\{(x,t) : x < g(t)\}$ intercepted with a neighborhood of $(0,0)$. Then, we can to see that there exists $\omega \in \mathbb{R} \setminus \{0,1\}$ such that $U \equiv 0$ in a neighborhood of $(0,0)$ intercepted with the region $\{(x,t) : x < h(t)\}$ where

$$h(t) = \begin{cases} \omega t & \text{if} \quad t \geq 0 \\ -\frac{1}{\omega} t & \text{if} \quad t < 0. \end{cases}$$

Now, we consider the following change of variables $(x,t) \to (X,T)$ with

$$\begin{aligned} X &= x - h(t) + |t| \\ T &= t. \end{aligned}$$

Notice that in the new variables, if $T \geq 0$ then the function

$$U = U(X,T) = (\eta(T,X), \Phi(X,T))$$

is a solution of the system

$$\begin{cases} (I - a\partial_X^2)\eta_T - b\partial_X^4\Phi + a(\omega-1)\partial_X^3\eta + (1 - \omega + F_1(X,T))\,\partial_X\eta + F_2(X,T)\partial_X^2\Phi = 0, \\ \\ (I - c\partial_X^2)\Phi_T + \eta - d\partial_X^2\eta + c(\omega-1)\partial_X^3\Phi + (1 - \omega + F_3(X,T))\,\partial_X\Phi = 0. \end{cases}$$

Then, $U \equiv 0$ in the region $\{(X,T) : X < T, \ T \geq 0\}$ intercepted with a neighborhood of $(0,0)$ and U satisfies

$$\mathcal{L}U = 0 \quad \text{if} \quad T \geq 0,$$

where

$$\mathcal{L} = \begin{pmatrix} \partial_T - a\partial_X^2\partial_T + c_1\partial_X^3 + f_1(X,T)\partial_X & f_2(X,T)\partial_X^2 - b\partial_X^4 \\ I - d\partial_X^2 & \partial_T - c\partial_X^2\partial_T + c_2\partial_X^3 + f_3(X,T)\partial_X \end{pmatrix}$$

with

$$c_1 = a(\omega-1), \quad c_2 = c(\omega-1), \quad f_1 = 1-\omega+F_1, \quad f_2 = F_2, \quad f_3 = 1-\omega+F_3.$$

So, using Lemma 3.1 with the previous differential operator $\mathcal{L}$, we obtain that there exists a neighborhood $\mathcal{O}_1$ of $(0,0)$ in the plane XT where $U \equiv 0$.

In a similar fashion, $U \equiv 0$ in the region $\{(X,T) : X < -T, \ T < 0\}$ intercepted with a neighborhood of $(0,0)$ and U satisfies

$$\mathcal{L}U = 0 \quad \text{if} \quad T < 0,$$

where

$$c_1 = a\left(1 - \tfrac{1}{\omega}\right), \quad c_2 = c\left(1-\tfrac{1}{\omega}\right), \quad f_1 = \tfrac{1}{\omega} - 1 + F_1, \quad f_2 = F_2, \quad f_3 = \tfrac{1}{\omega} - 1 + F_3.$$

Then, from Lemma 3.2 we have that there exists a neighborhood $\mathcal{O}_2$ of $(0,0)$ in the plane XT where $U \equiv 0$. Thus, returning to the original variables (x,t) we have the result. □

Now we have the main result on the unique continuation property for the system (1.1).

Theorem 3.2. *Let $T > 0$ and $(\eta, \Phi) = (\eta(x,t), \Phi(x,t))$ with*

$$\eta, \eta_t \in L^2(-T,T; H^2_{loc}(\mathbb{R})), \quad \Phi \in L^2(-T,T;H^4_{loc}(\mathbb{R})), \quad \Phi_t \in L^2(-T,T;H^2_{loc}(\mathbb{R}))$$

be a solution in $\mathbb{R}\times(-T,T)$ of the system (1.1). If $(\eta,\Phi) \equiv 0$ in an open subset Ω of $\mathbb{R}\times(-T,T)$, then $(\eta,\Phi)\equiv 0$ in the horizontal component of Ω.

Proof. By defining the functions

$$F_1(x,t) = \partial_x\Phi(x,t), \quad F_2(x,t) = 1 + \eta(x,t), \quad F_3(x,t) = \tfrac{1}{2}\partial_x\Phi(x,t),$$

the system (1.1) takes the form

$$\begin{cases} (I - a\partial_x^2)\eta_t - b\partial_x^4\Phi + F_1(x,t)\partial_x\eta + F_2(x,t)\partial_x^2\Phi = 0, \\ (I - c\partial_x^2)\Phi_t + \eta - d\partial_x^2\eta + F_3(x,t)\partial_x\Phi = 0. \end{cases} \tag{3.2}$$

with $F_1, F_2, F_3 \in L^\infty_{loc}(\mathbb{R}\times(-T,T))$. Then, we will show the result for the system (3.2).

Denote by Ω_1 the horizontal component of Ω and let

$$\Lambda = \{(x,t) \in \Omega_1 : (\eta,\Phi) \equiv 0 \quad \text{in a neighborhood of} \quad (x,t)\}.$$

Let $Q \in \Omega_1$ arbitrary. Choose $P \in \Lambda$ and let Γ be a continuous curve contained in Ω_1 joining P to Q, parametrized by a continuous function $f : [0,1] \to \Omega_1$ with $f(0) = P$ and $f(1) = Q$. Since $P \in \Lambda$, there exists $r > 0$ such that

$$(\eta,\Phi) \equiv 0 \quad \text{in} \quad B_r(P). \tag{3.3}$$

Taking $0 < r_0 < \min\{r, dist(\Gamma, \partial\Omega_1)\}$, where $\partial\Omega_1$ denotes the boundary of Ω_1, we have that

$$B_{r_0}(P) \subset \Lambda.$$

Now, if $r_1 < \frac{r_0}{4}$ we see that

$$B_{2r_1}(f(s)) \subset \Omega_1, \quad \text{for all } s \in [0,1]; \tag{3.4}$$

in fact, if $w \in B_{2r_1}(f(s))$ and $w \notin \Omega_1$ then

$$\|w - f(s)\| < 2r_1 < r_0 < dist(\Gamma, \partial\Omega_1) \leq \|w - f(s)\|,$$

which is a contradiction.

Next, let

$$\Lambda_1 = \{(x,t) \in \Lambda \, : \, (\eta, \Phi) \equiv 0 \quad \text{in} \quad B_{r_1}(x,t) \cap \Omega_1\}$$

and

$$S = \{0 \leq \ell \leq 1 \, : \, f(s) \in \Lambda_1 \quad \text{whenever} \quad 0 \leq s \leq \ell\}, \quad \ell_0 = \sup S.$$

We will prove that $f(\ell_0) \in \Lambda_1$. If $w \in B_{r_1}(f(\ell_0))$ and $r_2 = \|w - f(\ell_0)\|$ then there exists $0 < \delta < \ell_0$ such that $\|f(\ell_0) - f(\ell_0 - \delta)\| < r_1 - r_2$. Therefore

$$\|w - f(\ell_0 - \delta)\| \leq \|w - f(\ell_0)\| + \|f(\ell_0) - f(\ell_0 - \delta)\| < r_1,$$

and so $w \in B_{r_1}(f(\ell_0 - \delta))$. Now, from the definition of ℓ_0 there exists $\ell_\delta \in S$ such that $\ell_0 - \delta < \ell_\delta \leq \ell_0$, what implies $f(\ell_0 - \delta) \in \Lambda_1$. Then, using (3.4) we see that

$$(\eta, \Phi) \equiv 0 \quad \text{in} \quad B_{r_1}(f(\ell_0 - \delta)) \cap \Omega_1 = B_{r_1}(f(\ell_0 - \delta)). \tag{3.5}$$

Consequently we obtain that $(\eta(w), \Phi(w)) = 0$ and then

$$(\eta, \Phi) \equiv 0 \quad \text{in} \quad B_{r_1}(f(\ell_0)). \tag{3.6}$$

Hence, we have showed $f(\ell_0) \in \Lambda_1$.

If $\ell_0 = 1$ then from previous analysis we have that $Q = f(1) \in \Lambda_1 \subset \Lambda$. Thus, since Q was arbitrarily chosen we obtain that $(\eta, \Phi) \equiv 0$ in Ω_1, which proves Theorem 3.2. Then to finish the proof of Theorem 3.2 remains to prove that $\ell_0 = 1$. In fact, let us suppose that $\ell_0 < 1$ and let

$$G = \{Y \in \Omega_1 \, : \, \|Y - f(\ell_0)\| = r_1\}.$$

For $w = (x_1, t_1) \in G$ fixed, we consider the change of variable $(x,t) \to (X,T)$ where

$$X = x - x_1,$$
$$T = t - t_1.$$

Notice that $(0,0) \in G^* = \{Y = (X,T) \, : \, \|Y - (f(\ell_0) - w)\| = r_1\}$. Moreover, from (3.6) we see that

$$(\eta(X,T), \Phi(X,T)) = 0, \quad (X,T) \in B_{r_1}(f(\ell_0) - w).$$

So that, by using Corollary 3.1, there exists $r_w^* > 0$ such that

$$(\eta(X,T), \Phi(X,T)) = 0, \quad (X,T) \in B_{r_w^*}(0,0).$$

Returning to the original variables we have that for each $w \in G$ there exists $r_w^* > 0$ such that

$$(\eta, \Phi) \equiv 0 \quad \text{in} \quad B_{r_w^*}(w).$$

Then, using (3.6) and the compactness of G, we have that there is $\epsilon_1 > 0$ such that

$$(\eta, \Phi) \equiv 0 \quad \text{in} \quad B_{r_1+\epsilon_1}(f(\ell_0)). \tag{3.7}$$

Now, we note that there exists $0 < \delta_1 < 1 - \ell_0$ such that if $w \in B_{r_1}(f(\ell_0 + \delta_1))$ then

$$\|w - f(\ell_0)\| \leq \|w - f(\ell_0 + \delta_1)\| + \|f(\ell_0 + \delta_1) - f(\ell_0)\| < r_1 + \epsilon_1.$$

Thus, $w \in B_{r_1+\epsilon_1}(f(\ell_0))$ and so $B_{r_1}(f(\ell_0 + \delta_1)) \subset B_{r_1+\epsilon_1}(f(\ell_0))$. Therefore, using (3.7) we have that $(\eta, \Phi) \equiv 0$ in $B_{r_1}(f(\ell_0+\delta_1))$. Consequently $f(\ell_0+\delta_1) \in \Lambda_1$, which contradicts the definition of ℓ_0. So, $\ell_0 = 1$ and the proof of Theorem 3.2 is complete.

□

Conclusion. In this paper, using an appropriate Carleman-type estimate, we established a result of unique continuation for a special class of one-dimensional systems that model the evolution of long water waves with small amplitude in the presence of surface tension. We showed that if $(\eta, \Phi) = (\eta(x,t), \Phi(x,t))$ is a solution of the system (1.1) in a suitable function space and (η, Φ) vanishes on an open subset Ω of $\mathbb{R} \times [-T, T]$, then $(\eta, \Phi) \equiv 0$ in the horizontal component of Ω.

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