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Variational approach to impulsive Neumann problems with variable exponents and two parameters

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Abstract. Based on the variational methods and critical-point theory, we are concerned with the existence results for a second-order impulsive boundary value problem involving an ordinary differential equation with $p(x)$-Laplacian operator, and Neumann conditions.

Keywords. Multiple solutions, impulsive boundary value problems, variable exponent spaces, critical point theory, variational methods.

1 Introduction

In this paper, we consider the following boundary value problem

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x,u) + \mu g(x,u) & x \neq x_j, x \in (a,b), \\ \Delta(|u'(x_j)|^{p(x)-2}u'(x_j)) = I(u(x_j)) & j = 1,2,3,...,l, \\ u'(a) = u'(b) = 0, \end{cases} \tag{1.1}$$

where $p \in \mathbb{C}([a,b],\mathbb{R})$ with $1 < p^- := \min_{x\in[a,b]} p(x) \leq p^+ := \max_{x\in[a,b]} p(x)$, $\lambda > 0$ and $\mu \geq 0$ are real numbers, $a, b \in \mathbb{R}$ with $a < b$, $\alpha \in L^\infty([a,b])$ with $\alpha_- = essinf_{x\in[a,b]}\alpha(x) \geq 0$ and $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $a = x_0 < x_1 < x_2 < ... < x_l < x_{l+1} = b$. $I_j : \mathbb{R} \times \mathbb{R}$, $j = 1,2,...,l$ are continuous and

$$\Delta(|u'(x_j)|^{p(x)-2}u'(x_j)) = |u'(x_j^+)|^{p(x)-2}u'(x_j^+) - |u'(x_j^-)|^{p(x)-2}u'(x_j^-),$$

where $u'(x_j^\pm) = \lim_{x\to x_j^\pm} u'(x)$.

The $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. In recent years, the investigation of differential equations and variational problems with variable exponent has become a new and interesting topic. It arises from the nonlinear elasticity theory, the theory of electro-rheological fluids, etc (see [38]). Problems with variable exponent also have extensive applications in various research fields, such as the image-processing model (see, e.g., [19, 27]), stationary thermorheological viscous flows (see [4]), and the mathematical description of the processes of filtration of ideal barotropic gases through porous media (see [5]). The study of various mathematical problems with

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variable exponent has received considerable attention in recent years. For the background and results, we refer the reader to some recent contributions such as [24, 14, 15, 16, 22, 36, 26, 20, 29] and the references therein. For instance, D'Agui in [20] by using variational methods, established the existence of an unbounded sequence of weak solutions for following problem

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x,u(x)) \quad in \ \ (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\mu g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \mu h(u(1)), \end{cases} \tag{1.2}$$

where all the data are as in the problem (1.1). In [29] based on the variational methods and critical-point theory the existence of at least one solution for the problem

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x,u(x)) \quad in \ \ (0,1), \\ |u'(0)|^{p(0)-2}u'(0) = -\lambda g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = \lambda h(u(1)), \end{cases} \tag{1.3}$$

was established.

On the other hand, many dynamical systems describing models in applied sciences have an impulsive dynamical behaviour due to abrupt changes at certain instants during the evolution process. The rigorous mathematical description of these phenomena leads to impulsive differential equations; they characterize various processes of the real world described by models that are subject to sudden changes in their states. Essentially, impulsive differential equations correspond to a smooth evolution that may change instantaneously or even abruptly, as happens in various applications that describe mechanical or natural phenomena. These changes correspond to impulses in the smooth system, such as for example in the model of a mechanical clock. Impulsive differential equations also study models in physics, population dynamics, ecology, industrial robotics, biotechnology, economics, optimal control, chaos theory. Associated with this development, a theory of impulsive differential equations has been given extensive attention.
Recently, many researchers pay their attention to impulsive differential equations by variational method and critical point theory, and we refer the readers to the classical monographs [33, 2, 39]. Meanwhile, some people begin to study $p(x)$-Laplacian differential equations with impulsive effects. However, the existence of multiple solutions for $p(x)$-Laplacian problems with impulsive effects whose right-hand side nonlinear term is depending on two control parameters λ and μ has attracted less attention.

In the present paper, motivated by the above facts, we generalize the results obtained in [20] and [29]. The following result is a consequence of Theorem (2.3).

Theorem 1.1. *Assume that there exist two constants θ and $\overline{\eta}$ with*

$$p^+ m^{p^-} k_2 \eta^{p^+} > \theta^{p^-},$$

such that

(A_5) $\int_a^b F(x,\overline{\eta})dx > 0$;

(A_6) $\int_a^b F(x,\overline{\eta})dx > \int_a^b \sup_{|t|<\theta} \left(F(x,t) + \frac{\mu}{\lambda} G(x,t)\right) dx$;

(A_7) $\limsup\limits_{n\to+\infty} \frac{F(x,t)}{|t|^{p(x)}} \leq 0$;

(A_8) $\limsup\limits_{n\to+\infty} \frac{G(x,t)}{|t|^{p(x)}} < +\infty$.

Let $\lambda > \lambda_3$*, where*

$$\lambda_3 := \frac{1}{p^+ m^{p^-}} \frac{p^+ m^{p^-} k_2 \eta^{p^+} - \theta^{p^-}}{\int_a^b F(x, \overline{\eta}) dx - \int_a^b \sup_{|t|<\theta} \left(F(x,t) + \frac{\mu}{\lambda} G(x,t) \right) dx},$$

and $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ *is an* L^1*-Carathéodory function whose potential* $G(x,t) := \int_0^t g(x,\xi) d\xi$ *for all* $(x,t) \in [a,b] \times (0, +\infty)$*, is non-negative. Then for every* $\mu \in [0, \mu_3]$*, where*

$$\mu_3 := \frac{\theta^{p^-} - p^+ m^{p^-} k_2 \eta^{p^+} + \lambda p^+ m^{p^-} \int_a^b F(x, \overline{\eta}) dx - \lambda p^+ m^{p^-} \int_a^b sup_{|t|<\theta} F(x,t) dx}{p^+ m^{p^-} \int_a^b sup_{|t|<\theta} G(x,t) dx},$$

the problem (1.1) *admits at least one nontrivial weak solution* $u_3 \in W^{1,p(x)}([a,b])$*.*

The paper consists of four sections. Section 2 contains some background facts concerning the generalized Lebesgue-Sobolev spaces. The main results and their proofs are given in Section 3. Finally, Section 4 is devoted to some concrete applications.

2 Preliminaries

Our main tools are the following theorems.

Theorem 2.1. *[37, Theorem 2.5] Let* X *be a real Banach space,* $\Phi, \Psi : X \to \mathbb{R}$ *be two Gâteaux differentiable functionals such that* Φ *is sequentially weakly lower semicontinuous, strongly continuous and coercive, and* Ψ *is sequentially weakly upper semicontinuous, For every* $r > \inf_X \Phi$*, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\left(\sup_{v \in \Phi^{-1}((-\infty,r))} \Psi(v) \right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every* $r > \inf_X \Phi$ *and every* $\lambda \in (0, 1/\varphi(r))$*, the restriction of the functional*

$$I_\lambda = \Phi - \lambda \Psi,$$

to $\Phi^{-1}((-\infty, r))$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X*.*

(b) *If* $\gamma < +\infty$*, then for each* $\lambda \in (0, 1/\gamma)$*, the following alternative holds: either*

(1) I_λ *possesses a global minimum, or*

(2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that* $\lim_{n \to +\infty} \Phi(u_n) = +\infty$*.*

(c) *If* $\delta < +\infty$*, then for each* $\lambda \in (0, 1/\delta)$*, the following alternative holds: either*

(1) *there is a global minimum of* Φ *which is a local minimum of* I_λ*, or*

(2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, that converges weakly to a global minimum of Φ.*

For a given non-empty set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(r_1, r_2)} \frac{\sup_{u \in \Phi^{-1}(r_1, r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho_2(r_1, r_2) = \sup_{v \in \Phi^{-1}(r_1, r_2)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{\Phi(v) - r_1},$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$ and

$$\rho(r) = \sup_{v \in \Phi^{-1}(r, \infty)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u)}{\Phi(v) - r},$$

for all $r \in \mathbb{R}$.

Theorem 2.2. *[9, Theorem 5.1] Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in (\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)})$ there is $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ $\forall u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.3. *[9, Theorem 5.3] Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that*

$$\rho(r) > 0,$$

and for each $\lambda > \frac{1}{\rho(r)}$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then for each $\lambda \in (\frac{1}{\rho(r)}, +\infty)$ there is $u_{0,\lambda} \in \Phi^{-1}(r, +\infty)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ $\forall u \in \Phi^{-1}(r, +\infty)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

The variational exponent Lebesgue spaces are defined as follows

$$L^{p(x)}([a,b]) = \{u : [a,b] \to \mathbb{R} \ \text{ measurable and } \int_a^b |u(x)|^{p(x)} dx < +\infty\}.$$

On $L^{p(x)}([a,b])$, we consider the norm

$$||u||_{L^{p(x)}([a,b])} := \inf\{\beta > 0 : \int_a^b \Big|\frac{u(x)}{\beta}\Big|^{p(x)} dx \le 1\}.$$

Let X be generalized Lebesgue-Sobolev space $W^{1,p(x)}([a,b])$ defined by

$$W^{1,p(x)}([a,b]) := \{u \in L^{p(x)}([a,b]) : u' \in L^{p(x)}([a,b])\},$$

endowed with the norm

$$||u||_{W^{1,p(x)}([a,b])} := ||u||_{L^{p(x)}([a,b])} + ||u^{'}||_{L^{p(x)}([a,b])}. \tag{2.1}$$

It is well known (see[23]) that both $L^{p(x)}([a,b])$ and $W^{1,p(x)}([a,b])$ with the respective norms, are separable, reflexive and uniformly convex Banach space. Moreover, the norm

$$||u||_\alpha := \inf\{\sigma > 0 : \int_a^b \Big(\Big|\frac{u^{'}(x)}{\sigma}\Big|^{p(x)} + \alpha(x)\Big|\frac{u(x)}{\sigma}\Big|^{p(x)}\Big)dx \leq 1\},$$

on $W^{1,p(x)}([a,b])$ is equivalent to that introduced in (2.1). Next, we give an estimate on the embedding constant m of $W^{1,p(x)}([a,b])$ with norm $||.||_\alpha$ in $C([a,b])$.

Proposition 2.1. *[20, Proposition 2.1] For all $u \in W^{1,p(x)}([a,b])$, one has*

$$||u||_{C([a,b])} \leq m||u||_\alpha.$$

Proposition 2.2. *[18, Proposition 2.2] For all $u \in W^{1,p(x)}(\Omega)$,*

(j_1) *If* $||u||_\alpha < 1 \Longrightarrow \frac{1}{p^+}||u||_\alpha^{p^+} < \int_a^b \frac{1}{p(x)}(|u^{'}(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx < \frac{1}{p^-}||u||_\alpha^{p^-}$,

(j_2) *If* $||u||_\alpha > 1 \Longrightarrow \frac{1}{p^+}||u||_\alpha^{p^-} < \int_a^b \frac{1}{p(x)}(|u^{'}(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx < \frac{1}{p^-}||u||_\alpha^{p^+}$.

Corresponding to f and g we introduce the functions $F : [a,b]\times\mathbb{R} \to \mathbb{R}$ and $G : [a,b]\times\mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)d\xi, \quad \text{for all } (x,t) \in [a,b] \times \mathbb{R},$$

and

$$G(x,t) := \int_0^t g(x,\xi)d\xi, \quad \text{for all } (x,t) \in [a,b] \times \mathbb{R}.$$

We mean by a weak solution of problem (1.1), any function $u \in X$ such that

$$\int_a^b |u'(x)|^{p(x)-2}u'(x)v'(x)dx + \int_a^b \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)dx + \Sigma_{j=1}^l I(u(x_j))v(x_j)$$

$$-\lambda\int_a^b f(x,u(x))v(x)dx - \mu\int_a^b g(x,u(x))v(x)dx = 0,$$

for every $v \in X$.
Assume that there exists positive constant k_1 such that for each $u \in X$

$$0 \leq \Sigma_{j=1}^l \int_0^{u(x_j)} I_j(x)dx \leq k_1 \max_{j\in\{1,2,3,\ldots,l\}} |u(x_j)|^{p(x_j)}. \tag{2.2}$$

3 Main results

We use the following notations

$$k_2 := k_1 + \frac{||\alpha||_{L^1}}{p^-},$$

$$f_\infty := \liminf_{\xi\to+\infty} \frac{\int_a^b sup_{|v|<\xi} F(x, v(x))dx}{\xi^{p^-}},$$

$$g_\infty := \lim_{\xi\to+\infty} \frac{\int_a^b sup_{|v|<\xi} G(x, v(x))dx}{\xi^{p^-}},$$

$$B := \limsup_{\xi\to+\infty} \frac{\int_a^b F(x,\xi)dx}{\xi^{p^+}},$$

$$\lambda_1 := \frac{k_2}{B},$$

and

$$\lambda_2 := \frac{1}{p^+ m^{p^-} f_\infty}.$$

We now formulate our main result as follows.

Theorem 3.1. *Assume that*

(A_1) $\int_a^b F(x, \eta_n)dx \geq 0$ *for every* $x \in [a, b]$;

(A_2) $k_2 p^+ m^{p^-} f_\infty < B$.

Then, for each $\lambda \in \Lambda_1 = (\lambda_1, \lambda_2)$ *and for every* L^1*-Carathéodory function* $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ *whose potential* $G(x, t) := \int_0^t g(x, \xi)d\xi$ *for all* $(x, t) \in [a, b] \times [0, +\infty)$, *is a non-negative function satisfying the condition*

$$g_\infty < +\infty, \tag{3.1}$$

if we put

$$\mu_{g,\lambda} := \frac{1 - \lambda p^+ m^{p^-} f_\infty}{p^+ m^{p^-} g_\infty},$$

where $\mu_{g,\lambda} = +\infty$ *when* $g_\infty = 0$, *the problem* (1.1) *has an unbounded sequence of weak solutions for every* $\mu \in [0, \mu_{g,\lambda})$.

Proof. Our aim is to apply Theorem 2.1(b) to problem (1.1). Let $X = W^{1,p(x)}([a, b])$ be endowed with $||.||_\alpha$. To this end, fix $\overline{\lambda} \in (\lambda_1, \lambda_2)$ and g satisfying our assumptions. Since $\overline{\lambda} < \lambda_2$, we have

$$\mu_{g,\overline{\lambda}} := \frac{1 - \overline{\lambda} p^+ m^{p^-} f_\infty}{p^+ m^{p^-} g_\infty} > 0.$$

Now fix $\overline{\mu} \in (0, \mu_{g,\overline{\lambda}})$ and define the functionals $\Phi, \Psi : X \to \mathbb{R}$ as follows

$$\Phi(u) := \int_a^b \frac{1}{p(x)}(|u'(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx + \Sigma_{j=1}^l \int_0^{u(x_j)} I_j(x)dx, \tag{3.2}$$

$$\Psi(u) := \int_a^b \left(F(x, u(x)) + \frac{\overline{\mu}}{\overline{\lambda}} G(x, u(x)) \right) dx, \tag{3.3}$$

and put

$$I_{\overline{\lambda}}(u) := \Phi(u) + \overline{\lambda}\Psi(u),$$

for each $u \in X$.

Note that the weak solutions of (1.1) are exactly the critical points of $I_{\overline{\lambda}}$. Then the functionals Φ, Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_a^b |u'(x)|^{p(x)-2}u'(x)v'(x)dx + \int_a^b \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)dx + \Sigma_{j=1}^l I(u(x_j))v(x_j),$$

for any $v \in X$. Furthermore, $\Phi' : X \to X^*$ admits a continuous inverse. On the other hand, the fact that X is embedded into $C([a,b])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_a^b f(x,u(x))v(x)dx + \frac{\overline{\mu}}{\overline{\lambda}}\int_a^b g(x,u(x))v(x)dx.$$

Furthermore, $\lim_{||u||\to+\infty} \Phi(u) = +\infty$ for all $u \in X$ and so Φ is coercive.
First of all, we will show that $\overline{\lambda} < \frac{1}{\gamma}$. Hence, let $\{c_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} c_n = +\infty$. Put $r_n := \frac{c_n^{p^-}}{p^+m^{p^-}}$ for all $n \in \mathbb{N}$. Then, for all $u \in X$ with $\Phi(u) \le r_n$ taking Propositions 2.1 and 2.2 into account, one has

$$\begin{aligned} \Phi^{-1}(-\infty, r_n) &= \{u \in X; \Phi(u) < r_n\} \\ &= \{u \in X; \Phi(u) < \frac{c_n^{p^-}}{p^+m^{p^-}}\} \\ &= \{u \in X; |u| < c_n\}. \end{aligned}$$

Then, for all $n \in \mathbb{N}$,

$$\begin{aligned} \varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r_n))} \Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\ &\le \frac{\sup_{v\in\Phi^{-1}((-\infty,r_n))} \Psi(v)}{r_n} \\ &= p^+m^{p^-}\Big(\frac{\int_a^b sup_{|v|<c_n} F(x,v(x))dx}{c_n^{p^-}} + \frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_a^b sup_{|v|<c_n} G(x,v(x))dx}{c_n^{p^-}}\Big). \end{aligned}$$

Moreover, from the assumption (A_2) and (3.1)

$$\begin{aligned} \gamma &= \liminf_{r\to+\infty} \varphi(r) \le \liminf_{n\to+\infty} \varphi(r_n) \\ &\le p^+m^{p^-}\left(f_\infty + \frac{\overline{\mu}}{\overline{\lambda}} g_\infty\right) < +\infty. \end{aligned}$$

The assumption $\overline{\mu} \in (0, \mu_{g,\overline{\lambda}})$ immediately yields $\gamma < \frac{1}{\overline{\lambda}}$. Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}} < \frac{B}{k_2},$$

there exist a sequence $\{\eta_n\}$ and a positive constant τ such that $\lim_{n\to+\infty} \eta_n = +\infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{\int_a^b F(x,\eta_n)dx}{k_2\eta_n^{p^+}}. \tag{3.4}$$

Put $w_n(x) = \eta_n$, for all $x \in [a,b]$. Clearly $w_n(x) \in X$ for each $n \in \mathbb{N}$. Hence, from (2.2) we have

$$\begin{aligned}
\Phi(w_n) &= \int_a^b \frac{1}{p(x)}(|w_n^{'}(x)|^{p(x)} + \alpha(x)|w_n(x)|^{p(x)})dx + \Sigma_{j=1}^{l} \int_0^{w_n(x_j)} I_j(x)dx, \\
&\leq \int_a^b \frac{\alpha(x)}{p(x)}\eta_n^{p(x)}dx + k_1\eta_n^{p(x)} \\
&\leq \frac{\eta_n^{p^+}}{p^-}\int_a^b \alpha(x)dx + k_1\eta_n^{p^+} \\
&= \frac{\eta_n^{p^+}}{p^-}||\alpha||_{L^1} + k_1\eta_n^{p^+} \\
&= k_2\eta_n^{p^+}.
\end{aligned}$$

From (A_1) and since G is nonnegative, due to definition of Ψ, we infer

$$\Psi(w_n) \geq \int_a^b F(x,\eta_n)dx, \tag{3.5}$$

so

$$I_{\overline{\lambda}}(w_n) = \Phi(w_n) + \overline{\lambda}\Psi(w_n) \leq k_2\eta_n^{p^+} - \overline{\lambda}\int_a^b F(x,\eta_n)dx < k_2\eta_n^{p^+}(1-\overline{\lambda}\tau), \tag{3.6}$$

for every $n \in \mathbb{N}$ large enough. Since $\overline{\lambda}\tau > 1$ and $\lim_{n\to+\infty}\eta_n = +\infty$, we have

$$\lim_{n\to+\infty} I_{\overline{\lambda}}(w_n) = -\infty.$$

Then, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, by Theorem 2.1(b), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to+\infty}||u_n|| = +\infty$ and the conclusion is achieved. □

For a given non-negative constant θ and a given positive constant η, with $\theta^{p^-} \neq p^+m^{p^-}k_2\eta^{p^+}$, put

$$k_3 := k_1 + \frac{||\alpha||_{L^1}}{p^+},$$

$$a_\eta(\theta) := \frac{\int_a^b sup_{|t|<\theta}\left(F(x,t) + \frac{\mu}{\lambda}G(x,t)\right)dx - \int_a^b F(x,\eta)dx}{\theta^{p^-} - p^+m^{p^-}k_2\eta^{p^+}},$$

$$\mu_1 := \frac{\theta_1^{p^-} - p^+m^{p^-}k_2\eta^{p^+} - \lambda p^+m^{p^-}\int_a^b sup_{|t|<\theta_1}F(x,t)dx + \lambda p^+m^{p^-}\int_a^b F(x,\eta)dx}{p^+m^{p^-}\int_a^b sup_{|t|<\theta_1}G(x,t)dx},$$

and

$$\mu_2 := \frac{\theta_2^{p^-} - p^+m^{p^-}k_2\eta^{p^+} - \lambda p^+m^{p^-}\int_a^b sup_{|t|<\theta_2}F(x,t)dx + \lambda p^+m^{p^-}\int_a^b F(x,\eta)dx}{p^+m^{p^-}\int_a^b sup_{|t|<\theta_2}G(x,t)dx}.$$

Now, we present an application of Theorem 2.2 which we will use to obtain one nontrivial weak solution.

Theorem 3.2. *Assume that there exist a nonnegative constant θ_1 and two constants θ_2 and η with*

$$\theta_1^{p^-} < (p^+ m^{p^-} k_2)\eta^{p^+} < \theta_2^{p^-}, \tag{3.7}$$

such that

(A_3) $\int_a^b F(x,\eta)dx > 0$;

(A_4) $a_\eta(\theta_2) < a_\eta(\theta_1)$.

Moreover, $\lambda \in \frac{1}{p^+ m^{p^-}} \left] \frac{1}{a_\eta(\theta_1)}, \frac{1}{a_\eta(\theta_2)} \right[$ and $g : [a,b]\times\mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function whose potential $G(x,t) := \int_0^t g(x,\xi)d\xi$ for all $(x,t) \in [a,b]\times(0,+\infty)$, is non-negative. Then for every $\mu \in (\mu_1,\mu_2)$, the problem (1.1) admits at least one nontrivial weak solution $u_1 \in W^{1,p(x)}([a,b])$.

Proof. Let $X = W^{1,p(x)}([a,b])$ be endowed with $||.||_\alpha$. We introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) := \int_a^b \frac{1}{p(x)}(|u^{'}(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)})dx + \Sigma_{j=1}^l \int_0^{u(x_j)} I_j(x)dx,$$

and

$$\Psi(u) := \int_a^b \left(F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))\right)dx,$$

and put

$$I_\lambda(u) := \Phi(u) + \lambda\Psi(u).$$

Let us prove that the functionals Φ and Ψ satisfy the conditions. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_a^b f(x,u(x))v(x)dx + \frac{\mu}{\lambda}\int_a^b g(x,u(x))v(x)dx,$$

for every $v \in X$ as well as is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then u_n converges uniformly to u on $[a,b]$ as $n \to \infty$; see [41]. Since f, g are continuous functions in $\mathbb{R}$ for every $x \in [a,b]$, so

$$f(x,u_n) + \frac{\mu}{\lambda}g(x,u_n) \to f(x,u) + \frac{\mu}{\lambda}g(x,u),$$

as $n \to \infty$. Hence $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [41]. Furthermore, $\Phi' : X \to X^*$ admits a continuous inverse. Clearly, the weak solutions of the problem 1.1 are exactly the solutions of the equation $I'_\lambda(u) = 0$

Now, put $r_1 := \frac{\theta_1^{p^-}}{p^+ m^{p^-}}$ and $r_2 := \frac{\theta_2^{p^-}}{p^+ m^{p^-}}$ and $w(x) := \eta$. It is easy to verify that $w \in X$ and $k_3\eta^{p^-} \leq \Phi(w) \leq k_2\eta^{p^+}$. In particular, from (3.7), we conclude

$$r_1 < \Phi(w) < r_2.$$

On the other hand, for all $u \in X$, we have

$$\Phi^{-1}(-\infty, r_2) = \{u \in X; \Phi(u) < r_2\}$$

$$= \{u \in X; |u| < \theta_2\}.$$

From which it follows

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u) &= \sup_{u\in\Phi^{-1}(-\infty,r_2)} \int_a^b \Big(F(x,t) + \frac{\mu}{\lambda}G(x,t)\Big)\,dx \\ &\le \int_a^b \sup_{|u|<\theta_2} \Big(F(x,t) + \frac{\mu}{\lambda}G(x,t)\Big)\,dx.\end{aligned}$$

Arguing as before, we obtain

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u) &= \sup_{u\in\Phi^{-1}(-\infty,r_1)} \int_a^b \Big(F(x,t) + \frac{\mu}{\lambda}G(x,t)\Big)\,dx \\ &\le \int_a^b \sup_{|u|<\theta_1} \Big(F(x,t) + \frac{\mu}{\lambda}G(x,t)\Big)\,dx,\end{aligned}$$

since $0 \le w(x) \le \eta$ for each $x \in [a,b]$, assumption (A_3) ensures that

$$\Psi(w) \ge \int_a^b F(x,\eta)dx.$$

Then, due to $G \ge 0$, we get

$$\int_a^b \sup_{|u|<\theta_2} \Big(F(x,t) + \frac{\mu}{\lambda}G(x,t)\Big)\,dx \ge \int_a^b F(x,\eta)dx,$$

and thus $a_\eta(\theta_2) \ge 0$. At this point, one has

$$\begin{aligned}\beta(r_1,r_2) &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\int_a^b sup_{|t|<\theta_2}\left(F(x,t) + \frac{\mu}{\lambda}G(x,t)\right)dx - \int_a^b F(x,\eta)dx}{\frac{\theta_2^{p^-}}{p^+m^{p^-}} - k_2\eta^{p^+}} \\ &= p^+m^{p^-}\frac{\int_a^b sup_{|t|<\theta_2}\left(F(x,t) + \frac{\mu}{\lambda}G(x,t)\right)dx - \int_a^b F(x,\eta)dx}{\theta_2^{p^-} - p^+m^{p^-}k_2\eta^{p^+}} \\ &= p^+m^{p^-}a_\eta(\theta_2).\end{aligned}$$

Since $a_\eta(\theta_2) \ge 0$, hypothesis (A_4) implies that

$$\int_a^b \sup_{|u|<\theta_1} \Big(F(x,t) + \frac{\mu}{\lambda}G(x,t)\Big)\,dx < \int_a^b F(x,\eta)dx.$$

So, one has

$$\begin{aligned}\rho_2(r_1,r_2) &\ge \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u)}{\Phi(w) - r_1} \\ &\ge \frac{\int_a^b F(x,\eta)dx - \int_a^b \sup_{|u|<\theta_1}\left(F(x,t) + \frac{\mu}{\lambda}G(x,t)\right)dx}{k_2\eta^{p^+} - \frac{\theta_1^{p^-}}{p^+m^{p^-}}}\end{aligned}$$

$$
\begin{aligned}
&= p^+m^{p^-}\frac{\int_a^b F(x,\eta)dx - \int_a^b \sup_{|u|<\theta_1}\left(F(x,t)+\frac{\mu}{\lambda}G(x,t)\right)dx}{p^+m^{p^-}k_2\eta^{p^+}-\theta_1^{p^-}} \\
&= p^+m^{p^-}a_\eta(\theta_1).
\end{aligned}
$$

Hence, from assumption (A_4), one has $\beta(r_1,r_2) < \rho_2(r_1,r_2)$. Therefore, from Theorem (2.2), for each $\lambda \in \frac{1}{p^+m^{p^-}}\left]\frac{1}{a_\eta(\theta_1)}, \frac{1}{a_\eta(\theta_2)}\right[$ the functional I_λ admits at least one critical point u_1 such that

$$r_1 < \Phi(u_1) < r_2.$$

□

Now, we prove Theorem (1.1) in Introduction:

Proof. Let $X = W^{1,p(x)}([a,b])$ be endowed with $||.||_\alpha$. The functional Φ and Ψ defined in the proof of Theorem 3.1 satisfy all regularity assumptions requested in Theorem 2.3. Moreover, by standard computations, from (A_8), we can fix $l > 0$ such that $\limsup_{|t|\to\infty}\frac{\sup_{x\in\Omega}G(x,t)}{t^{p(x)}} < l$. Therefore, there exists a function $h \in L^1([a,b])$ such that

$$G(x,t) \le lt^{p(x)} + h(x), \tag{3.8}$$

for every $x \in [a,b]$ and $t \in \mathbb{R}$.
Now, fix $0 < \epsilon < \frac{1}{p^+C\lambda} - \frac{\mu l}{\lambda}$. From (A_7) there is a function $h_\epsilon \in L^1([a,b])$ such that

$$F(x,t) \le \epsilon t^{p(x)} + h_\epsilon(x), \tag{3.9}$$

for every $x \in [a,b]$ and $t \in \mathbb{R}$.
Taking Proposition 2.2 (j_2) into account, there exist $C > 0$ such that $||u||_{W^{1,p(x)}} \le C||u||_\alpha$. It follows that, for each $u \in X$,

$$
\begin{aligned}
I_\lambda(u) &\ge \frac{1}{p^+}||u||_\alpha^{p^-} - \lambda\epsilon\int_a^b |u(x)|^{p(x)}dx - \mu l\int_a^b |u(x)|^{p(x)}dx - \lambda||h_\epsilon||_{L^1} - \mu||h||_{L^1} \\
&\ge \frac{1}{p^+}||u||_\alpha^{p^-} - \lambda\epsilon||u||_{L^{p(x)}} - \mu l||u||_{L^{p(x)}} - \lambda||h_\epsilon||_{L^1} - \mu||h||_{L^1} \\
&\ge \frac{1}{p^+}(\frac{1}{C})^{p^-}||u||_{W^{1,p(x)}}^{p^-} - \lambda\epsilon||u||_{W^{1,p(x)}} - \mu l||u||_{W^{1,p(x)}} - \lambda||h_\epsilon||_{L^1} - \mu||h||_{L^1} \\
&\ge ||u||_{W^{1,p(x)}}(\frac{1}{p^+}(\frac{1}{C})^{p^-} - \lambda\epsilon - \mu l) - \lambda||h_\epsilon||_{L^1} - \mu||h||_{L^1},
\end{aligned}
$$

and thus

$$\lim_{||u||_{W^{1,p(x)}}\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty,$$

which means the functional I_λ is coercive. To this end, put $r := \frac{\theta^{p^-}}{p^+m^{p^-}}$ and $\overline{w}(x) = \overline{\eta}$.
Owing to (A_5) and (A_6), we obtain that

$$\rho(r) \ge p^+m^{p^-}\frac{\int_a^b F(x,\overline{\eta})dx - \int_a^b \sup_{|t|<\theta}\left(F(x,t)+\frac{\mu}{\lambda}G(x,t)\right)dx}{p^+m^{p^-}k_2\overline{\eta}^{p^+} - \theta^{p^-}}.$$

So, from our assumption it follows that $\rho(r) > 0$. Hence, from Theorem 2.3 for each $\lambda > \lambda_3$, the functional I_λ admits at least one local minimum u_3 such that

$$||u_3|| > \frac{\theta^{p^-}}{p^+ m^{p^-}},$$

and the conclusion is achieved. □

4 Applications

In this section, we point out some consequences and applications of the results previously obtained. First, we present the following consequence of Theorem 3.1 with $\mu = 0$.

Theorem 4.1. *Assume that all the assumptions in the Theorem 3.1 hold. Then, for each $\lambda \in \Lambda_1$ the problem*

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x,u) & x \in (a,b), \\ \Delta(|u'(x_j)|^{p(x)-2}u'(x_j)) = I(u(x_j)) & j = 1,2,3,...,l, \\ u'(a) = u'(b) = 0, \end{cases} \tag{4.1}$$

has an unbounded sequence of weak solutions in $W^{1,p(x)}([a,b])$.

Corollary 4.2. *Assume that the assumption (A_1) in Theorem 3.1 holds. Suppose that*

$$p^+ m^{p^-} k_2 f_\infty < k_2 < B.$$

Then, the problem

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u = f(x,u) & x \in (a,b), \\ \Delta(|u'(x_j)|^{p(x)-2}u'(x_j)) = I(u(x_j)) & j = 1,2,3,...,l, \\ u'(a) = u'(b) = 0, \end{cases} \tag{4.2}$$

has an unbounded sequence of weak solutions in $W^{1,p(x)}([a,b])$.

Remark 1. We notice that instead of Assumption (A_2) in Theorem 3.1 we are allowed to assume the more general condition

(A_9) there exist two sequence $\{\alpha_n\}$ and $\{\beta_n\}$ with

$$p^+ m^{p^-} k_2 \alpha_n^{p^+} < \beta_n^{p^-},$$

for every $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} \beta_n = +\infty$ such that

$$\liminf_{n\to+\infty} \frac{\beta_n^{p^-} - p^+ m^{p^-} k_2 \alpha_n^{p^+}}{\int_a^b sup_{|v|<\beta_n} F(x,v(x))dx - \int_a^b F(x,\alpha_n)dx} < \frac{1}{p^+ m^{p^-} k_2} \limsup_{n\to+\infty} \frac{\alpha_n^{p^+}}{\int_a^b F(x,\alpha_n)dx}.$$

Obviously, Assumption (A_2) follows from Assumption (A_9) by choosing $\alpha_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (A_9), instead of (A_2) and set $r_n = \frac{\beta_n^{p^-}}{p^+ m^{p^-}}$ for all $n \in \mathbb{N}$, by the same reasoning as in Theorem 3.1, we obtain

$$\varphi(r_n) = \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r_n))} \Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)}$$

$$\leq \frac{\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)-\Psi(w_n)}{r_n-\Phi(w_n)}$$
$$\leq p^+m^{p^-}\frac{\int_a^b sup_{|v|<\beta_n}F(x,v(x))dx-\int_a^b F(x,\alpha_n)dx}{\beta_n^{p^-}-p^+k_2m^{p^-}\alpha_n^{p^+}},$$

where $w_n(x)=\eta_n$, for $x\in[a,b]$ with α_n instead of η_n. We then have the same conclusion as in Theorem 3.1 with λ_2 replaced by

$$\lambda_2^{'} := \left(p^+m^{p^-}\liminf_{n\to+\infty}\frac{\int_a^b sup_{|v|<\xi_n}F(x,v(x))dx-\int_a^b F(x,\alpha_n)dx}{\beta_n^{p^-}-p^+m^{p^-}k_2\alpha_n^{p^+}}\right)^{-1}.$$

We want to point out a simple consequence of Theorem 3.1, in which the function f has separated variables.

Corollary 4.3. *Let $f_1\in L^1([a,b])$ and $f_2\in C(\mathbb{R})$ be two functions. Put $\tilde{F}(t)=\int_a^b f_2(\xi)d\xi$ for all $t\in\mathbb{R}$ and assume that*

(A_{10}) *$f_1(x)>0$ for each $x\in[a,b]$ and $f_2(t)\geq 0$ for each $t\in\mathbb{R}$;*

(A_{11}) $\liminf\limits_{n\to+\infty}\frac{\tilde{F}(\xi_n)}{\xi_n^{p^-}}<+\infty;$

(A_{12}) $\limsup\limits_{n\to+\infty}\frac{\tilde{F}(\xi_n)}{\xi_n^{p^+}}=+\infty.$

Then, for every $\lambda\in\left(0,\frac{1}{p^+m^{p^-}\int_a^b f_1(x)dx\liminf\limits_{n\to+\infty}\frac{\tilde{F}(\xi_n)}{\xi_n^{p^-}}}\right)$ the problem

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))'+\alpha(x)|u(x)|^{p(x)-2}u(x)=\lambda f_1(x)f_2(t) & x\in(a,b),t\in\mathbb{R},\\ \Delta(|u'(x_j)|^{p(x)-2}u'(x_j))=I(u(x_j)) & j=1,2,3,...,l,\\ u'(a)=u'(b)=0, \end{cases}\tag{4.3}$$

has an unbounded sequence of weak solution in $W^{1,p(x)}([a,b])$.

Proof. Set $f(x,u)=f_1(x)f_2(u)$ for each $(x,u)\in[a,b]\times\mathbb{R}$. Since

$$F(x,t)=f_1(x)\tilde{F}(t),$$

from (A_{10}), (A_{11}) and (A_{12}) we obtain (A_1) and (A_2), respectively. □

Here, we present a simple consequence of Theorem 3.1 in the case when f does not depend upon x.

Theorem 4.4. *Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous function such that:*

(A_{13}) *$H(t)=\int_0^t h(s)ds\geq 0$ for every $t\in[0,+\infty)$;*

(A_{14}) *Putting $f_\infty^{'}:=\liminf\limits_{t\to+\infty}\frac{\max_{|\xi|\leq t}H(\xi)}{t^{p^-}}$ and $B^{'}:=\limsup\limits_{t\to+\infty}\frac{H(t)}{t^{p^+}}$ one has $p^+m^{p^-}k_2f_\infty^{'}<B^{'}$.*

Then, for each $\lambda \in \left(\frac{k_2}{B'}, \frac{1}{p^+ m^{p^-} f'_\infty}\right)$ *and for every* $q \in \mathbb{C}(\mathbb{R})$ *such that*

$$Q(t) = \int_0^t q(s)ds \geq 0, \tag{4.4}$$

for every $t \in [0, +\infty)$,

$$Q_\infty := \limsup_{t \to +\infty} \frac{\max_{|\xi| \leq t} Q(\xi)}{t^{p^-}} < +\infty,$$

if we put $\mu^* := \frac{1 - \lambda p^+ m^{p^-} f'_\infty}{p^+ m^{p^-} Q_\infty}$, *for every* $\mu \in [0, \mu^*)$ *the problem*

$$\begin{cases} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u = \lambda h(u(x)) + \mu q(u(x)) & x \in (a,b), \\ \Delta(|u'(x_j)|^{p(x)-2}u'(x_j)) = I(u(x_j)) & j = 1,2,3,...,l, \\ u'(a) = u'(b) = 0, \end{cases} \tag{4.5}$$

admits an unbounded sequence of weak solutions.

Proof. Put $f(x,t) = h(t)$ and $g(x,t) = q(t)$ for every $x \in [a,b] \times \mathbb{R}$. obviously (A_{13}) implies (A_1). Moreover, (A_{14}) that is (A_2) holds and conclusion follows directly from Theorem 3.1 upon observing that $G(x,t) = Q(t)$ for every $x \in [a,b] \times \mathbb{R}$. □

Corollary 4.5. *Let* $h : \mathbb{R} \to \mathbb{R}$ *be a continuous and non-negative function such that:*

$$p^+ m^{p^-} k_2 \liminf_{t \to +\infty} \frac{H(t)}{t^{p^-}} < \limsup_{t \to +\infty} \frac{H(t)}{t^{p^+}}.$$

Moreover, $\lambda \in \left(\frac{k_2}{\limsup\limits_{t \to +\infty} \frac{H(t)}{t^{p^+}}}, \frac{1}{p^+ m^{p^-} \liminf\limits_{t \to +\infty} \frac{H(t)}{t^{p^-}}} \right)$, *and for every* $q \in \mathbb{C}(\mathbb{R})$ *one has*

$$tq(t) \geq 0, \tag{4.6}$$

$$\lim_{|t| \to +\infty} \frac{q(t)}{|t|^{p-1}} = 0. \tag{4.7}$$

Then for every $\mu \geq 0$, *problem (4.5) admits an unbounded sequence of weak solutions.*

Proof. It follows from Theorem 4.4 that, in view of the non-negativity of h, (A_{13}) holds and $f'_\infty = \liminf\limits_{t \to +\infty} \frac{H(t)}{t^{p^-}}$, and also (4.6) implies (4.4). Moreover, by (4.7) one has

$$0 \leq \limsup_{t \to +\infty} \frac{\max_{|\xi| \leq t} Q(\xi)}{t^{p^-}} = \limsup_{t \to +\infty} \frac{\{Q(t), Q(-t)\}}{t^{p^-}}.$$

Owing to the L'Hôpital's rule we have

$$\lim_{t \to +\infty} \frac{Q(t)}{t^{p^-}} = \lim_{t \to +\infty} \frac{Q(-t)}{t^{p^-}} = \pm \lim_{t \to +\infty} \frac{q(\pm t)}{|t|^{p^- - 1}} = 0.$$

Hence $Q_\infty = 0$ and our conclusion follows. □

Now, put

$$f_0 := \liminf_{\xi\to 0^+} \frac{\int_a^b sup_{|v|<\xi} F(x, v(x))dx}{\xi^{p^-}},$$

$$B_0 := \limsup_{\xi\to 0^+} \frac{\int_a^b F(x,\xi)dx}{\xi^{p^+}},$$

$$\lambda_4 := \frac{k_2}{B_0},$$

and

$$\lambda_5 := \frac{1}{p^+ m^{p^-} f_0}.$$

Using Theorem 2.1(c) and arguing as in the proof of Theorem 3.1, we can obtain the following result.

Theorem 4.6. *Assume that* (A_1) *holds and*

(A_{15}) $\quad p^+ m^{p^-} k_2 f_0 < B_0.$

Then, for every $\lambda \in (\lambda_4, \lambda_5)$ *and for every* L^1*-Carathéodory function* g*, such that, there exists* $d > 0$ *such that* $G(x,t) \geq 0$ *for every* $(x,t) \in [a,b] \times [0,d]$*,*

$$g_0 := \lim_{t\to 0^+} \frac{\int_a^b sup_{|\xi|\leq t} G(x,\xi)dx}{t^{p^-}} < +\infty, \tag{4.8}$$

if we put

$$\mu'_{g,\lambda} := \frac{1 - \lambda p^+ m^{p^-} f_0}{p^+ m^{p^-} g_0},$$

where $\mu'_{g,\lambda} = +\infty$ *when* $g_0 = 0$*, then for every* $\mu \in [0, \mu'_{g,\lambda})$ *problem* (1.1) *has a sequence of weak solutions, which converges strongly to zero* $\in W^{1,p(x)}([a,b])$*.*

Proof. Let $X = W^{1,p(x)}([a,b])$ be endowed with $||.||_\alpha$. Fix $\overline{\lambda} \in (\lambda_4, \lambda_5)$ and let g be a function that satisfies the condition (4.8). Since $\overline{\lambda} < \lambda_5$, we obtain

$$\mu'_{g,\overline{\lambda}} := \frac{1 - \overline{\lambda} p^+ m^{p^-} f_0}{p^+ m^{p^-} g_0} > 0.$$

Now fix $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$ and set

$$I_{\overline{\lambda}}(u) := \Phi(u) + \overline{\lambda}\Psi(u),$$

for all $(x,t) \in [a,b] \times \mathbb{R}$. We take Φ, Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Theorem 2.1. As first step, we will prove that $\overline{\lambda} < \frac{1}{\delta}$. Then, let $\{c_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} c_n = 0$ and

$$\lim_{n\to+\infty} \frac{\int_a^b sup_{|t|<c_n} F(x,t)dx}{c_n^{p^-}} = f_0.$$

By the fact that $inf_X \Phi = 0$ and the definition of δ, we have $\delta = \liminf_{r\to 0^+} \varphi(r)$.

Putting $r_n := \frac{c_n^{p^-}}{p^+m^{p^-}}$. Then, as in showing (4.8) in the proof of Theorem 3.1, we can prove that $\delta < +\infty$. From $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$, the following inequalities hold

$$\delta \leq p^+m^{p^-}\left(f_0 + \frac{\overline{\mu}}{\overline{\lambda}}g_0\right) < p^+m^{p^-}f_0 + \frac{1-\overline{\lambda}p^+m^{p^-}f_0}{\overline{\lambda}}.$$

Therefore, $\overline{\lambda} < \frac{1}{\delta}$. Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. Since

$$\frac{1}{\overline{\lambda}} < \frac{B_0}{k_2},$$

there exist a sequence $\{\eta_n\}$ of positive number and $\eta > 0$ such that $\lim_{n\to+\infty} \eta_n = 0^+$ and

$$\frac{1}{\overline{\lambda}} < \eta < \frac{1}{k_2}\frac{\int_a^b F(x,\eta_n)dx}{\eta_n^{p^+}},$$

for each $n \in \mathbb{N}$ large enough. Let $w_n = \{\eta_n\}$ be the sequence in X. From (A_1) one has (3.5) holds. Note that $\overline{\lambda}\eta > 1$. Then, as in showing (3.6), we can obtain

$$I_{\overline{\lambda}}(w_n) < k_2\eta_n^{p^+}(1-\overline{\lambda}\eta) \leq 0 = \Phi(0) + \overline{\lambda}\Psi(0),$$

for each $n \in \mathbb{N}$ large enough. Then, we see that zero is not a local minimum of $I_{\overline{\lambda}}$. Thus, together with the fact zero is the only global minimum of Φ, we deduce that the energy functional $I_{\overline{\lambda}}$ has not a local minimum at the unique global minimum of Φ. Therefore, by Theorem 2.1, there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ which converges weakly to zero. In view of the fact that the embedding $X \hookrightarrow C([a,b])$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Remark 2. Under the condition $f_0 = 0$ and $B_0 = +\infty$, Theorem 4.6 ensures that for every $\lambda > 0$ and for each $\mu = [0, \frac{1}{p^+m^{p^-}g_0})$, problem (1.1) admits a sequence of weak solutions which strongly converges to 0 in X. Moreover, if $g = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Remark 3. Applying Theorem 4.6, results similar to Theorem 4.1, Corollaries 4.2 and 4.3, can be obtained. We omit the discussions here.

Theorem 4.7. *Assume that there exist two positive constants θ and η with*

$$(p^+m^{p^-}k_2)\eta^{p^+} < \theta^{p^-},$$

such that assumption (A_3) in Theorem 3.2 holds. Furthermore, suppose that

(A_{16}) $\quad \frac{\int_a^b sup_{|t|<\theta}F(x,t)dx}{\theta^{p^-}} < \frac{\int_a^b F(x,\eta)dx}{p^+m^{p^-}k_2\eta^{p^+}}.$

Then, for each

$$\lambda \in \left] \frac{k_2\eta^{p^+}}{\int_a^b F(x,\eta)dx}, \frac{\theta^{p^-}}{p^+m^{p^-}\int_a^b sup_{|t|<\theta}F(x,t)dx} \right[,$$

problem (4.1) admits at least one nontrivial weak solution $\overline{u}$ such that $|\overline{u}(x)| < \theta$ for all $x \in [a,b]$.

Proof. The conclusion follows from Theorem 3.2, by taking $\theta_1 = 0$, $\theta_2 = \theta$ and $\mu = 0$ Indeed, owing to assumption (A_{16}), one has

$$\begin{aligned} a_\eta(\theta) &= \frac{\int_a^b sup_{|t|<\theta} F(x,t)dx - \int_a^b F(x,\eta)dx}{\theta^{p^-} - p^+ m^{p^-} k_2 \eta^{p^+}}, \\ &< \frac{\left(1 - \frac{p^+ m^{p^-} k_2 \eta^{p^+}}{\theta^{p^-}}\right) \int_a^b sup_{|t|<\theta} F(x,t)dx}{\theta^{p^-} - p^+ m^{p^-} k_2 \eta^{p^+}} \\ &= \frac{1}{\theta^{p^-}} \int_a^b sup_{|t|<\theta} F(x,t)dx. \end{aligned}$$

On the other hand, one has

$$a_\eta(0) = \frac{\int_a^b F(x,\eta)dx}{p^+ m^{p^-} k_2 \eta^{p^+}}.$$

Hence, taking assumption (A_{16}) and Proposition 2.1 into account, Theorem 3.2 ensures the conclusion. □

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