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A novel iterative algorithm for solving variational inequality, finite family of monotone inclusion and fixed point problems

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Abstract. In this paper, we introduce a method for finding common solution of variational inequality, finite family of monotone inclusion and fixed point problems of demicontractive mappings in a real Hilbert space. We prove strong convergence result of proposed method. We also provide a numerical example to show that our method is efficient from the numerical point of view.

Keywords. Iterative method, fixed point problem, variational inequality problem, monotone inclusion problem.

1 Introduction

Monotone inclusion problem (MIP) plays a crucial role in nonlinear analysis and optimization. MIP is the problem of finding a point ζ in a Hilbert space H such that

$$0 \in T\zeta, \tag{1.1}$$

where $T : H \to 2^H$ is a monotone operator. Mathematically, monotone inclusion problem includes image processing problem, variational inequality problem, split feasibility problem, convex minimization problem, equilibrium problem etc. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The first method, namely proximal point method, for solving MIP was proposed by Martinet in 1970. It was defined as

$$\zeta_{n+1} = (I + \lambda_n T)^{-1}\zeta_n. \tag{1.2}$$

Rockafeller [14] in 1976 and Bruck and Reich [15] in 1977, further developed this algorithm. But the evaluation of resolvent operator in proximal point algorithm was difficult in many cases. Consequently, to solve this issue, the operator T is divided into the sum of maximal monotone operator A and monotone operator B. As the resolvent operators $(I + \lambda_n A)^{-1}$ and $(I + \lambda_n B)^{-1}$ is simpler to calculate than the full resolvent $(I + \lambda_n T)^{-1}$. The problem (1.1) is equivalent to the following problem:

$$\text{Find } \zeta \in H \text{ such that } 0 \in (A + B)\zeta. \tag{1.3}$$

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The solution set of the problem (1.3) is given by $(A+B)^{-1}(0)$. The first method for solving problem (1.3) was forward-backward splitting algorithm. Many iterative algorithms have been designed to solve MIP, for instance, Douglas-Rachford splitting method [18], Peaceman-Rachford splitting method [19] and many more. Tseng [20] in 2000, introduced the modified forward-backward algorithm and proved its weak convergence. Gibali and Thong [21] in 2018, obtained a modified version of Tseng's splitting algorithm and proved its strong convergence.

Polyak [22] developed an inertial extrapolation method which is based on heavy ball method to speed up the convergence of iterative algorithms. Later, inertial extrapolation technique was used to solve MIP and numerous authors considerably enhanced it. For instance, Lorenz and Pock [23] in 2015, introduced the following inertial forward-backward algorithm for solving monotone inclusion problem and proved weak convergence in a real Hilbert space.

$$\begin{cases} \varphi_n = \zeta_n + \theta_n(\zeta_n - \zeta_{n-1}), \\ \zeta_{n+1} = (I + \lambda_n A)^{-1}(I - \lambda_n B)\varphi_n. \end{cases}$$

The above algorithm has better rate of convergence than some existing algorithms present in the literature. Thong and Cholamjiak [25] in 2019, introduced modified forward-backward splitting algorithm and proved strong convergence.

On the other hand, variational inequality is used in solving various class of problems like transportation, economics, engineering, optimization, elasticity and control theory [26, 27, 28, 29, 30]. Several numerical techniques have been devised for solving variational inequality problems (VIP). A VIP is to find a point ζ in a convex, closed and nonempty subset Q of Hilbert space H such that

$$\langle P(\zeta), \vartheta - \zeta \rangle \geq 0 \text{ for all } \vartheta \in Q, \tag{1.4}$$

where $P : Q \to H$ is a nonlinear mapping. The set of solutions of VIP is denoted by $VI(Q, P)$. Eslamian and Kamandi [31] in 2020, developed iterative algorithm for finding common solution of fixed point and monotone inclusion problem in Hilbert space. Recently, Olona and Narain [32] in 2022, introduced a method for approximating a common solution of fixed point problem for finite families of multivalued demicontractive mappings and finite families of variational inequality problem in a real Hilbert space.

Inspired and encouraged by the above results, we introduce a method for finding common solution of variational inequality problem, finite families of monotone inclusion and fixed point problems of demicontractive mappings in a real Hilbert space and prove strong convergence of proposed algorithm. Also, we provide a numerical example to show its applicability.

2 Preliminaries

We now give some definitions and lemmas which we will use in proving our main result. Suppose H denotes a real Hilbert space having inner product $\langle .,. \rangle$ and norm $\|.\|$. Assume that Q is a nonempty, closed and convex subset of H. In addition, $Fix(U)$ denotes the collection of all fixed points of mapping U.

Definition 1. [31] "The operator $U : H \to H$ is called

(i) Nonexpansive, if
$\|U\zeta - U\vartheta\| \leq \|\zeta - \vartheta\|$ for all $\zeta, \vartheta \in H$.

(ii) Demicontractive, if $Fix(U) \neq \phi$ there exists $k \in [0,1)$ such that $\|U\zeta - \vartheta\|^2 \leq \|\zeta - \vartheta\|^2 + k\|\zeta - U\zeta\|^2$ for all $\zeta \in H$ and $\vartheta \in Fix(U)$.

(iii) Contractive, if there exists a constant $0 \leq \theta < 1$ such that $\|U\zeta - U\vartheta\| \leq \theta\|\zeta - \vartheta\|$ for all $\zeta, \vartheta \in H$."

Definition 2. [32] " Let Q be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $U : Q \to Q$ is said to be

(i) monotone, if $\langle U\zeta - U\vartheta, \zeta - \vartheta\rangle \geq 0$ for all $\zeta, \vartheta \in H$,

(ii) α-inverse strongly monotone (ism), if there exists $\alpha > 0$ such that $\langle U\zeta - U\vartheta, \zeta - \vartheta\rangle \geq \alpha\|U\zeta - U\vartheta\|^2$ for all $\zeta, \vartheta \in Q$,

(iii) firmly nonexpansive, if $\langle U\zeta - U\vartheta, \zeta - \vartheta\rangle \geq \|U\zeta - U\vartheta\|^2$ for all $\zeta, \vartheta \in Q$,

(iv) Lipschitz, if there exists a constant $K > 0$ such that $\|U\zeta - U\vartheta\| \leq K\|\zeta - \vartheta\|$ for all $\zeta, \vartheta \in Q$."

Definition 3. [32] "If U is a multi-valued mapping, that is, $U : H \to 2^H$, then U is called monotone, if $\langle \zeta - \vartheta, \psi - \upsilon\rangle \geq 0$ for all $\zeta, \vartheta \in H, \psi \in U\zeta, \upsilon \in U\vartheta$ and U is maximal monotone, if the graph $G(U)$ of U defined by $G(U) = \{(\zeta, \vartheta) \in H \times H : \vartheta \in U(\zeta)\}$ is not properly contained in the graph of any other monotone mapping. It is generally known that U is maximal if and only if for $(\zeta, \vartheta) \in H \times H, \langle \zeta - \psi, \vartheta - \upsilon\rangle \geq 0$ for all $(\psi, \vartheta) \in G(U)$ implies $\vartheta \in U(\zeta)$."

Definition 4. [32] "The metric projection P_Q is a map defined on H onto Q which assign to each $\zeta \in H$, the unique point in Q, denoted by $P_Q\zeta$ such that $\|\zeta - P_Q\zeta\| = \inf\{\|\zeta - \vartheta\| : \vartheta \in Q\}$. It is well known that $P_Q\zeta$ is characterized by the inequality $\langle \zeta - P_Q(\zeta), \vartheta - P_Q(\zeta)\rangle \leq 0$, for all $\vartheta \in Q$ and P_Q is a firmly non-expansive mapping."

Definition 5. [33] "The resolvent mapping $J_\gamma^U : H \to H$ associated with the set-value mapping U is defined by $J_\gamma^U(\zeta) = (I + \gamma U)^{-1}(\zeta) \quad \forall \quad \zeta \in H$, for some $\gamma > 0$, where I stands for the identity operator on H."

Definition 6. [31] "Assume that $T : H \to H$ is a nonlinear mapping with $Fix(T) \neq \phi$. Then $I - T$ is said to be demiclosed at zero if for any $\{\chi_n\}$ in H, the following implications holds: $\zeta_n \rightharpoonup \iota$ and $(I - T)\zeta_n \to 0 \implies \iota \in Fix(T)$."

Lemma 2.1. *[34] "Assume that $\{t_n\} \subset [0, \infty)$ is a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n t_n \text{ for all } n \in \mathbb{N},$$

where $\{\gamma_n\} \subset [0,1]$ and $\{t_n\} \subset (-\infty, \infty)$ satisfying following conditions:

1. $\sum_{n=1}^{\infty} \gamma_n = \infty$,
2. $\limsup_{n\to\infty} t_n \leq 0$.

Then $\lim_{n\to\infty} s_n = 0$."

Proposition 2.1. *[35] "Let $L : Q \to H$ be an inverse strongly monotone(ism) mapping. Then, $z \in VI(Q, L) \iff z = P_Q(z - \lambda Lz), \lambda > 0$."*

Proposition 2.2. *[35] "Let L be an ism mapping of Q into H. Let N_Q be the normal cone to Q at $z \in Q$, i.e. $N_Q z = \{w \in H : \langle z - u, w\rangle \geq 0, \text{ for all } u \in Q\}$, and define*

$$Tz = \begin{cases} Lz + N_Q z, & z \in Q \\ \phi, & z \in H \backslash Q. \end{cases}$$

Then, T is maximal monotone and $0 \in Tz$ if and only if $z \in VI(Q, L)$."

Lemma 2.2. *[36, 38] "Assume that H is a real Hilbert space. Let mapping $g : H \to H$ be Lipschitz continuous monotone and mapping $A : H \to 2^H$ be maximal monotone. Then the mapping $K = g + A$ is maximal monotone."*

Lemma 2.3. *[32, 37] "Let H be a real Hilbert space. Then for all $\zeta, \vartheta \in H$ and $\gamma \in (0, 1)$, we have*

(i) $2\langle \zeta, \vartheta\rangle = \|\zeta\|^2 + \|\vartheta\|^2 - \|\zeta - \vartheta\|^2 = \|\zeta + \vartheta\|^2 - \|\zeta\|^2 - \|\vartheta\|^2$.

(ii) $\|\gamma\zeta + (1 - \gamma)\vartheta\|^2 = \gamma\|\zeta\|^2 + (1 - \gamma)\|\vartheta\|^2 - \gamma(1 - \gamma)\|\zeta - \vartheta\|^2$.

(iii) $\|\zeta + \vartheta\|^2 \leq \|\zeta\|^2 + 2\langle \vartheta, \zeta + \vartheta\rangle$."

Lemma 2.4. *[38] "Assume H is a real Hilbert space, $B : H \to H$ is a mapping and $A : H \to 2^H$ is maximal monotone mapping. Define the fixed point set of the mapping U as $Fix(U) = \{\zeta : \zeta = U\zeta\}$ and $U_\mu = (I + \mu A)^{-1}(I - \mu B), \mu > 0$. Then, $Fix(U_\mu) = (A + B)^{-1}(0)$, for all $\mu > 0$."*

Lemma 2.5. *[38] "Assume that the sequences $\{\varphi_n\}$ and $\{\vartheta_n\}$ are created by the following:*

$$\begin{cases} \varphi_n = \zeta_n + \alpha_n(\zeta_n - \zeta_{n-1}); \\ \vartheta_n = (I + \tau A)^{-1}(I - \tau B)\varphi_n. \end{cases}$$

If $\lim_{n\to\infty} \|\varphi_n - \vartheta_n\| = 0$ and $\{\varphi_{n_k}\}$ converges weakly to some $\iota \in H$, then $\iota \in (A + B)^{-1}(0)$."

Lemma 2.6. *[39] "Let H be a real Hilbert space. Let $\{\zeta_i, i = 1, 2, ...m\} \subset H$. For $\{\beta_i\} \subset (0, 1), i = 1, 2, ...m$ such that $\sum_{i=1}^m \beta_i = 1$, the following identity holds:*

$$\left\|\sum_{i=1}^{m} \beta_i\zeta_i\right\|^2 = \sum_{i=1}^{m} \beta_i\|\zeta_i\|^2 - \sum_{j,i=1,j\neq i} \beta_i\beta_j\|\zeta_i - \zeta_j\|^2."$$

Lemma 2.7. *[25, 40] "Let $\{\zeta_n\}$ be a sequence of nonnegative real numbers, such that there exists a subsequence $\{\zeta_{n_j}\}$ of $\{\zeta_n\}$ such that $\zeta_{n_j} < \zeta_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} n_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$: $\zeta_{n_k} \leq \zeta_{n_k+1}$ and $\zeta_k \leq \zeta_{n_k+1}$. In fact, n_k is the largest number n in the set $\{1, 2, ..., k\}$, such that $\zeta_n < \zeta_{n+1}$."*

3 Main Result

In this section, we introduce a method for finding common solution of variational inequality, finite family of monotone inclusion and fixed point problems of demicontractive mappings in a real Hilbert space. Suppose Q is nonempty, convex and closed subset of a real Hilbert space H and

S_i $(i = 1, 2, ...m)$ is finite family of demicontractive mappings having constant $k_i \in (0,1)$ such that $I - S_i$ are demiclosed at origin. Assume that $L : Q \to H$ is σ-inverse strongly monotone mapping, $A_j : H \to 2^H$ are maximal monotone mappings, $B_j : H \to H$ are ξ-cocoercive mappings, g is contractive mapping on H having constant $\theta \in (0,1)$ and constants τ_j such that $0 < \tau_j \le 2\xi$ for all $j = 1, 2, ...N$.

Algorithm 3.1. Consider $\beta_{n,i}, \gamma_n \in (0,1)$ such that $\sum_{i=0}^{m} \beta_{n,i} = 1, \beta_{n,0} \in (k,1)$, $\limsup\limits_{n\to\infty} \beta_{n,i}(\beta_{n,0} - k) \ge 0, \lim_{n\to\infty} \gamma_n = 0, \sum_{n=1}^{\infty} \gamma_n = \infty$, where $k = \sup_{i\ge1}\{k_i\} < 1$. Select $\eta_n \in (\epsilon, 2\sigma - \epsilon)$,where $\epsilon > 0$ and $\alpha_n \subseteq [0,\infty)$ such that $\lim_{n\to\infty} \frac{\alpha_n}{\gamma_n} = 0$. For $\zeta_1 \in H$, calculate $\{\zeta_{n+1}\}$ using the sequences $\{\epsilon_n\}, \{\varphi_n\}, \{\upsilon_n\}, \{\psi_n\}, \{\kappa_n\}$ as follows:

$$\epsilon_n = \begin{cases} min\left\{\frac{\alpha_n}{\|\zeta_n - \zeta_{n-1}\|}, \epsilon\right\}, if \quad \zeta_n \ne \zeta_{n-1}, \\ \epsilon, otherwise \end{cases}$$

$$\begin{cases} \varphi_n = \zeta_n + \epsilon_n(\zeta_n - \zeta_{n-1}); \\ \upsilon_n = \gamma_n g(\varphi_n) + (1-\gamma_n)\varphi_n; \\ \psi_n = \beta_{n,0}\upsilon_n + \sum_{i=1}^{m} \beta_{n,i} S_i \upsilon_n; \\ \kappa_n = P_Q(\psi_n - \eta_n L\psi_n); \\ \zeta_{n+1} = J_{\tau_N}^{A_N}(I - \tau_N B_N) J_{\tau_{N-1}}^{A_{N-1}}(I - \tau_{N-1}B_{N-1}) J_{\tau_{N-2}}^{A_{N-2}}(I - \tau_{N-2}B_{N-2}) ... J_{\tau_1}^{A_1}(I - \tau_1 B_1)\kappa_n. \end{cases} \tag{3.1}$$

Where $J_{\tau_j}^{A_j}$ $(j = 1, 2, ...N)$ represents the resolvent mapping of the mapping A_j and I represents the identity operator on H.

Remark 1. As $\epsilon_n\|\zeta_n - \zeta_{n-1}\| \le \alpha_n$ for all n and from our assumption $\lim_{n\to\infty} \frac{\alpha_n}{\gamma_n} = 0$, we have

$$\lim_{n\to\infty} \frac{\epsilon_n}{\gamma_n}\|\zeta_n - \zeta_{n-1}\| \le \lim_{n\to\infty} \frac{\alpha_n}{\gamma_n} = 0.$$

Hence,

$$\lim_{n\to\infty} \frac{\epsilon_n}{\gamma_n}\|\zeta_n - \zeta_{n-1}\| = 0. \tag{3.2}$$

Theorem 3.2. *Suppose that the solution set* $\Omega = \left\{\bigcap_{i=1}^{m} Fix(S_i)\right\} \cap VI(L,Q) \cap \left\{\bigcap_{j=1}^{N} (A_i + B_i)^{-1}(0)\right\}$ *is nonempty. Then the sequence* $\{\zeta_n\}$ *generated by Algorithm (3.1) converges strongly to an element in* Ω*.*

Proof. Firstly we will prove $\{\zeta_n\}$ is bounded. Suppose $\iota \in \Omega$ and $\phi^N = J_{\tau_N}^{A_N}(I - \tau_N B_N) J_{\tau_{N-1}}^{A_{N-1}}(I - \tau_{N-1}B_{N-1}) J_{\tau_{N-2}}^{A_{N-2}}(I - \tau_{N-2}B_{N-2}) ... J_{\tau_1}^{A_1}(I - \tau_1 B_1)$, where $\phi^0 = I$. As the resolvent mapping is nonexpansive and using equation (3.1), we have

$$\|\zeta_{n+1} - \iota\|^2 = \|J_{\tau_N}^{A_N}(I - \tau_N B_N)\phi^{N-1}\kappa_n - \iota\|^2 \le \|\phi^{N-1}\kappa_n - \iota\|^2 \le \|\phi^{N-2}\kappa_n - \iota\|^2 \le \|\kappa_n - \iota\|^2. \tag{3.3}$$

Since $\iota \in VI(L,Q)$. Also, from equation (3.1) and the fact that projection mapping is nonexpansive , we obtain

$$\|\kappa_n - \iota\|^2 = \|P_Q(\psi_n - \eta_n L\psi_n) - \iota\|^2$$

$$\begin{aligned}
&\leq \|(I-\eta_n L)(\psi_n)-(I-\eta_n L)\iota\|^2 \\
&= \|(\psi_n-\iota)-\eta_n(L\psi_n-L\iota)\|^2 \\
&= \|\psi_n-\iota\|^2+\eta_n^2\|L\psi_n-L\iota\|^2-2\eta_n\langle\psi_n-\iota, L\psi_n-L\iota\rangle \\
&\leq \|\psi_n-\iota\|^2+\eta_n^2\|L\psi_n-L\iota\|^2-2\eta_n\sigma\|L\psi_n-L\iota\|^2 \\
&= \|\psi_n-\iota\|^2-\eta_n(2\sigma-\eta_n)\|L\psi_n-L\iota\|^2. \qquad (3.4)
\end{aligned}$$

Since $\eta_n \in (\epsilon, 2\sigma-\epsilon)$, so we have

$$\|\kappa_n-\iota\|^2 \leq \|\psi_n-\iota\|^2. \qquad (3.5)$$

Using Lemmas (2.3), (2.6) and equation (3.1), we have

$$\begin{aligned}
\|\psi_n-\iota\|^2 &\leq \left\|\beta_{n,0}\upsilon_n+\sum_{i=1}^m \beta_{n,i}S_i\upsilon_n-\iota\right\|^2 \\
&= \left\|\beta_{n,0}(\upsilon_n-\iota)+\sum_{i=1}^m \beta_{n,i}(S_i\upsilon_n-\iota)\right\|^2 \\
&= \beta_{n,0}\|\upsilon_n-\iota\|^2+\sum_{i=1}^m \beta_{n,i}\|S_i\upsilon_n-\iota\|^2-\sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|\upsilon_n-S_i\upsilon_n\|^2 \\
&\leq \beta_{n,0}\|\upsilon_n-\iota\|^2+\sum_{i=1}^m \beta_{n,i}\left[\|\upsilon_n-\iota\|^2+k\|\upsilon_n-S_i\upsilon_n\|^2\right]-\sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|\upsilon_n-S_i\upsilon_n\|^2 \\
&= \|\upsilon_n-\iota\|^2+(k-\beta_{n,0})\sum_{i=1}^m \beta_{n,i}\|\upsilon_n-S_i\upsilon_n\|^2. \qquad (3.6)
\end{aligned}$$

Since $\beta_{n,0} \in (k,1)$. From equation (3.6), we have

$$\|\psi_n-\iota\|^2 \leq \|\upsilon_n-\iota\|^2. \qquad (3.7)$$

Again from equation (3.1), we have

$$\begin{aligned}
\|\varphi_n-\iota\| &= \|\zeta_n+\epsilon_n(\zeta_n-\zeta_{n-1})-\iota\| \\
&\leq \|\zeta_n-\iota\|+\gamma_n\frac{\epsilon_n}{\gamma_n}\|\zeta_n-\zeta_{n-1}\|. \qquad (3.8)
\end{aligned}$$

By Remark (1), we have $\lim_{n\to\infty}\frac{\epsilon_n}{\gamma_n}\|\zeta_n-\zeta_{n-1}\|=0$, therefore there exist a constant $M_1>0$ such that $\frac{\epsilon_n}{\gamma_n}\|\zeta_n-\zeta_{n-1}\|\leq M_1$. Therefore, equation (3.8) implies

$$\|\varphi_n-\iota\| \leq \|\zeta_n-\iota\|+\gamma_n M_1. \qquad (3.9)$$

Using equations (3.3), (3.5), (3.7) and (3.9) we deduce that

$$\begin{aligned}
\|\zeta_{n+1}-\iota\| &\leq \|\upsilon_n-\iota\| \qquad (3.10) \\
&= \|\gamma_n g(\varphi_n)+(1-\gamma_n)\varphi_n-\iota\| \\
&= \|\gamma_n(g(\varphi_n)-\iota)+(1-\gamma_n)(\varphi_n-\iota)\| \\
&= \|\gamma_n(g(\varphi_n)-g(\iota))+\gamma_n(g(\iota)-\iota)+(1-\gamma_n)(\varphi_n-\iota)\| \\
&\leq \gamma_n\|g(\varphi_n)-g(\iota)\|+\gamma_n\|g(\iota)-\iota\|+(1-\gamma_n)\|\varphi_n-\iota\|
\end{aligned}$$

$$\begin{aligned}
&\leq \gamma_n\theta\|\varphi_n - \iota\| + (1-\gamma_n)\|\varphi_n - \iota\| + \gamma_n\|g(\iota) - \iota\| \\
&= [1-\gamma_n(1-\theta)]\|\varphi_n - \iota\| + \gamma_n\|g(\iota)-\iota\| \\
&\leq [1-\gamma_n(1-\theta)][\|\zeta_n - \iota\| + \gamma_n M_1] + \gamma_n(1-\theta)\left(\frac{\|g(\iota)-\iota\|}{1-\theta}\right) \\
&\leq [1-\gamma_n(1-\theta)][\|\zeta_n - \iota\| + M_1] + \gamma_n(1-\theta)\left(\frac{\|g(\iota)-\iota\|}{1-\theta}\right).
\end{aligned}$$

Continuing like this,

$$\|\zeta_{n+1} - \iota\| \leq \max\left\{\|\zeta_0 - \iota\| + M_1, \frac{\|g(\iota)-\iota\|}{1-\theta}\right\}.$$

Thus $\{\zeta_n\}$ is bounded and hence $\{\psi_n\}$, $\{\varphi_n\}$, $\{\upsilon_n\}$ and $\{\kappa_n\}$ are also bounded. Now, using equation (3.1) and Lemma (2.3), we obtain

$$\begin{aligned}
\|\upsilon_n - \iota\|^2 &= \|\gamma_n g(\varphi_n) + (1-\gamma_n)\varphi_n - \iota\|^2 \\
&= \|\gamma_n(g(\varphi_n) - \iota) + (1-\gamma_n)(\varphi_n - \iota)\|^2 \\
&\leq (1-\gamma_n)\|\varphi_n - \iota\|^2 + 2\gamma_n\langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n(g(\varphi_n) - \varphi_n)\rangle. \quad (3.11)
\end{aligned}$$

From equations (3.3), (3.5), (3.6) and (3.11), we obtain

$$\begin{aligned}
\|\zeta_{n+1} - \iota\|^2 &\leq \|\psi_n - \iota\|^2 \\
&\leq \|\upsilon_n - \iota\|^2 + (k - \beta_{n,0})\sum_{i=1}^{m}\beta_{n,i}\|\upsilon_n - S_i\upsilon_n\|^2 \\
&\leq (1-\gamma_n)\|\varphi_n - \iota\|^2 + 2\gamma_n\langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n(g(\varphi_n) - \varphi_n)\rangle \\
&+ (k - \beta_{n,0})\sum_{i=1}^{m}\beta_{n,i}\|\upsilon_n - S_i\upsilon_n\|^2. \quad (3.12)
\end{aligned}$$

Since, $\beta_{n,0} \in (k, 1)$, so equation (3.12) gives

$$\|\zeta_{n+1} - \iota\|^2 \leq (1-\gamma_n)\|\varphi_n - \iota\|^2 + 2\gamma_n\langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n(g(\varphi_n) - \varphi_n)\rangle. \quad (3.13)$$

Now, from equation (3.1), we have

$$\begin{aligned}
\|\varphi_n - \iota\|^2 &= \|\zeta_n + \epsilon_n(\zeta_n - \zeta_{n-1}) - \iota\|^2 \\
&= \|\zeta_n - \iota\|^2 + \epsilon_n^2\|\zeta_n - \zeta_{n-1}\|^2 + 2\epsilon_n\langle \zeta_n - \iota, \zeta_n - \zeta_{n-1}\rangle \\
&\leq \|\zeta_n - \iota\|^2 + \epsilon_n^2\|\zeta_n - \zeta_{n-1}\|^2 + 2\epsilon_n\|\zeta_n - \iota\|\|\zeta_n - \zeta_{n-1}\|. \quad (3.14)
\end{aligned}$$

Now, we prove $\zeta_n \to \iota$. For this, we study two cases.

Case 1: Suppose that there exists a number $N \in \mathbb{N}$ such that $\|\zeta_{n+1} - \iota\|^2 \leq \|\zeta_n - \iota\|^2$ for any $n \geq N$. As sequence $\{\|\zeta_n - \iota\|\}^2$ is bounded and monotonic, this gives $\{\|\zeta_n - \iota\|^2\}$ is convergent. From equations (3.12) and (3.14), we have

$$\begin{aligned}
(\beta_{n,0} - k)\sum_{i=1}^{m}\beta_{n,i}\|\upsilon_n - S_i\upsilon_n\|^2 &\leq (1-\gamma_n)\{\|\zeta_n - \iota\|^2 + \epsilon_n^2\|\zeta_n - \zeta_{n-1}\|^2 + 2\epsilon_n\|\zeta_n - \iota\|\|\zeta_n - \zeta_{n-1}\|\} \\
&+ 2\gamma_n\langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n(g(\varphi_n) - \varphi_n)\rangle - \|\zeta_{n+1} - \iota\|^2
\end{aligned}$$

$$\leq \|\zeta_n - \iota\|^2 + \epsilon_n^2 \|\zeta_n - \zeta_{n-1}\|^2 + 2\epsilon_n \|\zeta_n - \iota\| \|\zeta_n - \zeta_{n-1}\| + 2\gamma_n \langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n (g(\varphi_n) - \varphi_n) \rangle - \|\zeta_{n+1} - \iota\|^2,$$

which implies

$$(\beta_{n,0} - k) \sum_{i=1}^{m} \beta_{n,i} \|\upsilon_n - S_i \upsilon_n\|^2 \leq \|\zeta_n - \iota\|^2 + \frac{\epsilon_n^2}{\gamma_n^2} \gamma_n^2 \|\zeta_n - \zeta_{n-1}\|^2 + \frac{2\epsilon_n}{\gamma_n} \gamma_n \|\zeta_n - \iota\| \|\zeta_n - \zeta_{n-1}\| + 2\gamma_n \langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n (g(\varphi_n) - \varphi_n) \rangle - \|\zeta_{n+1} - \iota\|^2. \tag{3.15}$$

Taking limit $n \to \infty$ in equation (3.15), by using Remark (1) and $\lim_{n \to \infty} \gamma_n = 0$, we obtain

$$\lim_{n \to \infty} \|\upsilon_n - S_i \upsilon_n\| = 0. \tag{3.16}$$

Using equation (3.1), we have

$$\|\upsilon_n - \varphi_n\| = \|\gamma_n g(\varphi_n) + (1 - \gamma_n) \varphi_n - \varphi_n\| = \gamma_n \|g(\varphi_n) - \varphi_n\|. \tag{3.17}$$

Taking limit $n \to \infty$ in equation (3.17) and using $\lim_{n \to \infty} \gamma_n = 0$, we obtain

$$\lim_{n \to \infty} \|\upsilon_n - \varphi_n\| = 0. \tag{3.18}$$

From equation (3.1), we have

$$\|\psi_n - \upsilon_n\| \leq \sum_{i=1}^{m} \beta_{n,i} \|S_i \upsilon_n - \upsilon_n\|. \tag{3.19}$$

Taking limit $n \to \infty$ in above equation and using equation (3.16), we get

$$\lim_{n \to \infty} \|\psi_n - \upsilon_n\| = 0. \tag{3.20}$$

Using triangle inequality, equations (3.18) and (3.20), we have

$$\lim_{n \to \infty} \|\psi_n - \varphi_n\| = 0. \tag{3.21}$$

From equations (3.3), (3.4) and (3.7), we have

$$\begin{aligned} \|\zeta_{n+1} - \iota\|^2 &\leq \|\kappa_n - \iota\|^2 \\ &\leq \|\psi_n - \iota\|^2 - \eta_n (2\sigma - \eta_n) \|L\psi_n - L\iota\|^2 \\ &\leq \|\upsilon_n - \iota\|^2 - \eta_n (2\sigma - \eta_n) \|L\psi_n - L\iota\|^2, \end{aligned} \tag{3.22}$$

which implies

$$\eta_n (2\sigma - \eta_n) \|L\psi_n - L\iota\|^2 \leq \|\upsilon_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2. \tag{3.23}$$

Using equations (3.11) and (3.23), we deduce

$$\eta_n (2\sigma - \eta_n) \|L\psi_n - L\iota\|^2 \leq (1 - \gamma_n) \|\varphi_n - \iota\|^2 + 2\gamma_n \langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n (g(\varphi_n) - \varphi_n) \rangle - \|\zeta_{n+1} - \iota\|^2. \tag{3.24}$$

Consequently, from equation (3.14) and (3.24), we have

$$\eta_n (2\sigma - \eta_n) \|L\psi_n - L\iota\|^2 \leq \|\zeta_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2 + 2\gamma_n \langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n (g(\varphi_n) - \varphi_n) \rangle + \epsilon_n^2 \|\zeta_n - \zeta_{n-1}\|^2 + 2\epsilon_n \|\zeta_n - \iota\| \|\zeta_n - \zeta_{n-1}\|. \tag{3.25}$$

Taking limit $n \to \infty$ in equation (3.25) and using Remark (1) and $\lim_{n \to \infty} \gamma_n = 0$, we have

$$\lim_{n \to \infty} \|L\psi_n - L\iota\| = 0. \tag{3.26}$$

Using equation (3.1) and Lemma (2.3), we obtain

$$\begin{aligned}\|\kappa_n-\iota\|^2 &= \|P_C(\psi_n-\eta_n L\psi_n)-\iota\|^2\\ &\le \langle \kappa_n-\iota,(I-\eta_n L)\psi_n-(I-\eta_n L)\iota\rangle\\ &= \frac{1}{2}\{\|\kappa_n-\iota\|^2+\|(I-\eta_n L)\psi_n-(I-\eta_n L)\iota\|^2-\|\kappa_n-\psi_n+\eta_n(L\psi_n-L\iota)\|^2.\end{aligned}$$

After rearranging the terms, we have

$$\|\kappa_n-\iota\|^2 \le \|\psi_n-\iota\|^2-\|\kappa_n-\psi_n\|^2-\eta_n^2\|L\psi_n-L\iota\|^2-2\eta_n\|\kappa_n-\psi_n\|\|L\psi_n-L\iota\|. \tag{3.27}$$

Now, from equation (3.13) and (3.14), we have

$$\begin{aligned}\|\psi_n-\iota\|^2 &\le (1-\gamma_n)\|\varphi_n-\iota\|^2+2\gamma_n\langle g(\varphi_n)-\iota,\varphi_n-\iota+\gamma_n(g\varphi_n-\varphi_n)\rangle\\ &\le (1-\gamma_n)\{\|\zeta_n-\iota\|^2+\epsilon_n^2\|\zeta_n-\zeta_{n-1}\|^2+2\epsilon_n\|\zeta_n-\iota\|\|\zeta_n-\zeta_{n-1}\|\}\\ &\quad+2\gamma_n\langle g(\varphi_n)-\iota,\varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle.\end{aligned} \tag{3.28}$$

Again, using equations (3.3), (3.27) and (3.28), we deduce

$$\begin{aligned}\|\zeta_{n+1}-\iota\|^2 &\le \|\kappa_n-\iota\|^2\\ &\le (1-\gamma_n)\{\|\zeta_n-\iota\|^2+\epsilon_n^2\|\zeta_n-\zeta_{n-1}\|^2+2\epsilon_n\|\zeta_n-\iota\|\|\zeta_n-\zeta_{n-1}\|\}-\|\kappa_n-\psi_n\|^2\\ &\quad+2\eta_n\|\kappa_n-\psi_n\|\|L\psi_n-L\iota\|\\ &\quad+2\gamma_n\langle g(\varphi_n)-\iota,\varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle-\eta_n^2\|L\psi_n-L\iota\|^2\\ &\le \|\zeta_n-\iota\|^2+\epsilon_n^2\|\zeta_n-\zeta_{n-1}\|^2+2\epsilon_n\|\zeta_n-\iota\|\|\zeta_n-\zeta_{n-1}\|-\|\kappa_n-\psi_n\|^2\\ &\quad+2\eta_n\|\psi_n-\iota\|\|L\psi_n-L\iota\|\\ &\quad+2\gamma_n\langle g(\varphi_n)-\iota,\varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle-\eta_n^2\|L\psi_n-L\iota\|^2,\end{aligned}$$

which implies

$$\begin{aligned}\|\kappa_n-\psi_n\|^2 &\le \|\zeta_n-\iota\|^2-\|\zeta_{n+1}-\iota\|^2+2\eta_n\|\psi_n-\iota\|\|L\psi_n-L\iota\|-\eta_n^2\|L\psi_n-L\iota\|^2\\ &\quad+2\gamma_n\langle g(\varphi_n)-\iota,\varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle\\ &\quad+\epsilon_n^2\|\zeta_n-\zeta_{n-1}\|^2+2\epsilon_n\|\zeta_n-\iota\|\|\zeta_n-\zeta_{n-1}\|.\end{aligned} \tag{3.29}$$

Taking limit $n\to\infty$ in equation (3.29) and using equation (3.26), remark (1) and $\lim_{n\to\infty}\gamma_n=0$, we get

$$\lim_{n\to\infty}\|\kappa_n-\psi_n\|=0. \tag{3.30}$$

From equations (3.21), (3.30) and triangle inequality, we have

$$\lim_{n\to\infty}\|\kappa_n-\varphi_n\|=0. \tag{3.31}$$

Consider

$$\|\zeta_n-\varphi_n\|=\|\epsilon_n(\zeta_n-\zeta_{n-1})\|. \tag{3.32}$$

By using Remark (1) and taking limit $n\to\infty$ in equation (3.32), we get

$$\lim_{n\to\infty}\|\zeta_n-\varphi_n\|=0. \tag{3.33}$$

By using equations (3.31), (3.33) and triangle inequality, we deduce

$$\lim_{n\to\infty}\|\kappa_n-\zeta_n\|=0. \tag{3.34}$$

From equations (3.30) and (3.34), we have

$$\lim_{n\to\infty}\|\psi_n-\zeta_n\|=0. \tag{3.35}$$

Again using equations (3.20), (3.30) and (3.34), we get

$$\lim_{n\to\infty} \|\zeta_n - \upsilon_n\| = 0. \tag{3.36}$$

Now, consider

$$\begin{aligned}\|\zeta_{n+1} - \iota\|^2 &= \|J_{\tau_N}^{A_N}(I - \tau_N B_N) J_{\tau_N}^{A_{N-1}}(I - \tau_{N-1}B_{N-1}) J_{\tau_{N-2}}^{A_{N-2}}(I - \tau_{N-2}B_{N-2})...J_{\tau_1}^{A_1}(I - \tau_1 B_1)\kappa_n - \iota\|^2 \\ &= \|J_{\tau_N}^{A_N}(I - \tau_N B_N)\phi^{N-1}\kappa_n - \iota\|^2.\end{aligned}$$

As $J_{\tau_N}^{A_N}(I - \tau_N B_N)$ is firmly nonexpansive, we have

$$\|\zeta_{n+1} - \iota\|^2 \le \langle \zeta_{n+1} - \iota, \phi^{N-1}\kappa_n - \iota\rangle. \tag{3.37}$$

Using Lemma (2.3) and equation (3.37), we obtain

$$\|\zeta_{n+1} - \iota\|^2 \le \frac{1}{2}\left[\|\zeta_{n+1} - \iota\|^2 + \|\phi^{N-1}\kappa_n - \iota\|^2 - \|\zeta_{n+1} - \phi^{N-1}\kappa_n\|^2\right]. \tag{3.38}$$

From equation (3.38), we have

$$\|\zeta_{n+1} - \iota\|^2 \le \|\phi^{N-1}\kappa_n - \iota\|^2 - \|\zeta_{n+1} - \phi^{N-1}\kappa_n\|^2,$$

which implies

$$\|\zeta_{n+1} - \phi^{N-1}\kappa_n\|^2 \le \|\phi^{N-1}\kappa_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2. \tag{3.39}$$

Taking limit $n \to \infty$ in equation (3.39) and using equations (3.3) and (3.34), we have

$$\begin{aligned}\lim_{n\to\infty} \|\zeta_{n+1} - \phi^{N-1}\kappa_n\|^2 &\le \lim_{n\to\infty}\left(\|\phi^{N-1}\kappa_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2\right) \\ &\le \lim_{n\to\infty}\left(\|\kappa_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2\right) \\ &= \lim_{n\to\infty}(\|\kappa_n - \zeta_n\|^2 + 2\|\kappa_n - \zeta_n\|\|\zeta_n - \iota\| \\ &+ \|\zeta_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2) \\ &\le \lim_{n\to\infty}\left(\|\kappa_n - \zeta_n\|^2 + 2\|\kappa_n - \zeta_n\|\|\zeta_n - \iota\|\right) \\ &+ \lim_{n\to\infty}\left(\|\zeta_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2\right) \\ &\le 0.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|\zeta_{n+1} - \phi^{N-1}\kappa_n\| = 0. \tag{3.40}$$

Using the similar argument as in (3.39) , we get

$$\|\phi^{N-1}\kappa_n - \phi^{N-2}\kappa_n\|^2 \le \left(\|\phi^{N-2}\kappa_n - \iota\|^2 - \|\phi^{N-1}\kappa_n - \iota\|^2\right). \tag{3.41}$$

Taking limit $n \to \infty$ in equation (3.41) and using equations (3.3), (3.34) and (3.40), we have

$$\begin{aligned}\lim_{n\to\infty} \|\phi^{N-1}\kappa_n - \phi^{N-2}\kappa_n\|^2 &\le \lim_{n\to\infty}\left(\|\phi^{N-2}\kappa_n - \iota\|^2 - \|\phi^{N-1}\kappa_n - \iota\|^2\right) \\ &\le \lim_{n\to\infty}\left(\|\kappa_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2\right) \\ &= \lim_{n\to\infty}(\|\kappa_n - \zeta_n\|^2 + 2\|\kappa_n - \zeta_n\|\|\zeta_n - \iota\| \\ &+ \|\zeta_n - \iota\|^2 - \|\zeta_{n+1} - \iota\|^2) \\ &\le \lim_{n\to\infty}\left(\|\kappa_n - \zeta_n\|^2 + 2\|\kappa_n - \zeta_n\|\|\zeta_n - \iota\|\right)\end{aligned}$$

$$+\lim_{n\to\infty}\left(\|\zeta_n-\iota\|^2-\|\zeta_{n+1}-\iota\|^2\right)$$
$$\le 0. \tag{3.42}$$

Hence, $\lim_{n\to\infty}\|\phi^{N-1}\kappa_n-\phi^{N-2}\kappa_n\|=0$.

Continuing like this,

$$\lim_{n\to\infty}\|\phi^j\kappa_n-\phi^{j-1}\kappa_n\|=0 \qquad \text{for all} \quad j=1,2,...N. \tag{3.43}$$

Also, using triangle inequality, we obtain

$$\|\zeta_{n+1}-\kappa_n\|\le\|\phi^N\kappa_n-\phi^{N-1}\kappa_n\|+\|\phi^{N-1}\kappa_n-\phi^{N-2}\kappa_n\|+...+\|\phi^1\kappa_n-\kappa_n\|. \tag{3.44}$$

Taking limit $n\to\infty$ in equation (3.44) and using equation (3.43), we have

$$\lim_{n\to\infty}\|\zeta_{n+1}-\kappa_n\|=0. \tag{3.45}$$

From equations (3.30), (3.35) and (3.45), we obtain

$$\lim_{n\to\infty}\|\zeta_{n+1}-\zeta_n\|=0. \tag{3.46}$$

As the sequence $\{\zeta_n\}$ is bounded, so there exists a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $\{\zeta_{n_k}\}$ converges to q. Using equations (3.35) and (3.36), there exists subsequences $\{\upsilon_{n_k}\}$ of $\{\upsilon_n\}$ and $\{\psi_{n_k}\}$ of $\{\psi_n\}$ that converges weakly to q. Hence, by demiclosedness of S_i at zero and using equation (3.16), we have $q\in Fix(S_i)$ for each $i=1,2,...m$. We next show that $q\in VI(L,Q)$. Let the mapping M' be defined by

$$M'(z^*)=\begin{cases}L(z^*)+N_Q(z^*) & \text{if } z^*\in Q\\ \phi & \text{if } z^*\notin Q\end{cases},$$

where $N_Q(z^*)=\{t'\in H:\langle z^*-p',t'\rangle\ge 0 \text{ for all } p'\in Q\}$ is the normal cone at Q and M' is maximal monotone, which implies $0\in M'(z^*)$ if and only if $z^*\in VI(L,Q)$. Let us assume that $(z^*,t')\in G(M')$, where $G(M')$ denotes graph of mapping M'. So, $t'\in M'(z^*)=L(z^*)+N_Q(z^*)$ which implies $t'-L(z^*)\in N_Q(z^*)$.

Hence, we obtain $\langle z^*-p',t'-L(z^*)\rangle\ge 0$ for all $p'\in Q$. Also, from equation (3.1) and using that $z^*\in Q$, we get $\langle\psi_n-\eta_nL\psi_n-\kappa_n,\kappa_n-z^*\rangle\ge 0$. Hence, $\langle z^*-\kappa_n,\frac{\kappa_n-\psi_n}{\eta_n}+L\psi_n\rangle\ge 0$. As $\langle z^*-p,t'-Lz^*\rangle\ge 0$ for all $p'\in Q,\kappa_n\in Q$ and monotonicity of L, we have

$$\begin{aligned}\langle z^*-\kappa_n,t'\rangle &\ge\langle z^*-\kappa_n,Lz^*\rangle\\ &\ge\langle z^*-\kappa_n,Lz^*\rangle-\langle z^*-\kappa_n,\frac{\kappa_n-\psi_n}{\eta_n}+L\psi_n\rangle\\ &=\langle z^*-\kappa_n,Lz^*-L\kappa_n\rangle-\langle z^*-\kappa_n,\frac{\kappa_n-\psi_n}{\eta_n}\rangle+\langle z^*-\kappa_n,L\kappa_n-L\psi_n\rangle\\ &\ge\langle z^*-\kappa_n,L\kappa_n-L\psi_n\rangle-\langle z^*-\kappa_n,\frac{\kappa_n-\psi_n}{\eta_n}\rangle.\end{aligned}$$

As L is continuous, taking limit $n\to\infty$, we obtain $\langle z^*-q,t'\rangle\ge 0$. Since, M' is maximal monotone, so we obtain $q\in M'^{-1}(0)$ and therefore $0\in M'(q)$ and hence $q\in VI(L,Q)$. Now, we prove $q\in(A_j+B_j)^{-1}(0)$. Let us denote $T_j=(I+\tau_jA_j)^{-1}(I-\tau_jB_j)$ for all $j=1,2,...N$. Putting $j=1$ in equation (3.43), we obtain

$$\lim_{n\to\infty}\|\phi^1\kappa_n-\kappa_n\|=0, \tag{3.47}$$

which implies

$$\lim_{n\to\infty}\|T_1\kappa_n-\kappa_n\|=0. \tag{3.48}$$

As T_1 is nonexpansive, we have

$$\begin{aligned}\|T_1\varphi_n - \varphi_n\| &= \|T_1\varphi_n - T_1\kappa_n + T_1\kappa_n - \varphi_n\| \\ &\leq \|T_1\varphi_n - T_1\kappa_n\| + \|T_1\kappa_n - \varphi_n\| \\ &\leq \|\varphi_n - \kappa_n\| + \|T_1\kappa_n - \varphi_n\| \\ &\leq \|\varphi_n - \kappa_n\| + \|T_1\kappa_n - \kappa_n + \kappa_n - \varphi_n\| \\ &\leq \|\varphi_n - \kappa_n\| + \|T_1\kappa_n - \kappa_n\| + \|\kappa_n - \varphi_n\|. \end{aligned} \tag{3.49}$$

Taking limit $n \to \infty$ in equation (3.49), we get

$$\lim_{n\to\infty} \|T_1\varphi_n - \varphi_n\| = 0. \tag{3.50}$$

From equation (3.50) and Lemmas (2.4) and (2.5), we deduce that

$$q \in (A_1 + B_1)^{-1}(0) = Fix(T_1).$$

Putting $j = 2$ in equation (3.43), we get

$$\lim_{n\to\infty} \|\phi^2\kappa_n - \phi^1\kappa_n\| = 0,$$

which implies

$$\lim_{n\to\infty} \|T_2T_1\kappa_n - T_1\kappa_n\| = 0. \tag{3.51}$$

Consider

$$\begin{aligned}\|T_2\kappa_n - \kappa_n\| &= \|T_2\kappa_n - T_2T_1\kappa_n + T_2T_1\kappa_n - \kappa_n\| \\ &\leq \|T_2\kappa_n - T_2T_1\kappa_n\| + \|T_2T_1\kappa_n - \kappa_n\| \\ &\leq \|\kappa_n - T_1\kappa_n\| + \|T_2T_1\kappa_n - \kappa_n\| \\ &= \|\kappa_n - T_1\kappa_n\| + \|T_2T_1\kappa_n - T_1\kappa_n + T_1\kappa_n - \kappa_n\| \\ &\leq \|\kappa_n - T_1\kappa_n\| + \|T_2T_1\kappa_n - T_1\kappa_n\| + \|T_1\kappa_n - \kappa_n\|. \end{aligned} \tag{3.52}$$

Taking limit $n \to \infty$ in equation (3.52) and using equations (3.48) and (3.51), we have

$$\lim_{n\to\infty} \|T_2\kappa_n - \kappa_n\| = 0. \tag{3.53}$$

Similarly, as equation (3.50), we obtain

$$\lim_{n\to\infty} \|T_2\varphi_n - \varphi_n\| = 0. \tag{3.54}$$

From equation (3.54), Lemmas (2.4) and (2.5), we obtain

$$q \in Fix(T_2) = (A_2 + B_2)^{-1}(0).$$

Continuing like this, we have $q \in Fix(T_j)$ for all $j = 1, 2, ...N$. Hence,

$$q \in \bigcap_{j=1}^{N} Fix(T_j) = \bigcap_{j=1}^{N} (A_j + B_j)^{-1}(0).$$

From equations (3.13) and (3.14), we obtain

$$\begin{aligned}\|\zeta_{n+1} - \iota\|^2 \leq (1 - \gamma_n)\{\|\zeta_n - \iota\|^2 + \epsilon_n^2\|\zeta_n - \zeta_{n-1}\|^2 + 2\epsilon_n\|\zeta_n - \iota\|\|\zeta_n - \zeta_{n-1}\|\} \\ + 2\gamma_n\langle g(\varphi_n) - \iota, \varphi_n - \iota + \gamma_n(g(\varphi_n) - \varphi_n)\rangle. \end{aligned} \tag{3.55}$$

Hence

$$\|\zeta_{n+1}-\iota\|^2 \le (1-\gamma_n)\|\zeta_n-\iota\|^2 + \gamma_n\{2\langle g(\varphi_n)-\iota, \varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle + \frac{\epsilon_n^2\gamma_n}{{\gamma_n}^2}\|\zeta_n-\zeta_{n-1}\|^2 + \frac{2\epsilon_n}{\gamma_n}\|\zeta_n-\iota\|\|\zeta_n-\zeta_{n-1}\|\}, \tag{3.56}$$

which gives

$$r_{n+1} \le (1-\gamma_n)r_n + \gamma_n s_n, \tag{3.57}$$

where $r_n = \|\zeta_n-\iota\|^2$ and
$s_n = 2\langle g(\varphi_n)-\iota, \varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle + \frac{\epsilon_n^2\gamma_n}{{\gamma_n}^2}\|\zeta_n-\zeta_{n-1}\|^2 + \frac{2\epsilon_n}{\gamma_n}\|\zeta_n-\iota\|\|\zeta_n-\zeta_{n-1}\|$.
Now, it remains to prove that $\limsup_{n\to\infty}\langle g(\varphi_n)-\iota, \varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle \le 0$.

Let $\{\varphi_{n_k}\}$ be a subsequence of $\{\varphi_n\}$. Using equation (3.33) , $\{\varphi_{n_k}\}$ weakly converges to q. Consider

$$\begin{aligned}\limsup_{n\to\infty}\langle g(\varphi_n)-\iota, \varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle &\le \limsup_{n\to\infty}\langle g(\varphi_n)-\iota, \varphi_n-\iota\rangle \\ &+ \limsup_{n\to\infty}\langle g(\varphi_n)-\iota, \gamma_n(g(\varphi_n)-\varphi_n)\rangle \\ &\le \limsup_{n\to\infty}\langle g(\varphi_n)-\iota, \varphi_n-\iota\rangle \\ &+ \limsup_{n\to\infty}\gamma_n\langle g(\varphi_n)-\iota, g(\varphi_n)-\varphi_n\rangle \\ &\le \limsup_{n\to\infty}\langle g(\varphi_n)-\iota, \varphi_n-\iota\rangle \\ &\le \langle g(q)-\iota, q-\iota\rangle \le 0.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\langle g(\varphi_n)-\iota, \varphi_n-\iota+\gamma_n(g(\varphi_n)-\varphi_n)\rangle \le 0. \tag{3.58}$$

Using equation (3.58), Remark (1) and Lemma (2.1), $\{r_n\}$ converges to zero as $n\to\infty$. Thus $\zeta_n\to\iota$.

Case 2: Suppose that there exists a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ such that $r_{n_j}\le r_{n_j+1}$ for any $j\in\mathbb{N}$. From Lemma (2.7), we observe that there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} n_k=\infty$ which satisfy following inequalities for all $k\in\mathbb{N}$.

$$r_{n_k}\le r_{n_k+1} \tag{3.59}$$

and

$$r_k\le r_{n_k+1}. \tag{3.60}$$

Using equation (3.3), we obtain

$$\begin{aligned}\|\upsilon_{n_k}-\rho_{n_k}\| &= \|\gamma_{n_k}g(\rho_{n_k})+(1-\gamma_{n_k})\rho_{n_k}-\rho_{n_k}\| \\ &= \gamma_{n_k}\|g(\rho_{n_k})-\rho_{n_k}\|.\end{aligned} \tag{3.61}$$

Taking limit $k\to\infty$ in equation (3.61), we get

$$\lim_{n\to\infty}\|\upsilon_{n_k}-\rho_{n_k}\|=0. \tag{3.62}$$

From equation (3.19), we have

$$\|\psi_{n_k}-\upsilon_{n_k}\| \le \sum_{i=1}^{m}\beta_{n_k,i}\|S_i\upsilon_{n_k}-\upsilon_{n_k}\|. \tag{3.63}$$

Taking limit $k \to \infty$ in above equation , we get

$$\lim_{k\to\infty} \|\psi_{n_k} - \upsilon_{n_k}\| = 0. \tag{3.64}$$

Similarly as in Case 1, we can prove

$$\lim_{k\to\infty} \|\kappa_{n_k} - \psi_{n_k}\| = 0 \text{ and } \lim_{k\to\infty} \|\kappa_{n_k} - \zeta_{n_k}\| = 0. \tag{3.65}$$

Also,

$$\lim_{k\to\infty} \|\zeta_{n_k} - \upsilon_{n_k}\| = 0 \text{ and } \lim_{k\to\infty} \|\zeta_{n_k+1} - \kappa_{n_k}\| = 0. \tag{3.66}$$

Similar to Case 1, $q \in VI(L,Q)$ and $q \in (A_j + B_j)^{-1}(0)$. Hence, $q \in \Omega$. Using the same arguments as in Case 1, we have

$$\limsup_{k\to\infty} \langle g(\rho_{n_k}) - \iota, \rho_{n_k} - \iota + \gamma_{n_k}(g(\rho_{n_k}) - \rho_{n_k})\rangle \leq 0. \tag{3.67}$$

Using equation (3.57), we obtain

$$r_{n_{k+1}} \leq (1-\gamma_{n_k})r_{n_k} + \gamma_{n_k} s_{n_k}. \tag{3.68}$$

Using equation (3.59) in equation (3.68), we get

$$\gamma_{n_k} r_{n_k} \leq \gamma_{n_k} s_{n_k}. \tag{3.69}$$

As $\gamma_{n_k} > 0$, we have $r_{n_k} \leq s_{n_k}$. So, we have

$$\begin{aligned} \|\zeta_{n_k} - \iota\|^2 &\leq 2\langle g(\rho_{n_k}) - \iota, \rho_{n_k} - \iota + \gamma_{n_k}(g(\rho_{n_k}) - \rho_{n_k})\rangle + \frac{\epsilon_{n_k}^2 \gamma_{n_k}}{{\gamma_{n_k}}^2}\|\zeta_{n_k} - \zeta_{n_k-1}\|^2 \\ &+ \frac{2\epsilon_{n_k}}{\gamma_{n_k}}\|\zeta_{n_k} - \iota\|\|\zeta_{n_k} - \zeta_{n_k-1}\|. \end{aligned} \tag{3.70}$$

As $\lim_{k\to\infty} \gamma_{n_k} = 0$, $\lim_{k\to\infty} \frac{\epsilon_{n_k}}{\gamma_{n_k}}\|\zeta_{n_k} - \zeta_{n_{k-1}}\| = 0$ and from equation (3.67), we obtain $r_{n_k} \to 0$ as $k \to \infty$. Also, using equation (3.68), we have $\lim_{k\to\infty} r_{n_k+1} = 0$ and from equation (3.60), we get $r_{n_k+1} \geq r_k$, which gives $\lim_{k\to\infty} r_k = 0$ that is, $\|\zeta_n - \iota\| \to 0$ as $n \to \infty$. Hence $\zeta_n \to \iota$ as $n \to \infty$. □

4 Numerical Example

In this section, we give numerical example and compare the convergence of algorithms [23, 24] with the Algorithm (3.1).

Example 1. Let $H = \mathbb{R}^4$ and $Q = \{\zeta \in \mathbb{R}^4 : \zeta_1 + \zeta_2 - 3\zeta_3 + \zeta_4 \leq 0\}$, where $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$. Suppose $A_j : H \to 2^H$ are maximal monotone mappings defined by $A_j(\zeta) = N_j^* N_j(\zeta)$ for all $\zeta \in H$, $j = 1, 2$, where $N_j : \mathbb{R}^4 \to \mathbb{R}^4$ are created from a normal distribution with unit variance and mean zero and the mappings $B_1, B_2 : H \to H$ are defined by $B_1(\zeta) = \frac{\zeta}{5}$ for all $\zeta \in H$, $B_2(\zeta) = \frac{\zeta}{7}$ for all $\zeta \in H$. Clearly, B_1 and B_2 are 1- inverse strongly monotone mappings. The mappings $S_i : H \to H$ are defined as $S_i(\zeta) = \frac{-3i\zeta}{i+1}$ for all $\zeta \in H, i = 1, 2, ...7$. It can be easily shown that S_i are $\frac{2i-1}{4i+1}$ demicontractive mappings. Further, $L : Q \to H$ is a mapping given by $L(\zeta) = \zeta$ for all $\zeta \in Q$, where L is 1- cocoercive mapping and the mapping $g : H \to H$ is defined as $g(\zeta) = \frac{\zeta}{2}$ for all $\zeta \in H$. Let $\alpha_n = \frac{1}{n^5}, \gamma_n = \frac{1}{n+6}, \epsilon = 0.4, \beta_{n,0} = \frac{57k+3}{67k}, \beta_{n,i} = \frac{10k-3}{469k}$ for all $i = 1, 2, ...7, \tau_1 = 0.5$ and $\tau_2 = 1$ and we choose $\|\zeta_n - \zeta_{n-1}\| \leq 10^{-4}$ as stopping criterion.

We take following cases for various initial values of ζ_0 and ζ_1 and plot the graphs of errors $E_n = \|\zeta_n - \zeta_{n-1}\|$ against number of iterations n.

Case 1. $\zeta_0 = (1, 1, 1, 1), \zeta_1 = (2, 2, 2, 2)$;
Case 2. $\zeta_0 = (10, 10, 10, 10), \zeta_1 = (20, 20, 20, 20)$;
Case 3. $\zeta_0 = (100, 100, 100, 100), \zeta_1 = (200, 200, 200, 200)$.
We also show that Algorithm (3.1) is more effective than Lorenz algorithm [23] and Cholamjiak algorithm [24].

Cases	iteration number	cpu time in seconds
Case 1.	7	0.02158
Case 2.	10	0.02419
Case 3.	38	0.03042

Table 1: Numerical analysis of Algorithm (3.1) for various cases

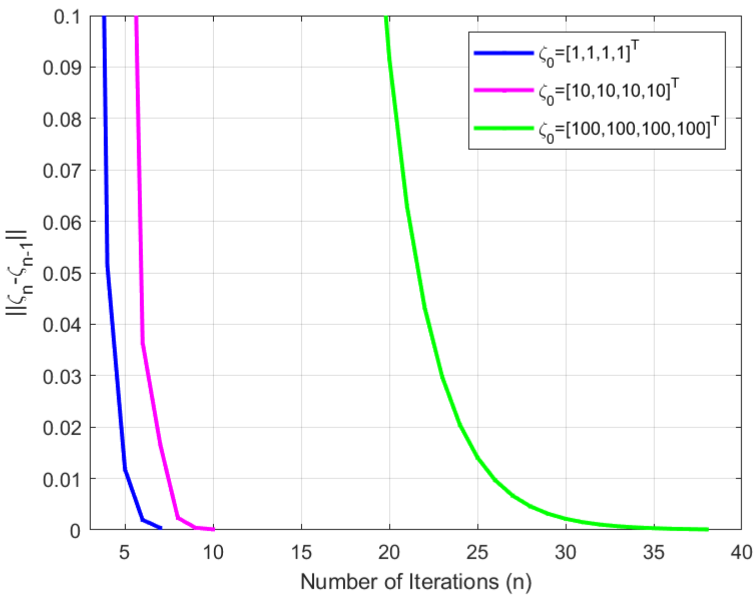

Figure 1: Numerical study of Algorithm (3.1) for various values of ζ_0 and $\zeta 1$

Algorithm	iteration number	cpu time (in seconds)
Algorithm (3.1)	7	0.01726
Lorenz Algorithm	38	0.01849
Cholamjiak Algorithm	431	0.01854

Table 2: Comparison of Algorithm (3.1) with Lorenz Algorithm [23] and Cholamjiak Algorithm [24]

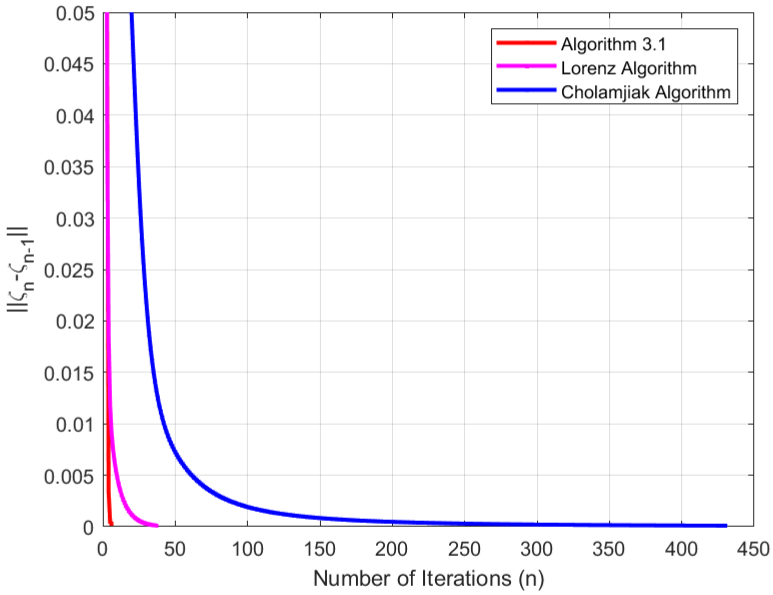

Figure 2: Comparison of Algorithm (3.1) with Lorenz Algorithm [23] and Cholamjiak Algorithm [24]

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Conflict of Interest

The authors declare that there is no conflict of interest.

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