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Asymptotic behaviour of the Einstein-Yang-Mills-Higgs system in a Bianchi type I model

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Abstract. We study the Einstein-Yang-Mills-Higgs (EYMH) system with a positive cosmological constant in the Bianchi type I space-time with locally rotational symmetry (LRS). In particular, we consider the nonlinear interaction of the Higgs field with the Yang-Mills field coupled to an unknown gravitational field. For the considered model, from certain additional conditions (the temporal gauge and some symmetries), we derive the conservation laws for the field equations and we then deduce the exact formulation of equations in the geometric framework. Furthermore, using an iterative approch and some mathematical analysis tools, we study the above system of equations. We then establish a global existence result for the homogeneous solution and we analyse its asymptotic behaviour.

Keywords. Einstein equations, Yang-Mills field, Higgs field, Bianchi type I model, global solution, asymptotic behaviour.

1 Introduction

In general relativity theory, cosmology plays one of the central roles, by coupling various matter fields to the Einstein equations in order to propose explanations in response to new astrophysical observations. In fact, the evolution of the universe since the time of the dominant radiation until the current cosmic acceleration can be explained by the homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) model [12]. One of the pillars of cosmology is that, the universe is homogeneous and isotropic on a large scale.

To date, the inflation is the main mechanism to explain the isotropization of the observable universe. This mechanism is based on the existence of a scalar field known as "Inflaton" [8]. During a brief inflation period, the scalar field energy density drives the universe towards a locally isotropic and homogeneous form that leaves only very small residual anisotropies. These anisotropies are observed in the cosmic microware background, which support the idea that space-times become isotropic by evolving in time [10].

The spatial homogeneous but anisotropic space-times are known as either Kantowski-Sachs or Bianchi cosmologies. These space-times contain many important models that have been used for discussions of the isotropy of the universe at late time [3]. To this list of important models, is

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also added that of FLRW in the limit of isotropization, e.g., Bianchi I isotropizes to flat FLRW models.

Bianchi space-times in the presence of a scalar field have been widely studied in the literature, see for instance [9, 10] and references therein. It has been shown in general in these works that, for specific initial conditions, an initial anisotropic universe can end in an isotropic universe. In this work, we are interested in the study of the gravitational field equations associated with the nonlinear action of the Higgs field coupled to the Yang-Mills field, in the universe model of Bianchi type I with LRS symmetry. The Einstein-Yang-Mills-Higgs (EYMH) system thus obtained form a basis for the interaction model of gravitational, gauge and scalar fields (see [2, 19]). It unifies two important trends in the theory of gravity. The first one is the Einstein-Yang-Mills model, which is a generalization to the non-Abelian case of the Einstein-Maxwell model of relativistic electromagnetism theory. The second trend is related to the investigation of interaction of gravitational and Higgs fields. The Higgs field is the generalization to the non-Abelian case of the scalar field. The present EYMH model also finds its applications in theories of dark matter and dark energy [2, 19].

We consider the Einstein equations with a cosmological constant, in fact, astrophysical observations have clarified that, the expansion of the universe is accelerating. It is well known that, a classical mathematical tool to model this phenomenon is to include the cosmological constant in the Einstein equations. For more details on the cosmological constant, we can consult [17] and references therein. Today, it is clearly established that, the gravitational field can propagate through space at the speed of the light, analogously to electromagnetic waves. The mathematical tool used to model this phenomenon consists to couple the scalar field or the Higgs field to the Einstein equations. We can found more details on this question in [4, 7, 16].

In the following, we derive the conservation laws for the field equations and we then deduce the exact formulation of equations in the geometric framework, thanks to some additional considerations. Furthermore, an iterative approach combined with some mathematical analysis tools is use to establish a global in time existence and uniqueness result for solutions of the EYMH system. This method is more convenient because it requires less sophisticated estimates, and offers the possibility of a numerical implementation of results. Moreover, the study of the asymptotic behaviour, shows that our model approches asymptotically the De Sitter model. We thus obtain an exact solution of general relativity equations, belonging to the solution class of FLRW. Physically, the De Sitter solution corresponds to a homogeneous, isotropic universe, with positive curvature and in which the only contribution to the energy density comes from the cosmological constant. Our results extend to the non-Abelian case the ones obtained in [15].

The rest of this paper is organized in three sections:
In section 2, we state the equations and we prove some preliminary results. Section 3 is devoted to local and global existence results for the system. In section 4, we study the asymptotic behaviour of the global solution at late times.

2 The Einstein-Yang-Mills-Higgs system

Throughout the paper, unless otherwise specified, Greek indices $\alpha,\ \beta,\dots$ range from 0 to 3 and Latin ones $i,\ j,\dots$ from 1 to 3. We adopt the standard convention of summing over repeated indices, i.e., $a_\alpha b^\alpha = \sum\limits_{\alpha=0}^{3} a_\alpha b^\alpha$. We first introduce some more comprehensible geometrical tools that are necessary for the understanding of the deep structure of the EYMH equations. We then derive equations in the geometric framework, solve the constraint problem, and we finally deduce the associated equivalent system of nonlinear differential equations.

2.1 Geometric tools and notations

The basic geometric framework is a Bianchi type I space-time $\left(\mathbb{R}^4, g\right)$ with locally rotational symmetric, i.e., a $4-$dimensional manifold equipped with a Lorentzian metric $g = (g_{\alpha\beta})$ of signature $(-,+,+,+)$ and given by:

$$g = -dt^2 + u^2(t)dx_1^2 + v^2(t)\left(dx_2^2 + dx_3^2\right), \tag{2.1}$$

where $u > 0$ and $v > 0$ are two continuously differentiable functions of the time t.

A Yang-Mills potential is represented by a $1-$form $A = (A_\alpha)$ defined on $\mathbb{R}^4$ with values in the Lie algebra $(\mathcal{G}, [,])$ of a Lie group G. We assume that, the Lie algebra $(\mathcal{G}, [,])$ admits an Ad-invariant scalar product, denoted by a dot, which enjoys the property:

$$f \cdot [k, l] = [f, k] \cdot l, \quad \forall f, k, l \in \mathcal{G}. \tag{2.2}$$

We assume that, $\mathcal{G}$ is an $N-$dimensional $\mathbb{R}-$based Lie algebra, $(\varepsilon_a)_{a=1}^N$ denotes an orthogonal basis of $\mathcal{G}$. The Yang-Mills potential is locally defined by:

$$A = (A_\alpha) \quad with \quad A_\alpha : \mathbb{R}^4 \longrightarrow \mathcal{G}. \tag{2.3}$$

The temporal gauge condition for the Yang-Mills potential states that: $A_0 = 0$.

The Yang-Mills field is the curvature of the Yang-Mills potential. It is represented by a $\mathcal{G}-$valued antisymmetric $2-$form $F = (F_{\lambda\mu})$ defined on $\mathbb{R}^4$ by:

$$F = dA + \frac{1}{2}[A, A]. \tag{2.4}$$

In the local coordinates (x^α), the above relation (2.4) is written:

$$F^a_{\lambda\mu} = \bigtriangledown_\lambda A^a_\mu - \bigtriangledown_\mu A^a_\lambda + [A_\lambda, A_\mu]^a \quad in\ which\ [A_\lambda, A_\mu]^a = C^a_{bc} A^b_\lambda A^c_\mu, \tag{2.5}$$

where $\bigtriangledown$ denotes the covariant derivative with respect to the space-time metric, and C^a_{bc} are the structure constants of the Lie Algebra $\mathcal{G}$.

In addition to the Yang-Mills field, many theories consider a Higgs field or a scalar multiplet which is represented by a $\mathcal{G}-$valued function ϕ defined on $\mathbb{R}^4$. In the basis (ε_a), ϕ is defined by:

$$\phi = \phi^a \varepsilon_a \quad with \quad \phi^a : \mathbb{R}^4 \longrightarrow \mathbb{R}. \tag{2.6}$$

2.2 The Einstein-Yang-Mills-Higgs equations

On its general form, the EYMH system with a positive cosmological constant Λ, can be written (see [7, 19]):

$$R_{\alpha\beta} - \left(\frac{1}{2}R - \Lambda\right) g_{\alpha\beta} = 8\pi\left(\tau_{\alpha\beta} + T_{\alpha\beta}\right), \tag{2.7}$$

$$\hat{\bigtriangledown}_\alpha F^{\alpha\beta} = J^\beta, \tag{2.8}$$

$$\hat{\bigtriangledown}_\alpha \hat{\bigtriangledown}^\alpha \phi = H. \tag{2.9}$$

Here, $\hat{\bigtriangledown}$ is the gauge covariant derivative or the Yang-Mills operator acting on ϕ and $F^{\alpha\beta}$ as follows:

$$\hat{\bigtriangledown}_\alpha\phi = \bigtriangledown_\alpha\phi + [A_\alpha, \phi], \quad \hat{\bigtriangledown}_\alpha F^{\alpha\beta} = \bigtriangledown_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}].$$

(2.7) is the Einstein system and determines the unknown metric tensor $g = (g_{\alpha\beta})$, $R_{\alpha\beta}$ and R are respectively, the Ricci curvature and the scalar curvature of the metric (here and throughout the work, indices are raised or lowered with respect to the space-time metric, i.e., $R^{\alpha\beta} = g^{\alpha\mu}g^{\beta\lambda}R_{\mu\lambda}$). $\tau_{\alpha\beta} + T_{\alpha\beta}$ is the stress-energy or the energy-momentum tensor. $T_{\alpha\beta}$ and $\tau_{\alpha\beta}$ whose expressions are given below (see [7]), are respectively the tensors associated to the coupled action of the Yang-Mills field and the Higgs field on the gravitational field.

$$\tau_{\alpha\beta} = -\frac{1}{4}g_{\alpha\beta}F_{\lambda\mu}\cdot F^{\lambda\mu} + F_{\alpha\lambda}\cdot F_\beta{}^\lambda, \tag{2.10}$$

$$T_{\alpha\beta} = \hat{\bigtriangledown}_\alpha\phi\cdot\hat{\bigtriangledown}_\beta\phi - \frac{1}{2}g_{\alpha\beta}\left(\hat{\bigtriangledown}_\lambda\phi.\hat{\bigtriangledown}^\lambda\phi + V(\phi^2)\right), \tag{2.11}$$

V being a C^∞ real valued function defined on $\mathbb{R}$ to $\mathcal{G}$ (often called the self interaction potential), and $\phi^2 = \phi\cdot\phi$.

(2.8) is the YM system, written in the covariant form for the Yang-Mills field $F = (E, \Phi)$ which is a closed antisymmetric 2-form, depending only on the time t. $E = \left(F^{0i}\right)$ and $\Phi = (F_{ij})$ are respectively its electric and magnetic parts. $J = (J^\beta)$ is the Yang-Mills current defined by:

$$J^\beta(A, \phi, \dot{\phi}) = [\phi, \hat{\bigtriangledown}^\beta\phi], \tag{2.12}$$

where $\dot{\phi} = \frac{\partial\phi}{\partial t}$. (2.9) is the Higgs equation and determines the Higgs field ϕ. H is the Higgs potential, it is a C^∞ $\mathcal{G}-$ valued function given by (see [7]):

$$H^I = \dot{V}(\phi^2)\phi^I, \; I = 1, ..., N. \tag{2.13}$$

Throughout the remainder of the paper the dot over function denotes the usual derivative with respect to time.

2.3 Derivation of equations

It is well-known that, the EYMH system is not an evolution system as it stands. In order to obtain such a system, we needs to impose on the unknown functions some supplementary conditions called gauge condition or to choose a spetial system of coordinates. In the present paper, we will use the temporal gauge condition and the locally rotational symmetry to the metric tensor see (2.1).

2.3.1 Initial data for the complete system

Let the following quantities called the initial data be given: $u^0 > 0$, $v^0 > 0$; $p^0 = \left(A_i^0\right)$ a constant Yang-Mills potential; $E^0 = \left(F^{0,0i}\right)$ a constant electric field; $\Phi^0 = \left(F_{ij}^0\right)$ a constant magnetic field; ϕ^0 a constant vector of $\mathbb{R}^N$. We look for u, v, $A = (0, A_i) = (0, p)$, $F = \left(F^{0i}, F_{ij}\right) = (E, \Phi)$ and ϕ solution of the EYMH system such that: $u(0) = u^0$, $v(0) = v^0$, $A(0) = (0, p^0)$, $F(0) = (E, \Phi)$ and $\phi(0) = \phi^0$.

2.3.2 The Yang-Mills equations in F and A

The YM system (2.8) is an incompatible system for its unknowns $E = (F^{0i})$ and $\Phi = (F_{ij})$. We then complete these equations by adding to them the Bianchi identities:

$$\hat{\bigtriangledown}_0 F_{ij} + \hat{\bigtriangledown}_i F_{0j} + \hat{\bigtriangledown}_j F_{0i} = 0, \quad \textit{and} \quad \hat{\bigtriangledown}_i F_{jk} + \hat{\bigtriangledown}_j F_{ki} + \hat{\bigtriangledown}_k F_{ij} = 0,$$

which can also be written:

$$\dot{F_{ij}} = -[A_j, F_{0i}] + [A_i, F_{0j}], \tag{2.14}$$
$$[A_i, F_{jk}] + [A_j, F_{ki}] + [A_k, F_{ij}] = 0. \tag{2.15}$$

Therefore, we then retain the system formed by equations (2.8) with $\beta = i$ and (2.14):

$$\begin{cases} \dot{F^{0i}} + \Gamma^l_{l0} F^{0i} + [A_j, F^{ji}] = J^i, \\ \dot{F_{ij}} = -[A_j, F_{0i}] + [A_i, F_{0j}], \end{cases} \tag{2.16}$$

and for which we show that, the unretained system formed from (2.8) with $\beta = 0$ and (2.15) is automatically satisfied by the solution of (2.16). In (2.16), $\Gamma^\alpha_{\lambda\mu}$ denote the Christoffel symbols associated to the metric tensor (2.1), which are computed using $\Gamma^\alpha_{\lambda\mu} = \frac{1}{2} g^{\alpha\beta}\left[\partial_\lambda g_{\beta\mu} + \partial\mu g_{\lambda\beta} - \partial\beta g_{\beta\mu}\right]$, and give for the non-zero cases:

$$\Gamma^1_{10} = \frac{\dot{u}}{u},\ \Gamma^2_{20} = \Gamma^3_{30} = \frac{\dot{v}}{v},\ \Gamma^0_{11} = u\dot{u},\ \Gamma^0_{22} = \Gamma^0_{33} = v\dot{v}.$$

Two authors of the present work have shown in [6] that, if the initial data for the EYMH system (2.7)-(2.9) satisfy

$$[A_i, E^i]_{t=0} = -J^0_{t=0} = -[\phi^0, \dot{\phi}^0], \tag{2.17}$$

then, the Yang-Mills system (2.8), in temporal gauge is equivalent to:

$$\begin{cases} \frac{dF^{0i}}{dt} = -\Gamma^l_{l0} F^{0i} + [A_j, F^{ij}] + J^i, \\ \frac{dF_{ij}}{dt} = -[A_j, F_{0i}] + [A_i, F_{0j}], \\ \frac{dA_i}{dt} = F_{0i}. \end{cases} \tag{2.18}$$

Concerning the Yang-Mills tensor $\tau = (\tau_{\alpha\beta})$ defined by (2.10), a direct computation leads to:

$$\begin{cases} \tau_{0i} = -F^{0j} \cdot F_{ij}, \\ \tau_{ij} = -g_{ii} g_{jj} F^{0i} \cdot F^{0j} + g^{kk} F_{ik} \cdot F_{jk} \quad i \neq j; \\ \tau_{\alpha\alpha} = \frac{1}{2}\left[g_{ll}\left(F^{0l}\right)^2 + g^{kk} g^{mm} \left(F_{km}\right)^2\right] - \left(g_{\alpha\alpha}\right)^2 \left(F^{0\alpha}\right)^2 + g^{kk}\left(F_{\alpha k}\right)^2, \end{cases} \tag{2.19}$$

where $\left(F^{0i}\right)^2 = F^{0i} \cdot F^{0i}$ and $\left(F_{ij}\right)^2 = F_{ij} \cdot F_{ij}$.

2.3.3 The Higgs system in ϕ

Using the Einstein summation convention, the Higgs system (2.9) is written:

$$\hat{\bigtriangledown}_0 \hat{\bigtriangledown}^0 \phi + \hat{\bigtriangledown}_i \hat{\bigtriangledown}^i \phi = H.$$

Taking into account the gauge covariant derivative, it follows that:

$$\ddot{\phi} + \left[A_i, \left[A^i, \phi\right]\right] = H. \tag{2.20}$$

The self interaction potential of a real Higgs field ϕ can be written as (see [14]):

$$V(\phi^2) = m^2\phi^2 + k_1\phi^4 + k_2, \tag{2.21}$$

where m is the mass parameter, k_1 is the Higgs self-coupling parameter ($k_1 \geqslant 0$) and k_2 is an arbitrary normalization constant ($k_2 \geqslant 0$).
The insertion of (2.21) in (2.13) yields:

$$H = 2m^2\dot{\phi}\phi^2 + 4k_1\dot{\phi}\phi^4, \tag{2.22}$$

finally, from (2.20) and (2.22), it comes that:

$$\ddot{\phi} - \left(2m^2\phi^2 + 4k_1\phi^4\right)\dot{\phi} + \left[A_i, \left[A^i, \phi\right]\right] = 0. \tag{2.23}$$

At this step, we set $\psi = \dot{\phi}$ and (2.23) is reduced to:

$$\begin{cases} \frac{d\phi}{dt} = \psi, \\ \frac{d\psi}{dt} = \left(2m^2\phi^2 + 4k_1\phi^4\right)\psi - \left[A_i, \left[A^i, \phi\right]\right]. \end{cases} \tag{2.24}$$

A direct calculation using (2.11) shows that components of the Higgs tensor $T = (T_{\alpha\beta})$ are given by:

$$\begin{cases} T_{0i} = \psi[A_i, \phi], \\ T_{ij} = [A_i, \phi] \cdot [A_j, \phi] \quad with \quad i \neq j, \\ T_{00} = \frac{3}{2}(\psi)^2 + \frac{1}{2}\{[A_k, \phi] \cdot [A^k, \phi] + V(\psi^2)\}, \\ T_{ii} = [A_i, \phi] \cdot [A_i, \phi] + \frac{g_{ii}}{2}\{(\psi)^2 + [A_k, \phi] \cdot [A^k, \phi] + V(\phi^2)\}. \end{cases} \tag{2.25}$$

2.3.4 The Einstein equations in u and v

By determine components of the Einstein tensor $S = (S_{\alpha\beta})$ defined by, $S_{\alpha\beta} = R_{\alpha\beta} - \left(\frac{1}{2}R - \Lambda\right)g_{\alpha\beta}$, and taking into account relations (2.19) and (2.25), we show that, the Einstein equations in u and v can be written:

$$-u^2\left[2\frac{\ddot{v}}{v} + \left(\frac{\dot{v}}{v}\right)^2 - \Lambda\right] = 8\pi\left(T_{11} + \tau_{11}\right), \tag{2.26}$$

$$-v^2\left[\frac{\ddot{u}}{u} + \frac{\ddot{v}}{v} + \frac{\dot{u}\dot{v}}{uv} - \Lambda\right] = 8\pi\left(T_{22} + \tau_{22}\right), \tag{2.27}$$

subject to the following constraints:

$$2\frac{\dot{u}\dot{v}}{uv} + \left(\frac{\dot{v}}{v}\right)^2 - \Lambda = 8\pi\left(T_{00} + \tau_{00}\right), \tag{2.28}$$

$$T_{0i} + \tau_{0i} = 0, \tag{2.29}$$

$$T_{ij} + \tau_{ij} = 0 \; if \; i \neq j, \tag{2.30}$$

$$T_{22} + \tau_{22} = T_{33} + \tau_{33}. \tag{2.31}$$

Constraints problem for the Einstein equations

Proposition 2.1. *The constraints (2.28)-(2.31) are satisfied in the whole existence domain of solutions of the evolution system (2.26)-(2.27), if and only if, they are satisfied for $t = 0$.*

Proof. We set:

$$Y^\alpha = S^{\alpha\beta} - 8\pi\left(T^{\alpha\beta} + \tau^{\alpha\beta}\right), \; B = T_{0i} + \tau_{0i}, \; C = T_{ij} + \tau_{ij}, \; D = T_{22} + \tau_{22} - T_{33} - \tau_{33}.$$

Due to the Bianchi identities, it is easy to see that $\bigtriangledown_\alpha Y^\alpha = 0$ which leads to:

$$\frac{dY^0}{dt} = -\left(\frac{\dot{u}}{u} + 2\frac{\dot{v}}{v}\right) Y^0(t), \tag{2.32}$$

integrating (2.32) over $[0,t], \; t > 0$, we have:

$$Y^0(t) = \frac{u_0 v_0^2}{uv^2} Y^0(0).$$

Next, the conservation laws $\bigtriangledown_\alpha\left(T^{\alpha\beta} + \tau^{\alpha\beta}\right) = 0$, leads to:

$$\partial_0\left(T^{0\beta} + \tau^{0\beta}\right) + \Gamma^l_{l0}\left(T^{0\beta} + \tau^{0\beta}\right) + \Gamma^\beta_{\alpha\lambda}\left(T^{\alpha\lambda} + \tau^{\alpha\lambda}\right) = 0. \tag{2.33}$$

For $\beta = i$, (2.33) becomes:

$$\partial_0\left(T^{0i} + \tau^{0i}\right) + \Gamma^l_{l0}\left(T^{0i} + \tau^{0i}\right) = 0 \quad \Rightarrow \quad \left(T^{0i} + \tau^{0i}\right)(t) = \frac{u_0 v_0^2}{uv^2}\left(T^{0i} + \tau^{0i}\right)(0).$$

In the sequel, we reduce the Einstein system (2.7) in the form

$$R_{\alpha\beta} = \Theta_{\alpha\beta}, \quad where \;\; \Theta_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} + \tau_{\alpha\beta}\right) + g_{\alpha\beta}\left(\frac{1}{2}R - \Lambda\right). \tag{2.34}$$

It is established in [5] (page 413), that (2.34) satisfy:

$$\partial_0 R_{ij} - \bigtriangledown_i R_{0j} - \bigtriangledown_j R_{0i} = \partial_0 \Theta_{ij} - \bigtriangledown_i \Theta_{0j} - \bigtriangledown_j \Theta_{0i}. \tag{2.35}$$

For $i \neq j$, since $g_{ij} = 0$, $R_{ij} = 0$ and $R_{0i} = 0$, (2.35) gives:

$$\partial_0\left(T_{ij} + \tau_{ij}\right) = 0 \quad \Rightarrow \quad C(t) = C(0).$$

Finally, we write (2.35) for $i = j = 2$ and for $i = j = 3$, knowing that $R_{22} = R_{33}$ and $R_{0i} = 0$, we obtain:

$$\partial_0\left(T_{22} + \tau_{22} - T_{33} - \tau_{33}\right) = 0 \quad \Rightarrow \quad D(t) = D(0).$$

Which completes the proof of Proposition 2.1. □

Remark 1. In all what follows, we suppose that the initial data for the EYMH system satisfy constraints problem (2.17) and (2.28)-(2.31).

Reformulation of the Einstein equations (2.26)-(2.27)

We note:

$$\rho = 8\pi\left(T_{00} + \tau_{00}\right), \quad P_i = 8\pi\left(\frac{T_{ii} + \tau_{ii}}{g_{ii}}\right) \;\; i = 1, 2, \tag{2.36}$$

ρ is called the energy density, P_1 and P_2 are components of the pressure in the energy-momentum tensor. With these new notations, after performing a good combination of equations (2.26), (2.27) and (2.28), the evolution system then becomes:

$$\frac{\ddot{u}}{u} = \frac{2}{3}\left[\left(\frac{\dot{v}}{v}\right)^2 - \frac{\dot{u}\dot{v}}{uv}\right] - \frac{\rho}{6} + \frac{1}{2}(P_1 - 2P_2) + \frac{\Lambda}{3}, \tag{2.37}$$

$$\frac{\ddot{v}}{v} = \frac{1}{3}\left[\frac{\dot{u}\dot{v}}{uv} - \left(\frac{\dot{v}}{v}\right)^2\right] - \frac{\rho}{6} - \frac{1}{2}P_1 + \frac{\Lambda}{3}. \tag{2.38}$$

In order to reduce the system (2.37)-(2.38) to a system of first order differential equations, we introduce the new variables below:

$$h = \frac{1}{3}\left(\frac{\dot{u}}{u} + 2\frac{\dot{v}}{v}\right),\ r = \frac{u^2v^2}{2u^2+v^2+u^2v^2},\ y = \frac{v^2}{2u^2+v^2},\ z = \frac{1}{h}\frac{\dot{v}}{v} - 1. \tag{2.39}$$

We also define:

$$R_1 = \frac{P_1+2P_2}{\rho},\ R_2 = \frac{P_2-P_1}{\rho},\ \Omega = \frac{\rho}{3h^2},\ \varrho = 2z^2 + \frac{\Omega}{2}\left(1+R_1\right). \tag{2.40}$$

Using these new notations, it follows that:

$$0<y<1,\ 0<r<1,\ u^2 = \frac{r}{y(1-r)},\ v^2 = \frac{2r}{(1-y)(1-r)},\ \Omega = 1 - z^2 - \frac{\Lambda}{3h^2}, \tag{2.41}$$

$$0 \le P_1 + 2P_2 \le \rho,\ 0 \le R_1 \le 1,\ \Omega \ge 0,\ 0 \le \varrho \le 2,\ -1 \le R_2 \le 1. \tag{2.42}$$

Proceeding as in [1], we show that, the Einstein system (2.37)-(2.38) leads to the following system of first order non linear differential equations:

$$\begin{cases} \frac{dh}{dt} = -h^2(1+\varrho) + \frac{\Lambda}{3}, \\ \frac{dy}{dt} = 6y(1-y)zh, \\ \frac{dr}{dt} = 2r(1-r)(1+z-3yz)h, \\ \frac{dz}{dt} = -(2+\varrho)zh + \Omega R_2 h - \frac{\Lambda z}{3h}. \end{cases} \tag{2.43}$$

Variables h, y, r and z are chosen on the subset $\mathbb{B}$ of $\mathbb{R}^4$ defined by

$$\mathbf{B} = \{(h,y,r,z) \in \mathbf{R}^4, \sqrt{\frac{\Lambda}{3}} \le h(t) \le h^0 = h(0), 0<y<1, 0<r<1, -1<z<1\}. \tag{2.44}$$

3 Global existence and uniqueness of the solution

We now consider the equivalent EYMH system given by:

$$\begin{cases} \frac{dh}{dt} = -h^2(1+\varrho) + \frac{\Lambda}{3}, \\ \frac{dy}{dt} = 6y(1-y)zh, \\ \frac{dr}{dt} = 2r(1-r)(1+z-3yz)h, \\ \frac{dz}{dt} = -(2+\varrho)zh + \Omega R_2 h - \frac{\Lambda z}{3h}, \\ \frac{dF^{0i}}{dt} = -3hF^{0i} + [A_j, F^{ij}] + J^i, \\ \frac{dF_{ij}}{dt} = -[A_j, F_{0i}] + [A_i, F_{0j}],\ i \ne j, \\ \frac{dA_i}{dt} = F_{0i}, \\ \frac{d\phi}{dt} = \psi, \\ \frac{d\psi}{dt} = \left(2m^2\phi^2 + 4k_1\phi^4\right)\psi - \left[A_i, \left[A^i, \phi\right]\right], \end{cases} \tag{3.1}$$

with the initial data:

$$\left(h,y,r,z,E,\Phi,p,\psi,\phi\right)(0) = \left(h^0, y^0, r^0, z^0, E^0, \Phi^0, p^0, \psi^0, \phi^0\right). \tag{3.2}$$

We consider the Banach space

$$\mathbb{K} = \mathbb{R}^4 \times \mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^N \times \mathbb{R}^N, \tag{3.3}$$

endowed with the norm

$$\begin{gathered} \|(h, y, r, z, E, \Phi, p, \psi, \phi)\|_{\mathbb{K}} = \\ |h| + |y| + |r| + |z| + \|E\|_{\mathbb{R}^{3N}} + \|\Phi\|_{\mathbb{R}^{3N}} + \|p\|_{\mathbb{R}^{3N}} + \|\psi\|_{\mathbb{R}^N} + \|\phi\|_{\mathbb{R}^N} . \end{gathered} \tag{3.4}$$

3.1 Construction of the iterated sequence

We construct the sequence $(X_n)_{n\in\mathbb{N}}$ of iterations $X_n = (h_n, y_n, r_n, z_n, E_n, \Phi_n, p_n, \psi_n, \phi_n)$, as follows:

- We set $h_0 = h^0$, $y_0 = y^0$, $r_0 = r^0$, $z_0 = z^0$, $E_0 = E^0$, $\Phi_0 = \Phi^0$, $p_0 = p^0$, $\psi_0 = \psi^0$, $\phi_0 = \phi^0$.
- If h_n, y_n, r_n, z_n, E_n, Φ_n, p_n, ψ_n, ϕ_n are known, we define $T_{n,\alpha\beta}$ and $\tau_{n,\alpha\beta}$ by substituting h, y, r, z, E, Φ, p, ψ, ϕ in expressions (2.10), (2.11) by h_n, y_n, r_n, z_n, E_n, Φ_n, p_n, ψ_n, ϕ_n.
- We define $X_{n+1} = (h_{n+1}, y_{n+1}, r_{n+1}, z_{n+1}, E_{n+1}, \Phi_{n+1}, p_{n+1}, \psi_{n+1}, \phi_{n+1})$ as solution of the system of first order nonlinear differential equations obtained by substituting h, y, r, z, E, Φ, p, ψ, ϕ in the r.h.s. of system (3.1) by h_n, y_n, r_n, z_n, E_n, Φ_n, p_n, ψ_n and ϕ_n.

It is important to note that, for every $n \in \mathbb{N}$, the initial data for the corresponding system is the same initial data $\left(h^0, y^0, r^0, z^0, E^0, \Phi^0, p^0, \psi^0, \phi^0\right)$. We thus obtain recursively a sequence defined in a maximal interval $[0, T_n[$, $T_n > 0$.

3.2 Boundedness of the iterated sequence

Proposition 3.1. *There exists $T^1 > 0$ independent of n, such that, the iterated sequence $(X_n) = (h_n, y_n, r_n, z_n, E_n, \Phi_n, p_n, \psi_n, \phi_n)$ is defined and uniformly bounded over $[0, T^1[$.*

Proof. Let $N \in \mathbb{N}$, $N > 1$. Suppose that we have, for $n \leq N-1$, the inequalities:

$$\left\{ \begin{array}{l} |h_n - h_0| \leq K_1, \quad |y_n - y_0| \leq K_2, \quad |r_n - r_0| \leq K_3, \quad |z_n - z_0| \leq K_4 \\ \|E_n - E_0\|_{\mathbb{R}^{3N}} \leq K_5, \quad \|\Phi_n - \Phi_0\|_{\mathbb{R}^{3N}} \leq K_6, \quad \|p_n - p_0\|_{\mathbb{R}^{3N}} \leq K_7, \\ \|\psi_n - \psi_0\|_{\mathbb{R}^N} \leq K_8, \quad \|\phi_n - \phi_0\|_{\mathbb{R}^N} \leq K_9, \end{array} \right. \tag{3.5}$$

where $K_i > 0$, $i = 1, ..., 9$ are given constants.

We are going to prove that one can choose constants K_i such that (3.5) still holds for $n = N$ on $[0, T^1[$ with $T^1 > 0$ and sufficiently small.

Inegrating over $[0, t]$, $t > 0$, equations satisfied by h_N, y_N, r_N, z_N, E_N, Φ_N, p_N, ψ_N, ϕ_N and using (2.44), it comes that:

$$\left\{ \begin{array}{l} |h_N - h_0| \leq L_1 t, \quad |y_N - y_0| \leq L_2 t, \quad |r_N - r_0| \leq L_3 t, \quad |z_N - z_0| \leq L_4 t \\ \|E_N - E_0\|_{\mathbb{R}^{3N}} \leq L_5 t, \quad \|\Phi_N - \Phi_0\|_{\mathbb{R}^{3N}} \leq L_6 t, \quad \|p_N - p_0\|_{\mathbb{R}^{3N}} \leq L_7 t, \\ \|\psi_N - \psi_0\|_{\mathbb{R}^N} \leq L_8 t, \quad \|\phi_N - \phi_0\|_{\mathbb{R}^N} \leq L_9 t, \end{array} \right. \tag{3.6}$$

where L_i, $i = 1, ..., 9$ are constants depending only on K_i. Since

$$\forall i = 1, ..., 9, \quad L_i t \longrightarrow 0 \quad when \quad t \longrightarrow 0,$$

we can choose $T^1 > 0$, sufficiently small such that for all $t \leq T^1$,

$$L_i T^1 \leq K_i, \quad i = 1, ..., 9. \tag{3.7}$$

Then, by (3.7) X_N also satisfies (3.5) on $[0, T^1[$. Hence the iterated sequence (X_n) is uniformly bounded over $[0, T^1[$. $\square$

3.3 Local existence and uniqueness of the solution

Theorem 3.1. *The initial value problem (3.1)-(3.2) for the EYMH system has a unique local solution.*

Proof. We prove that, the iterated sequence (X_n) converges uniformly on each bounded interval $[0, \eta] \subset [0, T^1]$, $\eta > 0$. For this propose, we show that (X_n) is a Cauchy sequence in the Banach space $\mathbb{K}$ defined in (3.3).

Let $\varepsilon > 0$, we are looking for $N_\varepsilon \in \mathbb{N}^*$ such that, for all $n \geqslant N_\varepsilon$ and $m \in \mathbb{N}^*$ we have $\|X_{n+m} - X_n\|_{\mathbb{K}} \leq \varepsilon$.
Taking into account the fact that the sequence (X_n) is uniformly bounded, we deduce from the evolution system that, there exists a constant $\mathcal{K} > 0$ depending only on K_i such that $\forall t \in [0, T^1]$, we have

$$\|X_{n+1} - X_n\|_{\mathbb{K}} \leq \mathcal{K} \int_0^t \|X_n - X_{n-1}\|_{\mathbb{K}} \, d\tau \leq \mathcal{K} t \, \|X_n - X_{n-1}\|_{\mathbb{K}} \, . \tag{3.8}$$

By an immediate induction on n, the relation (3.8) gives, the existence of a strictly positive constant $\mathcal{K}_n$ depending only on K_i such that:

$$\|X_{n+1} - X_n\|_{\mathbb{K}} \leq \mathcal{K}_n t^n \, \|X_1 - X_0\|_{\mathbb{K}} \, . \tag{3.9}$$

Now, we choose a strictly positive real $T^2 \leq T^1$ and sufficiently small such that:

$$\forall t \in [0, T^2], \quad t^n \longrightarrow 0, \; when \; n \longrightarrow +\infty. \tag{3.10}$$

In the sequel, t is chosen such that (3.10) is satisfied. Returning to (3.9), $\forall n \in \mathbb{N}$ and $m \in \mathbb{N}^*$, we have:

$$\begin{aligned}
\|X_{n+m} - X_n\|_{\mathbb{K}} &\leq \|X_{n+m} - X_{n+m-1}\|_{\mathbb{K}} + \|X_{n+m-1} - X_{n+m-2}\|_{\mathbb{K}} + ... + \|X_{n+1} - X_n\|_{\mathbb{K}} \, , \\
&\leq \left(t^{n+m-1} \mathcal{K}_{n+m-1} + t^{n+m-2} \mathcal{K}_{n+m-2} + ... + t^n \mathcal{K}_n \right) \|X_1 - X_0\|_{\mathbb{K}} \, , \\
&\leq \left(\sum_{i=0}^{m-1} \mathcal{K}_{n+i} \right) \left(\frac{1 - t^m}{1 - t} \right) t^n \, \|X_1 - X_0\|_{\mathbb{K}} \, , \\
&\leq \left(\sum_{i=0}^{m-1} \mathcal{K}_{n+i} \right) t^n \, \|X_1 - X_0\|_{\mathbb{K}} \, .
\end{aligned} \tag{3.11}$$

Using (3.10) once more, $\left(\sum_{i=0}^{m-1} \mathcal{K}_{n+i} \right) t^n \, \|X_1 - X_0\|_{\mathbb{K}} \longrightarrow 0$, when $n \longrightarrow +\infty$, so there exists $N_0 \in \mathbb{N}^*$ such that $\forall n \geqslant N_0$, $\left(\sum_{i=0}^{m-1} \mathcal{K}_{n+i} \right) t^n \, \|X_1 - X_0\|_{\mathbb{K}} \leq \varepsilon$.

Taking $N_\varepsilon = N_0$, we conclude from (3.11) that (X_n) is a Cauchy sequence in the Banach space $\mathbb{K}$. Which then shows that each of sequences h_n, y_n, r_n, z_n, E_n, Φ_n, p_n, ψ_n and ϕ_n

converges uniformly on each bounded interval $[0, \eta]$, $0 < \eta \leq T^2$ and that their respective limits denoted: h, y, r, z, E, Φ, p, ψ and ϕ are continuous function of $t \in [0, T^2]$.

From the iterated equations, it appears immediately that there exists a constant $\mathcal{C} > 0$ such that:

$$\left\| \frac{dX_{n+1}}{dt} - \frac{dX_n}{dt} \right\|_{\mathbb{K}} \leq \mathcal{C} \left\| X_{n+1} - X_n \right\|_{\mathbb{K}}. \tag{3.12}$$

The convergence of sequence (X_n) then implies by (3.12), the convergence of the sequence $\left(\frac{dX_n}{dt}\right)$ and hence, that each of sequences $\dot{h}_n$, $\dot{y}_n$, $\dot{r}_n$, $\dot{z}_n$, $\dot{E}_n$, $\dot{\Phi}_n$, $\dot{p}_n$, $\dot{\psi}_n$ and $\dot{\phi}_n$ converges uniformly on each bounded interval $[0, \eta]$, $0 < \eta \leq T^2$. Consequently the limit functions h, y, r, z, E, Φ, p, ψ and ϕ are of C^1-class and $X = (h, y, r, z, E, \Phi, p, \psi, \phi)$ is a local solution of the EYMH system.

Uniqueness of the solution: Let X_1 and X_2 be two solutions of the Cauchy problem (3.1)-(3.2), with the same initial data. Using the evolution system, it comes that:

$$\|X_1 - X_2\|_{\mathbb{K}} \leq \mathcal{K} \int_0^t \|X_1 - X_2\|_{\mathbb{K}} \, d\tau, \tag{3.13}$$

which gives by Gronwall Lemma $\|X_1 - X_2\|_{\mathbb{K}} = 0$, hence $X_1 = X_2$. $\square$

Corollary 3.2. *Let $0 < T < +\infty$, for any solution (u, v) of the Einstein equations coupled to the Yang-Mills-Higgs system:*

i) Maps $t \in [0, T] \longmapsto v(t)$ and $t \in [0, T] \longmapsto uv^2(t)$ are increasing.

2i) Maps $t \in [0, T] \longmapsto \frac{\dot{u}}{u}$ and $t \in [0, T] \longmapsto \frac{\dot{v}}{v}$ are bounded.

Proof. i) By (2.39), we have: $\frac{\dot{v}}{v} = h(1 + z)$. Since $v > 0$, $h > 0$ and $z \in]-1, 1[$, we deduce that $\dot{v} > 0$. Furthermore,

$$\frac{d(uv^2)}{dt} = uv^2 \left(\frac{\dot{u}}{u} + 2\frac{\dot{v}}{v} \right) = 3uv^2 h \geqslant 0.$$

2i) By (2.39) once more, we have $\frac{\dot{u}}{u} = h(1 - 2z)$ and $\frac{\dot{v}}{v} = h(1 + z)$, which shows, thanks to (2.44), that functions $\frac{\dot{u}}{u}$ and $\frac{\dot{v}}{v}$ are bounded on $[0, T]$. $\square$

3.4 Global existence of the solution

Let us sketch out the method we adopt. Denote $[0, T^*)$, $T^* > 0$, the maximal existence interval of the solution of the Cauchy problem (3.1)-(3.2). Assume by contradiction that $T^* < +\infty$ (otherwise $T^* = +\infty$ and there is nothing to do). Then we will prove using a continuity type argument that the solution is uniformely bounded on $[0, T^*)$ by a constant depending only on the initial data. It will then follow by the continuation critrium that, this solution can be extended to a larger time interval $[0, \hat{T})$ thus contradicting the maximality of $\hat{T}$. This will imply that $T^* = +\infty$ and the solution is global. Before doing this, let us give some useful estimates on the obtained local solution.

Lemma 3.1. *Let $0 < T < +\infty$, be given. For all $t \in [0, T]$, the solution (h, y, r, z) of the Einstein equations coupled to the Yang-Mills-Higgs system satisfies:*

$$\frac{1}{M_0} \leq h(t) \leq h^0, \; \frac{1}{M_0} \leq y(t) \leq \frac{1}{1 + 2^{-2} u_0^2 v_0^4 M_0^{-6}}, \; \frac{1}{M_0} \leq r(t) \leq \frac{1}{1 + M_0^{-2}}, \; -1 \leq z(t) \leq 1,$$

where

$$M_0 = \left(\frac{1}{h^0} + \frac{1}{y^0} + \frac{1}{1-y^0} + \frac{1}{r^0} + \frac{1}{1-r^0} \right) e^{10h^0 T}$$

Proof. See [1]. □

Theorem 3.3. *The initial value problem (3.1)-(3.2) for the EYMH system has a unique global solution defined all over the interval* $[0, +\infty[$.

Proof. We use the standard theory on the first order differential system. We prove that, each solution of the evolution system (3.1)-(3.2) is uniformly bounded over every bounded interval $[0, T^*[$, where $T^* < +\infty$.

According to Lemma 3.1, h, y, r and z are bounded.

Following the $(3+1)$ formulation of the Einstein equations (see [15]), the Hamiltonian constraint (2.28) can be written:

$$\frac{2}{3}h^2 - 2\Lambda = 16\pi \left(T_{00} + \tau_{00} \right) - R. \tag{3.14}$$

We recall that in (3.14), $R = g^{ij} R_{ij} \leq 0$ (see [11]). We then deduce from (3.14) whose the left hand side is bounded and using $-R \geqslant 0$, $\tau_{00} \geqslant 0$, $T_{00} \geqslant 0$, that τ_{00} and T_{00} are bounded. From (2.10)-(2.11), one has:

$$0 \leq \frac{1}{2} g_{ij} F^{0i} \cdot F^{0j} \leq \tau_{00}, \quad 0 \leq (\psi)^2 \leq T_{00}. \tag{3.15}$$

Invoking (2.1), the first estimate in (3.15) becomes $0 \leq \frac{1}{2} g_{ii} F^{0i} \cdot F^{0i} \leq \tau_{00}$, and since $g_{ii} F^{0i} \cdot F^{0i}$ is bounded, we conclude that $\|E\|_{\mathbb{R}^{3N}}$ is also bounded. The second estimate in (3.15) shows that $\|\psi\|_{\mathbb{R}^N}$ is bounded. To complete the proof of Theorem 3.3, we show that Φ, p and ϕ are bounded by taking into account the boundedness of h, y, r, z, E, ψ and integrating the folowing equations over $[0, t]$, $t \leq T^* < +\infty$:

$$\frac{dA_i}{dt} = F_{0i}, \quad \frac{dF_{ij}}{dt} = -[A_j, F_{0i}] + [A_i, F_{0j}], \quad \frac{d\phi}{dt} = \psi.$$

□

4 Asymptotic behaviour of the solution

We consider the global solution over $[0, +\infty[$ and we study the asymptotic behaviour of the different elements at late times.

We note by: $\Sigma = (\Sigma_{ij})$ the second fundamental form defined by $\Sigma_{ij} = -\frac{1}{2} \frac{dg_{ij}}{dt}$ and $tr\,(\Sigma) = \Sigma_{11} + \Sigma_{22} + \Sigma_{33}$ his trace; $L = (L_{ij})$ the traceless tensor associated to $\Sigma = (\Sigma_{ij})$ and defined by $L^{ij} = \Sigma_{ij} + g_{ij} h$, which gives by direct computation

$$\Sigma_{ij} \Sigma^{ij} = L_{ij} L^{ij} + 3h^2. \tag{4.1}$$

Proposition 4.1. *The Hubble variable h satisfies:*

$$\frac{dh}{dt} = -\frac{1}{3} \left[R + (tr(\Sigma))^2 + 4\pi g^{ij} T_{ij} - 12\pi T_{00} - 8\pi \tau_{00} - 3\Lambda \right], \tag{4.2}$$

$$\frac{dh}{dt} = -\frac{1}{3}\left[\Sigma_{ij}\Sigma^{ij} - \Lambda + 4\pi g^{ij}T_{ij} + 4\pi\left(T_{00} + \tau_{00}\right) + 8\pi\tau_{00}\right], \tag{4.3}$$

$$\frac{dh}{dt} = \frac{1}{3}\left(-3h^2 + \Lambda - L_{ij}L^{ij} - 4\pi g^{ij}T_{ij} - 4\pi T_{00} - 8\pi\tau_{00}\right), \tag{4.4}$$

$$-6h^2 + 2\Lambda = -L_{ij}L^{ij} - 16\pi\left(T_{00} + \tau_{00}\right) + R, \tag{4.5}$$

$$\frac{dh}{dt} \leq \frac{1}{3}\left(-3h^2 + \Lambda\right), \quad \sqrt{\frac{\Lambda}{3}} \leq h(t) \leq h^0. \tag{4.6}$$

Proof. a) We have $h = -\frac{1}{3}g^{ij}\Sigma_{ij}$, which leads to:

$$\begin{aligned}
\frac{dh}{dt} &= -\frac{1}{3}\left[\Sigma_{11}\frac{dg^{11}}{dt} + 2\Sigma_{22}\frac{dg^{22}}{dt} + g^{11}\frac{d\Sigma_{11}}{dt} + 2g^{22}\frac{d\Sigma_{22}}{dt}\right],\\
&= -\frac{1}{3}\left[\left(\frac{\dot u}{u}\right)^2\frac{1}{u} + 2\left(\frac{\dot v}{v}\right)^2\frac{1}{v} - \left(\left(\frac{\dot u}{u}\right)^2 + \frac{\ddot u}{u} + 2\left(\frac{\dot v}{v}\right)^2 + 2\frac{\ddot v}{v}\right)\right],\\
&= -\frac{1}{3}\left[2\left(\frac{\ddot u}{u} + 2\frac{\ddot v}{v} + 2\frac{\dot u}{u}\frac{\dot v}{v} + \left(\frac{\dot v}{v}\right)^2\right) + \frac{1}{9}\left(\frac{\dot u}{u} + 2\frac{\dot v}{v}\right)^2 + \frac{P_1 + P_2}{2} - \frac{3}{4}\rho\right] + 3\Lambda,\\
&= -\frac{1}{3}\left[R + (tr(\Sigma))^2 + \frac{T_{11}}{u^2} + 2\frac{T_{22}}{v^2} + \frac{\tau_{11}}{u^2} + \frac{\tau_{22}}{v^2}\right] - \frac{1}{3}\left[-12\pi\left(T_{00} + \tau_{00}\right) - 3\Lambda\right],
\end{aligned}$$

which gives (4.2), taking into account $g^{ij}\tau_{ij} = \tau_{00}$.
b) Using the $(3+1)$ formulation, the Hamiltonian constraint (2.28) can still be written

$$R + (tr(\Sigma))^2 = \Sigma_{ij}\Sigma^{ij} + 2\Lambda + 16\pi\left(T_{00} + \tau_{00}\right).$$

To establish the relation (4.3), just insert in the above relation in the relation (4.2).
c) Using the traceless tensor $L = (L_{ij})$ associated to $\Sigma = (\Sigma_{ij})$, we introduce the relation (4.1) in (4.3) for establish (4.4).
d) Equaling (4.2) and (4.4), we get the relation (4.5).
e) Since $L_{ij}L^{ij} > 0$, $\tau_{00} > 0$, $g^{ij}T_{ij} > 0$ and $T_{00} > 0$, we deduce from (4.4) and (4.5) that:

$$\frac{dh}{dt} \leq \frac{1}{3}\left(-3h^2 + \Lambda\right), \quad -3h^2 + \Lambda \leq 0. \tag{4.7}$$

(4.7) imply that $\frac{dh}{dt} \leq 0$, so $h^2 \geqslant \frac{\Lambda}{3}$. Since $h \geqslant 0$, we then conclude that $\sqrt{\frac{\Lambda}{3}} \leq h(t) \leq h^0$. □

Theorem 4.1. *At late times, we have:*

$$h = \beta + O\left(e^{-2\beta t}\right), \tag{4.8}$$

$$g_{ij} = \left(C_{ij} + O\left(e^{-\beta t}\right)\right)e^{2\beta t}, \tag{4.9}$$

$$g^{ij} = \left(C^{ij} + O\left(e^{-\beta t}\right)\right)e^{-2\beta t}, \tag{4.10}$$

$$L_{ij}L^{ij} = O\left(e^{-2\beta t}\right), \tag{4.11}$$

$$\tau_{00} = O\left(e^{-2\beta t}\right), \tag{4.12}$$

$$T_{00} = O\left(e^{-2\beta t}\right), \tag{4.13}$$

$$|R| = O\left(e^{-2\beta t}\right), \tag{4.14}$$

$$F^{0i} \cdot F_{0i} = O\left(e^{-2\beta t}\right), \tag{4.15}$$

$$F_{ij} \cdot F^{ij} = O\left(e^{-2\beta t}\right), \tag{4.16}$$

$$A_i \cdot A^i = O\left(e^{-\beta t}\right), \tag{4.17}$$

$$(\psi)^2 = O\left(e^{-3\beta t}\right), \tag{4.18}$$

$$(\phi)^2 = O\left(e^{-3\beta t}\right), \tag{4.19}$$

$$L_{ij} = O\left(e^{\beta t}\right), \tag{4.20}$$

$$L^{ij} = O\left(e^{-3\beta t}\right), \tag{4.21}$$

where, C_{ij} and C^{ij} are positive constants independent of t and $\beta = \sqrt{\frac{\Lambda}{3}}$.

Proof. From inequality (4.6), we write:

$$\frac{dh}{dt} \leq \left(h - \sqrt{\frac{\Lambda}{3}}\right)\left(-h - \sqrt{\frac{\Lambda}{3}}\right), \tag{4.22}$$

and $-h \leq -\sqrt{\frac{\Lambda}{3}}$, (4.22) gives:

$$\frac{d\left(h - \sqrt{\frac{\Lambda}{3}}\right)}{dt} + \frac{2}{3}\sqrt{3\Lambda}\left(h - \sqrt{\frac{\Lambda}{3}}\right) \leq 0, \tag{4.23}$$

multiplying (4.23) by $e^{2\sqrt{\frac{\Lambda}{3}}t}$ and integrating over $[0,t]$, $t \geq 0$, we obtain:

$$e^{2\sqrt{\frac{\Lambda}{3}}t}\left(h - \sqrt{\frac{\Lambda}{3}}\right) \leq h(0) - \sqrt{\frac{\Lambda}{3}}.$$

Using (4.6) once more, we have $h \geq \sqrt{\frac{\Lambda}{3}}$, which implies:

$$0 \leq h - \sqrt{\frac{\Lambda}{3}} \leq \left(h(0) - \sqrt{\frac{\Lambda}{3}}\right)e^{-2\sqrt{\frac{\Lambda}{3}}}t.$$

Finally $h = \sqrt{\frac{\Lambda}{3}} + O\left(e^{-2\sqrt{\frac{\Lambda}{3}}t}\right)$, this proves (4.8) with $\beta = \sqrt{\frac{\Lambda}{3}}$.

Proof of relations (4.9) and (4.10).
In Cartesian coordinates, the metric of De-Sitter can be written as:

$$\hat{g}_d = -dt^2 + \lambda^2 ch^2\left(\frac{t}{\lambda}\right)\left[dx_1^2 + dx_2^2 + dx_3^2\right). \tag{4.24}$$

(4.9) shows that the metric $\tilde{g}$ defined by (2.1) can be written in the form:

$$\tilde{g} = -dt^2 + e^{2\beta t}\left(C_{11} + O\left(e^{-2\beta t}\right)\right)dx_1^2 + e^{2\beta t}\left(C_{22} + O\left(e^{-2\beta t}\right)\right)\left[dx_2^2 + dx_3^2\right], \tag{4.25}$$

where C_{11} and C_{22} are two positive constantes. By direct computations we have:

$$ch^2\left(\frac{t}{\lambda}\right) = \frac{1}{4}\left(e^{\frac{t}{\lambda}} + e^{-\frac{t}{\lambda}}\right)^2 = e^{2\frac{t}{\lambda}}\left(\frac{1}{4} + \frac{e^{-2\frac{t}{\lambda}}}{2} + \frac{e^{-4\frac{t}{\lambda}}}{4}\right),$$

which yields for large times t: $ch^2\left(\frac{t}{\lambda}\right) = e^{2\frac{t}{\lambda}}\left(\frac{1}{4} + O\left(e^{-\frac{t}{\lambda}}\right)\right)$. So, for large times t, the metric of De-Sitter (4.24) takes the form:

$$\hat{g}_d = -dt^2 + e^{2\frac{t}{\lambda}}\left(\frac{1}{4}\lambda^2 + O\left(e^{-\frac{t}{\lambda}}\right)\right)dx_1^2 + e^{2\frac{t}{\lambda}}\left(\frac{1}{4}\lambda^2 + O\left(e^{-\frac{t}{\lambda}}\right)\right)\left[dx_2^2 + dx_3^2\right]. \tag{4.26}$$

Hence, our model (4.25) approach asymptotically the De-Sitter model if we take $C_{11} = C_{22} = \frac{1}{4}\lambda^2$ and $\beta = \frac{1}{\lambda}$.

Finally, we have $g_{ij} = \left(C_{ij} + O\left(e^{-\beta t}\right)\right)e^{2\beta t}$ and the relation (4.10) is a direct consequence of the relation (4.9), since $g^{ij} = (g_{ij})^{-1}$.

Proof of relations (4.11)-(4.14). Noting that: $T_{00} > 0$, $\tau_{00} > 0$, $L_{ij}L^{ij} > 0$, and $R \leq 0$, we obtain from equality (4.5) the following inequalities:

$$L_{ij}L^{ij} \leq 6h^2 - 2\Lambda, \quad \tau_{00} \leq 6h^2 - 2\Lambda, \quad T_{00} \leq 6h^2 - 2\Lambda, \quad R \leq 6h^2 - 2\Lambda.$$

Now, using the relation (4.8) one has:

$$L_{ij}L^{ij} = O\left(e^{-2\beta t}\right), \quad \tau_{00} = O\left(e^{-2\beta t}\right), \quad T_{00} = O\left(e^{-2\beta t}\right), \quad |R| = O\left(e^{-2\beta t}\right).$$

Proof of relaions (4.15)-(4.17). From the formula (2.11), it comes that:

$$0 \leq \frac{1}{2}g_{ij}F^{0i} \cdot F^{0j} \leq \tau_{00} \quad \text{and} \quad 0 \leq \frac{1}{4}F_{ij} \cdot F_{ij} \leq \tau_{00},$$

thus (4.15) and (4.16) are consequences of (4.12).
Applying the temporal gauge condition $A_0 = 0$ to (2.5), we have: $F^a_{ij} = C^a_{bc}A^b_i A^c_j$, which yields to (4.17) using (4.16) and (4.10).
(4.18) comes immediately from (4.13), and to obtain (4.19), integrate equation $\frac{d\phi}{dt} = \psi$ and use (4.18). For the proof of relations (4.20) and (4.21), see [13]. □

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