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DEGREE OF APPROXIMATION OF A FUNCTION BELONGING TO WEIGHTED $(L_r, \xi(t))$ CLASS BY $(C,1)(E,q)$ MEANS

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Abstract. In this paper, a new theorem on degree of approximation of a function $f \in W(L_r, \xi(t))$ class by $(C,1)(E,q)$ product summability means of Fourier series has been proved.

1. Introduction

A good amount of work on degree of approximation of functions belonging to Lipα, Lip(α, r), Lip$(\xi(t), r)$ and $W(L_r, \xi(t))$ classes using Cesàro, Nörlund and generalized Nörlund single summability methods has been done by number of researchers like Alexits [1], Sahney and Geol [12], Qureshi and Neha [10], Qureshi [7, 8, 9], Chandra [2], Khan [4], Leindler [5] and Rhoades [11]. But till now nothing seems to have been done in the direction of present work. Therefore, in this paper, we establish a theorem on degree of approximation of function f belonging to weighted i.e. $W(L_r, \xi(t))$-class, $r \geq 1$, by $(E,q)(C,1)$ summability means of Fourier series.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n^{th} partial sum $\{s_n\}$.

The $(C,1)$ transform is defined as the n^{th} partial sum of $(C,1)$ summability and is given by

$$\begin{aligned} t_n &= \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \\ &= \frac{1}{n+1} \sum_{k=0}^{n} s_n \to s \text{ as } n \to \infty \end{aligned}$$

then the series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by $(C,1)$ method.

If

$$(E,q) = E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \to s \text{ as } n \to \infty$$

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then the infinite series $\sum_{n=0}^{\infty} u_n$ with partial sum s_n is said to be summable by (E,q) method to a definite number s (Hardy [3]).

A product of $(C,1)$ transform of (E,q) transform defines $(C,1)(E,q)$ transform and it can be denoted by $C_n^1 E_n^q$.

Thus if

$$C_n^1 E_n^q = \frac{1}{n+1}\sum_{k=0}^{n} E_k^q \to s \text{ as } n \to \infty$$

where E_n^q denotes the (E,q) transform, then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(C,1)(E,q)$ means or summable $(C,1)(E,q)$ to a definite number s.

Let $f(x)$ be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of $f(x)$ is given by

$$f(x) \backsim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

with n^{th} partial sum $s_n(f;x)$.

A function $f \in \mathrm{Lip}\alpha$ if

$$f(x+t) - f(x) = O(|t|^{\alpha}) \qquad \text{for } 0 < \alpha \le 1,$$

$f \in \mathrm{Lip}(\alpha, r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx\right)^{\frac{1}{r}} = O(|t|^{\alpha}), \quad 0 < \alpha \le 1,\ r \ge 1$$

(Definition 5.38 of Mc Fadden [6], 1942).

Given a positive increasing function $\xi(t)$ and an integer $r \ge 1$, $f \in \mathrm{Lip}(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx\right)^{\frac{1}{r}} = O(\xi(t)),$$

and that $f(t) \in W(L_r, \xi(t))$ if

$$\left(\int_0^{2\pi} \left|\{f(x+t) - f(x)\} \sin^{\beta} x\right|^r dx\right)^{\frac{1}{r}} = O(\xi(t)),\ \beta \ge 0.$$

If $\beta = 0$ then $W(L_r, \xi(t))$ coincides with the class $\mathrm{Lip}(\xi(t), r)$ and if $\xi(t) = t^{\alpha}$ then $\mathrm{Lip}(\xi(t), r)$ class coincides with the class $\mathrm{Lip}(\alpha, r)$ and if $r \to \infty$ then $\mathrm{Lip}(\alpha, r)$ class reduces to the class $\mathrm{Lip}\alpha$.

Now we define norm by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r \, dt\right)^{\frac{1}{r}}, \quad r \ge 1 \tag{1.2}$$

and let the degree of approximation $E_n(f)$ be given by

$$E_n(f) = \min \|f - T_n\|_r \quad \text{(Zygmund [14])} \tag{1.3}$$

where $T_n(x)$ is a trigonometric polynomial of degree n.

The following notations are used through out the paper:

$$\begin{aligned} \phi(t) &= f(x+t) + f(x-t) - 2f(x) \\ K_n(t) &= \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(1+q)^k} \sum_{\nu=0}^{k} \left\{ \binom{k}{\nu} q^{k-\nu} \frac{\sin(\nu+\frac{1}{2})t}{\sin(\frac{t}{2})} \right\} \right]. \end{aligned}$$

2. Main theorem

If f is a 2π-periodic function, Lebesgue integrable on $[0, 2\pi]$ and belongs to $W(L_r, \xi(t))$ class, then its degree of approximation is given by

$$\|C_n^1 E_n^q - f\|_r = O\left[(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right] \tag{2.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ be a decreasing sequence,} \tag{2.2}$$

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t\, dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{2.3}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\} \tag{2.4}$$

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1$, conditions (2.3) and (2.4) hold uniformly in x and $C_n^1 E_n^q$ is $(C,1)(E,q)$ means of the Fourier series (1.1).

3. Lemmas

Following lemmas are required for the proof of our theorem:

Lemma 1.

$$|K_n(t)| = O(n+1), \text{ for } 0 \le t \le \frac{1}{n+1}.$$

Proof. For $0 \le t \le \frac{1}{n+1}$, $\sin nt \le n \sin t$

$$|K_n(t)| \le \frac{1}{2\pi(n+1)} \left| \sum_{k=0}^{n} \left[\frac{1}{(1+q)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\nu} \frac{(2\nu+1)\sin(\frac{t}{2})}{\sin(\frac{t}{2})} \right] \right|$$

$$\leq \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}(2k+1)\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right]\right|$$
$$= \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}(2k+1)$$
$$= O(n+1) \qquad \square$$

Lemma 2.

$$|K_n(t)| = O\Big(\frac{1}{t}\Big), \text{ for } \frac{1}{n+1} \leq t \leq \pi.$$

Proof. For $\frac{1}{n+1} \leq t \leq \pi$, by applying Jordan's lemma, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $\sin nt \leq 1$

$$|K_n(t)| \leq \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{1}{\left(\frac{t}{\pi}\right)}\right]\right|$$
$$\leq \frac{1}{2t(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}(1+q)^k\right]$$
$$= \frac{1}{2t(n+1)}\sum_{k=0}^{n}1$$
$$= O\Big(\frac{1}{t}\Big). \qquad \square$$

4. Proof of main theorem

Following Titchmarsh [13] and using Riemann-Lebesgue theorem, $s_n\left(f;x\right)$ of the series (1.1) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

Therefore, using (1.1) the (E,q) transform $\left(E_n^q\right)$ of $s_n\left(f;x\right)$ is given by

$$E_n^q - f(x) = \frac{1}{2\pi(1+q)^k}\int_0^{\pi}\frac{\phi(t)}{\sin\left(\frac{t}{2}\right)}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin\left(k+\frac{1}{2}\right)t\right\}dt.$$

Now denoting $(C,1)\,(E,q)$ transform of $s_n\left(f;x\right)$ as $C_n^1E_n^q$, we write

$$C_n^1E_n^q - f(x) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\int_0^{\pi}\frac{\phi(t)}{\sin(\frac{t}{2})}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{n-k}\sin\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$
$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)\,dt$$
$$= I_1 + I_2 \quad \text{(say)}. \tag{4.1}$$

Now

$$|I_1| \le \int_0^{\frac{1}{n+1}} |\phi(t)||K_n(t)|dt.$$

We have

$$|\phi(x+t)-\phi(x)| \le |f(u+x+t)-f(u+x)|+|f(u-x-t)-f(u-x)|.$$

Hence, by Minkowiski's inequality,

$$\begin{aligned}
&\left[\int_0^{2\pi}\left\{\left|\phi(x+t)-\phi(x)\right\}\sin^\beta x\right|^r dx\right]^{\frac{1}{r}}\\
&\quad\le \left[\int_0^{2\pi}\left|\left\{f(u+x+t)-f(u+x)\right\}\sin^\beta x\right|^r dx\right]^{\frac{1}{r}}\\
&\quad\quad+\left[\int_0^{2\pi}\left|\left\{f(u-x-t)-f(u-x)\right\}\sin^\beta x\right|^r dx\right]^{\frac{1}{r}}\\
&\quad= O\{\xi(t)\}.
\end{aligned}$$

Then $f \in W(L_r, \xi(t)) \Rightarrow \phi \in W(L_r, \xi(t))$.

Using Hölder's inequality and the fact that $\phi(t) \in W(L_r, \xi(t))$, (2.3), Lemma 1 and second mean value theorem for integrals, we have

$$\begin{aligned}
|I_1| &\le \left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\left|\phi(t)\right|\sin^\beta t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)|K_n(t)|}{t\sin^\beta t}\right\}^s dt\right]^{\frac{1}{s}}\\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{(n+1)\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}}\\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_\epsilon^{\frac{1}{n+1}}\frac{dt}{t^{(1+\beta)s}}\right]^{\frac{1}{s}} \text{ for some } 0<\epsilon<\frac{1}{n+1}\\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1. \qquad (4.2)
\end{aligned}$$

Now using Hölder's inequality, $|\sin t| < 1$, $\sin t \ge \left(\frac{2t}{\pi}\right)$, Lemma 2, (2.2), (2.4) and mean value theorem,

$$\begin{aligned}
|I_2| &\le \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\left|\phi(t)\right|\sin^\beta t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)|K_n(t)|}{t^{-\delta}\sin^\beta t}\right\}^s dt\right]^{\frac{1}{s}}\\
&= O\left\{(n+1)^\delta\right\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{\beta+1-\delta}}\right\}^s dt\right]^{\frac{1}{s}}\\
&= O\left\{(n+1)^\delta\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}}\right\}^s\frac{dy}{y^2}\right]^{\frac{1}{s}}
\end{aligned}$$

$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{(n+1)^{s(1+\beta-\delta)-1}-\pi^{s(\delta-1-\beta)+1}}{s(1+\beta-\delta)-1}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1+\beta-\delta)-\frac{1}{s}}\right]$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1. \tag{4.3}$$

Now combining (4.1), (4.2) and (4.3), we get

$$\left|C_n^1E_n^q - f(x)\right| = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}$$

$$\left\|C_n^1E_n^q - f\right\|_r = \left\{\int_0^{2\pi} O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

This completes the proof of the main theorem. □

5. Corollaries

Following corollaries can be derived from our main theorem:

Corollary 1. *If $\beta = 0$ and $\xi(t) = t^{\alpha}$, then the degree of approximation of a function $f \in \mathrm{Lip}(\alpha, r)$, $0 < \alpha \le 1$, is given by*

$$\left\|C_n^1E_n^q - f(x)\right\|_r = O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}.$$

Corollary 2. *If $r \to \infty$ in Corollary 1, then for $0 < \alpha < 1$,*

$$\left\|C_n^1E_n^q - f(x)\right\|_\infty = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}.$$

Corollary 3. *If $\beta = 0$, $\xi(t) = t^{\alpha}$ and $q_n = 1\ \forall n$, then the degree of approximation of a function $f \in \mathrm{Lip}(\alpha, r)$, $0 < \alpha \le 1$, is given by*

$$\left\|C_n^1E_n^1 - f(x)\right\|_r = O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}.$$

Corollary 4. *If $r \to \infty$ in Corollary 3, then for $0 < \alpha < 1$, we have*

$$\left\| C_n^1 E_n^1 - f(x) \right\|_\infty = O\left\{ \frac{1}{(n+1)^\alpha} \right\}.$$

Remark. Independent proofs of Corollaries 1 and 3 can be obtained along the same lines of our theorem.

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