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Various new traveling wave solutions for conformable time-fractional Sasa-Satsuma equation

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Abstract. In this paper, the extended Jacobi elliptic function expansion method is applied to conformable time-fractional Sasa-Satsuma equation. Variety of new traveling wave solutions are constructed. Thanks to the *Mathematica* software package, to interprete the behivor of some particular exact solutions, surfaces and contour plots are ploted.

Keywords. Fractional derivative, Sasa-Satsuma equation, Jacobi elliptic function, traveling wave solution.

1 Introduction

Recently, a wide variety of analytic and approximate solution methods have been developed to construct exact solutions of nonlinear partial differential equations (PDEs) and in particular the Sasa-Satsuma equation, such as extended equation method (ETEM) and generalized Kudryashov method [4], new auxiliary method [15], bilinear forms method [19], the direct integration [22], Darboux transformation method [27, 28, 35], modified simple equation approach [31], trial equation approach [32], Painlevé -Bäcklund transformation [34] and many others [11, 12, 26, 29, 33]. In this work, the conformable time-fractional Sasa-Satsuma equation, which reads

$$i\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2}+u(x,t)\left|u(x,t)\right|^2 \\ +\mathrm{i}\epsilon\left(\frac{\partial^3 u(x,t)}{\partial x^3}+6\left|\psi(x,t)\right|^2\frac{\partial u(x,t)}{\partial x}+3u(x,t)\frac{\partial\left|u(x,t)\right|^2}{\partial x}\right)=0,$$

is investigated using the extended Jacobi elliptic function expansion method [30, 36]. Because of the generalized properties of the Jacobi elliptic functions, the double periodic and other solutions of conformable fractional Sasa-Satsuma equation have been obtained. We point that in the case $\alpha=1$, this model governs the interaction and propagation of the ultrashort pulses in the sub-picosecond or femtosecond regime as well as the propagation of femtosecond pulses in optical fibers [32].

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2 Preliminaries

2.1 The conformable fractional derivative

In literature, to investigate the exact solutions for fractional PDEs, the authors use many forms of fractional derivative, in particular, Riemann Liouville, Jumarie's modified Riemann-Liouville and Caputo derivatives [2, 5, 6, 7, 8, 16, 23].

In 2014, a new definition of fractional derivative is presented by R. Khalil, *et al.* [17]. Now let us give the definition and some properties of conformable fractional derivative [1, 3, 9, 10, 13, 14, 18, 20, 21, 24, 25].

Definition 1. [17] Let $f : [0, +\infty) \to \mathbb{R}$ be a function. α^{th} order "conformable fractional derivative" of f is defined by

$$D_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{2.1}$$

for all $t > 0, \alpha \in (0, 1)$. If f is α-differentiable in some $(0, a), a > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

Definition 2. [17] The conformable fractional integral of a function $f : [0, t] \to \mathbb{R}$ of order $0 < \alpha < 1$ is defined by

$$I^\alpha(f)(t) = \int_0^t s^{\alpha-1} f(s)\, ds. \tag{2.2}$$

This new definitions satisfies the properties which are given in the following theorems.

Theorem 2.1. *[17] Let $\alpha \in (0, 1]$ and f, g α-differentiable at point $t > 0$. then*

1. $D^\alpha(cf + dg) = cD^\alpha(f) + dD^\alpha(g)$ *for all* $c, d \in \mathbb{R}$.
2. $D^\alpha(t^p) = pt^{p-\alpha}, \quad \forall\, p \in \mathbb{R}$.
3. $D^\alpha(\lambda) = 0$ *for all constant function* $f(t) = \lambda$.
4. $D^\alpha(f(t)g(t)) = f(t)D^\alpha g(t) + g(t)D^\alpha f(t)$
5. $D_t^\alpha(\frac{f}{g})(t) = \dfrac{g(t)D^\alpha f(t) - f(t)D^\alpha g(t)}{g(t)^2}$
6. *If , in addition to f is differentiable, then* $D_t^\alpha(f)(t) = t^{1-\alpha} f'(t)$.
7. $D_t^\alpha(I^\alpha f(t)) = f(t)$.
8. $I_t^\alpha(D_t^\alpha f(t)) = f(t) - f(0)$ *on the interval* $[0, t]$.

Theorem 2.2. *[13] Suppose $f : (0, \infty) \to \mathbb{R}$ be a function such that f is differentiable and also α-differentiable. Let g be a function defined in the range of f and also differentiable. Then*

$$D_t^\alpha(fog)(t) = t^{1-\alpha} g'(t) f'(g(t)).$$

Remark 1. The use of the transformation $\tau = \frac{\omega t^\alpha}{\alpha}$, by use of (2) one can obtain $D_t^\alpha \tau = \omega$. Furthermore, by use of (2.2) one can deduce that $D_t^\alpha u = u'_\tau D_t^\alpha \tau = \omega u'_\tau$.

2.2 Summary of the extended Jacobi elliptic function expansion method

The general time-fractional nonlinear evolution equation, say in two independent variables x and t , is given by

$$P(u, D_t^{\alpha}u, u_x, D_{tt}^{2\alpha}u, u_{xx}, D_t^{\alpha}u_x, \ldots) = 0, \tag{2.3}$$

where $0 < \alpha \leq 1$, $u = u(x,t)$ is an unknown function and P is a polynomial of u and its partial fractional derivatives, in which the nonlinear terms and the highest order derivatives are included. To find the traveling wave solution of Eq. (2.3) by the extended Jacobi elliptic function expansion method, we follow the following steps:

- **Step 1:** To obtain exact traveling wave solution, the following fractional complex transformation [2]has been applied

$$u(x,t) = U(\xi)e^{i\theta},\ \xi = ik(x - \omega\frac{t^{\alpha}}{\alpha}),\ \ \theta = \lambda x + \mu\frac{t^{\alpha}}{\alpha}, \tag{2.4}$$

  where k, ω, λ and μ are constants to be discussed latter. Then, the Eq. (2.3) is reduced to the following nonlinear ordinary differential equation

$$P(U, -\omega U', kU', \omega^2 U'', k^2 U'', -k\omega U'', \lambda U, \mu U, \ldots) = 0, \tag{2.5}$$

  where $U^{(i)} = U_{i\xi}$.

- **Step 2:** Assuming that the solution of Eq. (2.5) can be expressed as a finite power series of the following form

$$U(\xi) = a_0 + \sum_{j=1}^{N} f_i^{j-1}(\xi)\left[a_j f_i(\xi) + b_j g_i(\xi)\right],\quad i = 1,2,3,4 \tag{2.6}$$

  with

$$\begin{aligned} f_1(\xi) &= \mathrm{sn}(\xi|m),\quad g_1(\xi) = \mathrm{cn}(\xi|m),\\ f_2(\xi) &= \mathrm{sn}(\xi|m),\quad g_2(\xi) = \mathrm{dn}(\xi|m),\\ f_3(\xi) &= \mathrm{sc}(\xi|m),\quad g_3(\xi) = \mathrm{cs}(\xi|m),\\ f_4(\xi) &= \mathrm{sd}(\xi|m),\quad g_4(\xi) = \mathrm{ds}(\xi|m), \end{aligned} \tag{2.7}$$

  where $\mathrm{sc}(\xi|m) = \dfrac{\mathrm{sn}(\xi|m)}{\mathrm{cn}(\xi|m)}$, $\mathrm{cs}(\xi|m) = \dfrac{\mathrm{cn}(\xi|m)}{\mathrm{sn}(\xi|m)}$, $\mathrm{sd}(\xi|m) = \dfrac{\mathrm{sn}(\xi|m)}{\mathrm{dn}(\xi|m)}$, $\mathrm{ds}(\xi|m) = \dfrac{\mathrm{dn}(\xi|m)}{\mathrm{sn}(\xi|m)}$ and m is a modulus satisfies $0 < m < 1$. Here, $\mathrm{sn}(\xi|m)$, $\mathrm{cn}(\xi|m)$ and $\mathrm{dn}(\xi|m)$ are Jacobi elliptic functions, which are double periodic and possess the following properties:

  1. Properties of triangular function:

$$\begin{aligned} &\mathrm{sn}(\xi|m)^2 = 1 - \mathrm{cn}(\xi|m)^2,\\ &\mathrm{dn}(\xi|m)^2 = 1 - m.\mathrm{sn}(\xi|m)^2 = 1 - m + m.\mathrm{cn}(\xi|m)^2 \end{aligned} \tag{2.8}$$

  2. Derivatives of the Jacobi elliptic functions:

$$\begin{aligned} \frac{\partial \mathrm{sn}(\xi|m)}{\partial \xi} &= \mathrm{cn}(\xi|m).\mathrm{dn}(\xi|m),\\ \frac{\partial \mathrm{cn}(\xi|m)}{\partial \xi} &= -\mathrm{dn}(\xi|m).(\mathrm{sn})(\xi|m),\\ \frac{\partial \mathrm{dn}(\xi|m)}{\partial \xi} &= -m.\mathrm{sn}(\xi|m).\mathrm{cn}(\xi|m), \end{aligned} \tag{2.9}$$

3. Some limits: we have

$$\begin{aligned}
&\lim_{m\to 1}(\mathrm{cn}(\xi|m)) = \mathrm{sech}(\xi),\ \lim_{m\to 0}(\mathrm{cn}(\xi|m)) = \cos(\xi),\\
&\lim_{m\to 1}(\mathrm{sn}(\xi|m)) = \tanh(\xi),\ \lim_{m\to 0}(\mathrm{sn}(\xi|m)) = \sin(\xi),\\
&\lim_{m\to 1}(\mathrm{dn}(\xi|m)) = \mathrm{sech}(\xi),\\
&\lim_{m\to 1}(\mathrm{sc}(\xi|m)) = \sinh(\xi),\ \lim_{m\to 0}(\mathrm{sc}(\xi|m)) = \tan(\xi),\\
&\lim_{m\to 1}(\mathrm{cs}(\xi|m)) = \mathrm{csch}(\xi),\ \lim_{m\to 0}(\mathrm{cs}(\xi|m)) = \cot(\xi),\\
&\lim_{m\to 1}(\mathrm{sd}(\xi|m)) = \sinh(\xi),\ \lim_{m\to 0}(\mathrm{sd}(\xi|m)) = \sin(\xi),\\
&\lim_{m\to 1}(\mathrm{ds}(\xi|m)) = \mathrm{csch}(\xi),\ \lim_{m\to 0}(\mathrm{ds}(\xi|m)) = \csc(\xi).
\end{aligned} \tag{2.10}$$

- **Step 3:** The degree N of the power series (2.6) is determined by considering the homogeneous balance between the nonlinear term in Eq. (2.5) and the highest-order derivative. We define the degree of $U(\xi)$ is $D\left[U(\xi)\right] = N$, which gives rise to the degree of other expressions as, for example,

$$D\left[\frac{\partial^\beta U\xi}{\partial\xi^\beta}\right] = \beta + N,\ D\left[\left(\frac{\partial^\beta U(\xi)}{\partial\xi^\beta}\right)^q\right] = q(\beta+N) \tag{2.11}$$

and

$$D\left[U^p(\xi)\left(\frac{\partial^\beta U(\xi)}{\partial\xi^\beta}\right)^q\right] = Np + q(\beta+N). \tag{2.12}$$

- **Step 4:** Substituting Eqs. (2.6)-(2.7) using (2.8)-(2.9) into Eq.(2.5). Then collecting the coefficients of like powers of $\mathrm{cn}(\xi|m)^\kappa \mathrm{sn}(\xi|m)^\ell$, $\mathrm{dn}(\xi|m)^\kappa \mathrm{sn}(\xi|m)^\ell$ where $\kappa = 0,1; \ell = 0,1,2,3\ldots n$. By setting them to zero, we get a system of algebraic equations, and solving the over-determined system of nonlinear algebraic equations by use of ***Mathematica*** software package, we would end up with explicit expressions for $k, \omega, a_j, b_j, (j = 0,1,2,\ldots,N)$.
- **Step 5:** Finally, substitute the values of k, ω, a_j, b_j into (2.6), we obtain the exact traveling wave solutions of the fractional nonlinear evolution equation (2.5).

3 Application of the method to the Conformable Time-fractional Sasa-Satsuma equation

Let's consider (CTFSSE) in the form

$$iD_t^\alpha u + \frac{1}{2}u_{2x} + u\left|u\right|^2 + i\epsilon\left(u_{3x} + 6\left|u\right|^2 u_x + 3u(\left|u\right|^2)_x\right) = 0, \tag{3.1}$$

where $0 < \alpha \le 1$ and ϵ is an arbitrary constant.

Using the fractional complex transformation (2.4), the (CTFSSE) (3.1) is converted to the nonlinear ODE

$$\begin{aligned}
-k^2\left(\frac{1}{2} - 3\lambda\epsilon\right)U'' - 12k\epsilon U^2 U' + k\left(-\lambda + \omega + 3\lambda^2\epsilon\right)U'\\
+ \lambda^3\epsilon U^{(3)} + \left(-\frac{\lambda^2}{2} - \mu + \lambda^3\epsilon\right)U + (1-6\lambda\epsilon)U^3 = 0
\end{aligned} \tag{3.2}$$

Balancing $U^{(3)}$ with U^2U' gives N + 3 = 3 N + 1, hence N = 1. We then suppose that (3.2) has the following formal solutions:

- 1^{st} **approach:** $U(\xi) = a_0 + a_1\mathrm{sn}(\xi|m) + b_1\mathrm{cn}(\xi|m)$,
- 2^{nd} **approach:** $U(\xi) = a_0 + a_1\mathrm{sn}(\xi|m) + b_1\mathrm{dn}(\xi|m)$,
- 3^{th} **approach:** $U(\xi) = a_0 + a_1\mathrm{sc}(\xi|m) + b_1\mathrm{cs}(\xi|m)$,
- 5^{th} **approach:** $U(\xi) = a_0 + a_1\mathrm{sd}(\xi|m) + b_1\mathrm{ds}(\xi|m)$

1. **Using 1^{st} approach:** Proceding as above (Step 4) yields a set of overdetermined algebraic equations with respect to $a_0, a_1, b_1, k, \omega, \lambda, \mu$

$$
\begin{aligned}
&-18a_0b_1^2\lambda\epsilon + 3a_0b_1^2 - \frac{a_0\lambda^2}{2} - a_0\mu + a_0\lambda^3\epsilon - 6a_0^3\lambda\epsilon + a_0^3 = 0,\\
&-18a_1b_1^2\lambda\epsilon + 3a_1b_1^2 - 3a_1k^2\lambda m\epsilon + \frac{1}{2}a_1k^2m - 3a_1k^2\lambda\epsilon - \frac{a_1\lambda^2}{2}\\
&+\frac{a_1k^2}{2} - a_1\mu + a_1\lambda^3\epsilon - 18a_0^2a_1\lambda\epsilon + 3a_0^2a_1 = 0,\\
&-18a_0^2b_1\lambda\epsilon + 3a_0^2b_1 - 3b_1k^2\lambda\epsilon - \frac{b_1\lambda^2}{2} + \frac{b_1k^2}{2} - b_1\mu + b_1\lambda^3\epsilon - 6b_1^3\lambda\epsilon + b_1^3 = 0,\\
&-24a_0a_1b_1k\epsilon = 0,\\
&18a_0b_1^2\lambda\epsilon - 3a_0b_1^2 - 18a_0a_1^2\lambda\epsilon + 3a_0a_1^2 = 0,\\
&18a_1b_1^2\lambda\epsilon - 3a_1b_1^2 + 6a_1k^2\lambda m\epsilon - a_1k^2m - 6a_1^3\lambda\epsilon + a_1^3 = 0,\\
&-12a_1b_1^2k\epsilon - a_1k\lambda + a_1k\omega + 3a_1k\lambda^2\epsilon - 12a_0^2a_1k\epsilon + a_1\lambda^3(-m)\epsilon - a_1\lambda^3\epsilon = 0,\\
&12a_0^2b_1k\epsilon - 24a_1^2b_1k\epsilon + b_1k\lambda - b_1k\omega - 3b_1k\lambda^2\epsilon + 12b_1^3k\epsilon + 4b_1\lambda^3m\epsilon + b_1\lambda^3\epsilon = 0,\\
&6a_0a_1b_1 - 36a_0a_1b_1\lambda\epsilon = 0,\\
&-18a_1^2b_1\lambda\epsilon + 3a_1^2b_1 + 6b_1k^2\lambda m\epsilon - b_1k^2m + 6b_1^3\lambda\epsilon - b_1^3 = 0,\\
&48a_0a_1b_1k\epsilon = 0,\\
&36a_1^2b_1k\epsilon - 12b_1^3k\epsilon - 6b_1\lambda^3m\epsilon = 0,\\
&24a_0b_1^2k\epsilon - 24a_0a_1^2k\epsilon = 0,\\
&36a_1b_1^2k\epsilon - 12a_1^3k\epsilon + 6a_1\lambda^3m\epsilon = 0.
\end{aligned}
\tag{3.3}
$$

By using ***Mathematica*** to solve the overdetermined algebraic equations, we get the following results:

$$
\begin{aligned}
&\left\{a_0 \to 0, a_1 \to \pm\frac{\sqrt{2m}}{12\epsilon}, b_1 \to 0, k \to \frac{1}{6\epsilon}, \lambda \to \frac{1}{6\epsilon}, \omega \to \frac{m+4}{36\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0 \to 0, a_1 \to 0, b_1 \to \pm\frac{i\sqrt{2m}}{12\epsilon}, k \to \frac{1}{6\epsilon}, \lambda \to \frac{1}{6\epsilon}, \omega \to \frac{4-2m}{36\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0 \to 0, a_1 \to \pm\frac{\sqrt{2m}}{24\epsilon}, b_1 \to \pm\frac{i\sqrt{2m}}{24\epsilon}, k \to \frac{1}{6\epsilon}, \lambda \to \frac{1}{6\epsilon}, \omega \to \frac{8-m}{72\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0 \to 0, a_1 \to \pm\frac{\sqrt{2m}}{24\epsilon}, b_1 \to \mp\frac{i\sqrt{2m}}{24\epsilon}, k \to \frac{1}{6\epsilon}, \lambda \to \frac{1}{6\epsilon}, \omega \to \frac{8-m}{72\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\}.
\end{aligned}
\tag{3.4}
$$

So we obtain the following elliptic traveling wave solutions (double periodic solutions) of EQ. (3.1)

$$u_{(1,1,2)} = \pm\frac{\sqrt{2m}}{12\epsilon}\mathrm{sn}\left(\frac{i}{6\epsilon}\left(x-\frac{(m+4)t^{\alpha}}{36\epsilon\alpha}\right)\middle| m\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^{\alpha}}{108\epsilon^{2}\alpha}\right)}, \tag{3.5}$$

$$u_{(1,3,4)} = \pm\frac{i\sqrt{2m}}{12\epsilon}\mathrm{cn}\left(\frac{i}{6\epsilon}\left(x-\frac{(4-2m)t^{\alpha}}{36\epsilon\alpha}\right)\middle| m\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^{\alpha}}{108\epsilon^{2}\alpha}\right)}, \tag{3.6}$$

$$u_{(1,5,6)} = \left\{\begin{array}{l}\pm\frac{\sqrt{2m}}{24\epsilon}\mathrm{sn}\left(\frac{i}{6\epsilon}\left(x-\frac{(8-m)t^{\alpha}}{72\epsilon\alpha}\right)\middle| m\right)\\ \pm\frac{i\sqrt{2m}}{24\epsilon}\mathrm{cn}\left(\frac{i}{6\epsilon}\left(x-\frac{(8-m)t^{\alpha}}{72\epsilon\alpha}\right)\middle| m\right)\end{array}\right\}e^{i\left(\frac{x}{6\epsilon}-\frac{t^{\alpha}}{108\epsilon^{2}\alpha}\right)}, \tag{3.7}$$

$$u_{(1,7,8)} = \left\{\begin{array}{l}\pm\frac{\sqrt{2m}}{24\epsilon}\mathrm{sn}\left(\frac{i}{6\epsilon}\left(x-\frac{(8-m)t^{\alpha}}{72\epsilon\alpha}\right)\middle| m\right)\\ \mp\frac{i\sqrt{2m}}{24\epsilon}\mathrm{cn}\left(\frac{i}{6\epsilon}\left(x-\frac{(8-m)t^{\alpha}}{72\epsilon\alpha}\right)\middle| m\right)\end{array}\right\}e^{i\left(\frac{x}{6\epsilon}-\frac{t^{\alpha}}{108\epsilon^{2}\alpha}\right)}. \tag{3.8}$$

In particular, if $m \to 1$, then we get the hyperbolic function solutions (solitary wave solutions) of Eq.(3.1)

$$u_{(1,1,2)} = \pm\frac{\sqrt{2}}{12\epsilon}\tanh\left(\frac{i}{6\epsilon}\left(x-\frac{5t^{\alpha}}{36\epsilon\alpha}\right)\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^{\alpha}}{108\epsilon^{2}\alpha}\right)}, \tag{3.9}$$

$$u_{(1,3,4)} = \pm\frac{i\sqrt{2}}{12\epsilon}\mathrm{sech}\left(\frac{i}{6\epsilon}\left(x-\frac{t^{\alpha}}{18\epsilon\alpha}\right)\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^{\alpha}}{108\epsilon^{2}\alpha}\right)}, \tag{3.10}$$

$$u_{(1,5,6)} = \left\{\begin{array}{l}\pm\frac{\sqrt{2}}{24\epsilon}\tanh\left(\frac{i}{6\epsilon}\left(x-\frac{7t^{\alpha}}{72\epsilon\alpha}\right)\right)\\ \\ \pm\frac{i\sqrt{2}}{24\epsilon}\mathrm{sech}\left(\frac{i}{6\epsilon}\left(x-\frac{7t^{\alpha}}{72\epsilon\alpha}\right)\right)\end{array}\right\}e^{i\left(\frac{x}{6\epsilon}-\frac{t^{\alpha}}{108\epsilon^{2}\alpha}\right)}, \tag{3.11}$$

$$u_{(1,7,8)} = \left\{\begin{array}{l}\pm\frac{\sqrt{2}}{24\epsilon}\tanh\left(\frac{i}{6\epsilon}\left(x-\frac{7t^{\alpha}}{72\epsilon\alpha}\right)\right)\\ \\ \mp\frac{i\sqrt{2}}{24\epsilon}\mathrm{sech}\left(\frac{i}{6\epsilon}\left(x-\frac{7t^{\alpha}}{72\epsilon\alpha}\right)\right)\end{array}\right\}e^{i\left(\frac{x}{6\epsilon}-\frac{t^{\alpha}}{108\epsilon^{2}\alpha}\right)}. \tag{3.12}$$

2. **Using 2^{nd} approach:** with the same process as above the following system of algebraic

equations are obtained

$$
\begin{aligned}
&-18a_0b_1^2\lambda\epsilon+3a_0b_1^2-\frac{a_0\lambda^2}{2}-a_0\mu+a_0\lambda^3\epsilon-6a_0^3\lambda\epsilon+a_0^3=0,\\
&-12a_1b_1^2k\epsilon-a_1k\lambda+a_1k\omega+3a_1k\lambda^2\epsilon-12a_0^2a_1k\epsilon+a_1\lambda^3(-m)\epsilon-a_1\lambda^3\epsilon=0,\\
&-18a_1b_1^2\lambda\epsilon+3a_1b_1^2-3a_1k^2\lambda m\epsilon+\frac{1}{2}a_1k^2m-3a_1k^2\lambda\epsilon-\frac{a_1\lambda^2}{2}+\frac{a_1k^2}{2}-a_1\mu\\
&+a_1\lambda^3\epsilon-18a_0^2a_1\lambda\epsilon+3a_0^2a_1=0,\\
&36a_1b_1^2km\epsilon-12a_1^3k\epsilon++6a_1\lambda^3m\epsilon=0,\\
&18a_1b_1^2\lambda m\epsilon-3a_1b_1^2m+6a_1k^2\lambda m\epsilon-a_1k^2m-6a_1^3\lambda\epsilon+a_1^3=0,\\
&24a_0b_1^2km\epsilon-24a_0a_1^2k\epsilon=0,\\
&18a_0b_1^2\lambda m\epsilon-3a_0b_1^2m-18a_0a_1^2\lambda\epsilon++3a_0a_1^2=0,\\
&-18a_0^2b_1\lambda\epsilon+3a_0^2b_1-3b_1k^2\lambda m\epsilon++\frac{1}{2}b_1k^2m-\frac{b_1\lambda^2}{2}-b_1\mu+b_1\lambda^3\epsilon-6b_1^3\lambda\epsilon+b_1^3=0,\\
&12a_0^2b_1km\epsilon-24a_1^2b_1k\epsilon+b_1+k\lambda m-b_1km\omega-3b_1k\lambda^2m\epsilon+12b_1^3km\epsilon+b_1\lambda^3m^2\epsilon+4b_1\lambda^3m\epsilon=0,\\
&-18a_1^2b_1\lambda\epsilon+3a_1^2b_1+6b_1k^2\lambda m\epsilon-b_1k^2m+6b_1^3\lambda m\epsilon+b_1^3(-m)=0,\\
&36a_1^2b_1km\epsilon-12b_1^3km^2\epsilon-6b_1\lambda^3m^2\epsilon=0,\\
&-24a_0a_1b_1k\epsilon=0,\\
&6a_0a_1b_1-36a_0a_1b_1\lambda\epsilon=0,\\
&48a_0a_1b_1km\epsilon=0.
\end{aligned}
\tag{3.13}
$$

By using ***Mathematica*** to solve ([3.13]), the following results are obtained

$$
\begin{aligned}
&\left\{a_0\to 0,a_1\to\pm\frac{\sqrt{2m}}{12\epsilon},b_1\to 0,k\to\frac{1}{6\epsilon},\omega\to\frac{m+4}{36\epsilon},\lambda\to\frac{1}{6\epsilon},\mu\to-\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0\to 0,a_1\to 0,b_1\to\pm\frac{i\sqrt{2}}{12\epsilon},k\to\frac{1}{6\epsilon},\omega\to\frac{m+1}{36\epsilon},\lambda\to\frac{1}{6\epsilon},\mu\to-\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0\to 0,a_1\to\pm\frac{\sqrt{2m}}{24\epsilon},b_1\to\pm\frac{i\sqrt{2}}{24\epsilon},k\to\frac{1}{6\epsilon},\omega\to\frac{2m+5}{72\epsilon},\lambda\to\frac{1}{6\epsilon},\mu\to-\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0\to 0,a_1\to\pm\frac{\sqrt{2m}}{24\epsilon},b_1\to\mp\frac{i\sqrt{2}}{24\epsilon},k\to\frac{1}{6\epsilon},\omega\to\frac{2m+5}{72\epsilon},\lambda\to\frac{1}{6\epsilon},\mu\to-\frac{1}{108\epsilon^2}\right\}.
\end{aligned}
\tag{3.14}
$$

So we obtain the following elliptic traveling wave solutions (double periodic solutions) of EQ. ([3.1])

$$
u_{(2,1,2)}=\pm\frac{\sqrt{2m}}{12\epsilon}\mathrm{sn}\left(\frac{i}{6\epsilon}\left(x-\frac{(m+4)t^\alpha}{36\epsilon\alpha}\right)\middle|\,m\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)},
\tag{3.15}
$$

$$
u_{(2,3,4)}=\pm\frac{i\sqrt{2}}{12\epsilon}\mathrm{dn}\left(\frac{i}{6\epsilon}\left(x-\frac{(m+1)t^\alpha}{36\epsilon\alpha}\right)\middle|\,m\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)},
\tag{3.16}
$$

$$u_{(2,5,6)} = \left\{ \begin{array}{l} \pm\frac{\sqrt{2m}}{24\epsilon}\mathrm{sn}\left(\frac{i}{6\epsilon}\left(x-\frac{(2m+5)t^\alpha}{72\epsilon\alpha}\right)\middle| m\right) \\ \pm\frac{i\sqrt{2}}{24\epsilon}\mathrm{dn}\left(\frac{i}{6\epsilon}\left(x-\frac{(2m+5)t^\alpha}{72\epsilon\alpha}\right)\middle| m\right) \end{array} \right\} e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \tag{3.17}$$

$$u_{(2,7,8)} = \left\{ \begin{array}{l} \pm\frac{\sqrt{2m}}{24\epsilon}\mathrm{sn}\left(\frac{i}{6\epsilon}\left(x-\frac{(2m+5)t^\alpha}{72\epsilon\alpha}\right)\middle| m\right) \\ \mp\frac{i\sqrt{2}}{24\epsilon}\mathrm{dn}\left(\frac{i}{6\epsilon}\left(x-\frac{(2m+5)t^\alpha}{72\epsilon\alpha}\right)\middle| m\right) \end{array} \right\} e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}. \tag{3.18}$$

In particular, if $m \to 1$, then we get the hyperbolic function solutions (solitary wave solutions) of Eq.(3.1)

$$u_{(2,1,2)} = \pm\frac{\sqrt{2}}{12\epsilon}\tanh\left(\frac{i}{6\epsilon}\left(x-\frac{5t^\alpha}{36\epsilon\alpha}\right)\right) e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \tag{3.19}$$

$$u_{(2,3,4)} = \pm\frac{i\sqrt{2}}{12\epsilon}\mathrm{sech}\left(\frac{i}{6\epsilon}\left(x-\frac{t^\alpha}{18\epsilon\alpha}\right)\right) e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \tag{3.20}$$

$$u_{(2,5,6)} = \left\{ \begin{array}{l} \pm\frac{\sqrt{2}}{24\epsilon}\tanh\left(\frac{i}{6\epsilon}\left(x-\frac{7t^\alpha}{72\epsilon\alpha}\right)\right) \\ \pm\frac{i\sqrt{2}}{24\epsilon}\mathrm{sech}\left(\frac{i}{6\epsilon}\left(x-\frac{7t^\alpha}{72\epsilon\alpha}\right)\right) \end{array} \right\} e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \tag{3.21}$$

$$u_{(2,7,8)} = \left\{ \begin{array}{l} \pm\frac{\sqrt{2}}{24\epsilon}\tanh\left(\frac{i}{6\epsilon}\left(x-\frac{7t^\alpha}{72\epsilon\alpha}\right)\right) \\ \mp\frac{i\sqrt{2}}{24\epsilon}\mathrm{sech}\left(\frac{i}{6\epsilon}\left(x-\frac{7t^\alpha}{72\epsilon\alpha}\right)\right) \end{array} \right\} e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}. \tag{3.22}$$

3. **Using $3^{\mathbf{rd}}$ approach:** By inserting the $3^{\mathbf{rd}}$ equation of Eq. (2.6) along with Eq. (2.7) in Eq. (3.1) and equating the coefficient of each power of $\mathrm{cn}(\xi|m)^\kappa\mathrm{sn}(\xi|m)^\ell$, $\mathrm{dn}(\xi|m)^\kappa\mathrm{sn}(\xi|m)^\ell$ where $\kappa = 0, 1; \ell = 0, 1, 2, 3\ldots$ to zero, we get a system of algebraic equations in different

parameters $a_0, a_1, b_1, k, \omega, \lambda, \mu$.

$$
\begin{aligned}
&24b_1^3k\epsilon - 12b_1\lambda^3\epsilon = 0,\\
&6a_0b_1^2 - 36a_0b_1^2\lambda\epsilon = 0,\\
&108a_0b_1^2\lambda\epsilon - 72a_0a_1b_1\lambda\epsilon - 18a_0b_1^2 + 12a_0a_1b_1 - a_0\lambda^2 - 2a_0\mu + 2a_0\lambda^3\epsilon - 12a_0^3\lambda\epsilon + 2a_0^3 = 0,\\
&-108a_0b_1^2\lambda\epsilon + 144a_0a_1b_1\lambda\epsilon + 18a_0b_1^2 - 24a_0a_1b_1 + 2a_0\lambda^2 + 4a_0\mu - 4a_0\lambda^3\epsilon\\
&+24a_0^3\lambda\epsilon - 36a_0a_1^2\lambda\epsilon - 4a_0^3 + 6a_0a_1^2 = 0,\\
&36a_0b_1^2\lambda\epsilon - 72a_0a_1b_1\lambda\epsilon - 6a_0b_1^2 + 12a_0a_1b_1 - a_0\lambda^2 - 2a_0\mu + 2a_0\lambda^3\epsilon - 12a_0^3\lambda\epsilon\\
&+36a_0a_1^2\lambda\epsilon + 2a_0^3 - 6a_0a_1^2 = 0,\\
&12b_1k^2\lambda\epsilon - 2b_1k^2 - 12b_1^3\lambda\epsilon + 2b_1^3 = 0,\\
&-36a_1b_1^2\lambda\epsilon - 36a_0^2b_1\lambda\epsilon + 6a_1b_1^2 + 6a_0^2b_1 - 6b_1k^2\lambda m\epsilon + b_1k^2m - 24b_1k^2\lambda\epsilon - b_1\lambda^2 + 4b_1k^2\\
&-2b_1\mu + 2b_1\lambda^3\epsilon + 36b_1^3\lambda\epsilon - 6b_1^3 = 0,\\
&72a_1b_1^2\lambda\epsilon + 72a_0^2b_1\lambda\epsilon - 36a_1^2b_1\lambda\epsilon - 12a_1b_1^2 - 12a_0^2b_1 + 6a_1^2b_1 - 6a_1k^2\lambda m\epsilon + a_1k^2m + 12a_1k^2\lambda\epsilon\\
&-a_1\lambda^2 - 2a_1k^2 - 2a_1\mu + 2a_1\lambda^3\epsilon - 36a_0^2a_1\lambda\epsilon + 6a_0^2a_1 + 12b_1k^2\lambda m\epsilon - 2b_1k^2m\\
&+12b_1k^2\lambda\epsilon + 2b_1\lambda^2 - 2b_1k^2 + 4b_1\mu - 4b_1\lambda^3\epsilon - 36b_1^3\lambda\epsilon + 6b_1^3 = 0,\\
&-36a_1b_1^2\lambda\epsilon - 36a_0^2b_1\lambda\epsilon + 36a_1^2b_1\lambda\epsilon + 6a_1b_1^2 + 6a_0^2b_1 - 6a_1^2b_1 - 6a_1k^2\lambda m\epsilon + a_1k^2m + a_1\lambda^2 + 2a_1\mu\\
&-2a_1\lambda^3\epsilon - 12a_1^3\lambda\epsilon + 36a_0^2a_1\lambda\epsilon + 2a_1^3 - 6a_0^2a_1 - 6b_1k^2\lambda m\epsilon + b_1k^2m - b_1\lambda^2\\
&-2b_1\mu + 2b_1\lambda^3\epsilon + 12b_1^3\lambda\epsilon - 2b_1^3 = 0,\\
&24a_1b_1^2k\epsilon + 24a_0^2b_1k\epsilon + 2b_1k\lambda - 2b_1k\omega - 6b_1k\lambda^2\epsilon - 72b_1^3k\epsilon + 2b_1\lambda^3m\epsilon + 32b_1\lambda^3\epsilon = 0,\\
&-48a_1b_1^2k\epsilon - 48a_0^2b_1k\epsilon - 24a_1^2b_1k\epsilon - 2a_1k\lambda + 2a_1k\omega + 6a_1k\lambda^2\epsilon - 24a_0^2a_1k\epsilon - 2a_1\lambda^3m\epsilon\\
&+4a_1\lambda^3\epsilon - 4b_1k\lambda + 4b_1k\omega + 12b_1k\lambda^2\epsilon + 72b_1^3k\epsilon - 4b_1\lambda^3m\epsilon - 28b_1\lambda^3\epsilon = 0,\\
&24a_1b_1^2k\epsilon + 24a_0^2b_1k\epsilon + 24a_1^2b_1k\epsilon + 2a_1k\lambda - 2a_1k\omega - 6a_1k\lambda^2\epsilon - 24a_1^3k\epsilon + 24a_0^2a_1k\epsilon\\
&-10a_1\lambda^3m\epsilon + 8a_1\lambda^3\epsilon + 2b_1k\lambda - 2b_1k\omega - 6b_1k\lambda^2\epsilon - 24b_1^3k\epsilon + 2b_1\lambda^3m\epsilon + 8b_1\lambda^3\epsilon = 0,\\
&48a_0b_1^2k\epsilon = 0,\\
&-96a_0b_1^2k\epsilon = 0, 48a_0b_1^2k\epsilon - 48a_0a_1^2k\epsilon = 0.
\end{aligned}
\tag{3.23}
$$

The resulting system of algebraic equations has been solved by ***Mathematica*** to obtain the values of the unknown parameters $a_0, a_1, b_1, k, \omega, \lambda, \mu$.

$$
\begin{aligned}
&\left\{a_0 \to 0, a_1 \to \pm\frac{\sqrt{2(1-m)}}{12\epsilon}, b_1 \to 0, k \to \frac{1}{6\epsilon}, \omega \to \frac{m+1}{36\epsilon}, \lambda \to \frac{1}{6\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0 \to 0, a_1 \to 0, b_1 \to \pm\frac{\sqrt{2}}{12\epsilon}, k \to \frac{1}{6\epsilon}, \omega \to \frac{m+1}{36\epsilon}, \lambda \to \frac{1}{6\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0 \to 0, a_1 \to \pm\frac{\sqrt{2(1-m)}}{12\epsilon}, b_1 \to \pm\frac{\sqrt{2}}{12\epsilon}, k \to \frac{1}{6\epsilon}, \omega \to \frac{m+6\sqrt{1-m}+1}{36\epsilon}, \lambda \to \frac{1}{6\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0 \to 0, a_1 \to \pm\frac{\sqrt{2(1-m)}}{12\epsilon}, b_1 \to \mp\frac{\sqrt{2}}{12\epsilon}, k \to \frac{1}{6\epsilon}, \omega \to \frac{m-6\sqrt{1-m}+1}{36\epsilon}, \lambda \to \frac{1}{6\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\}.
\end{aligned}
\tag{3.24}
$$

Finally, new elliptic trveling wave solutions (double periodic solutions) for the nonlinear Eq. (3.1) are reached.

$$u_{(3,1,2)} = \pm\frac{\sqrt{2(1-m)}}{12\epsilon}\mathrm{sc}\left(\frac{i}{6\epsilon}\left(x-\frac{(m+1)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \quad (3.25)$$

$$u_{(3,3,4)} = \pm\frac{\sqrt{2}}{12\epsilon}\mathrm{cs}\left(\frac{i}{6\epsilon}\left(x-\frac{(m+1)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \quad (3.26)$$

$$u_{(3,5,6)} = \left\{\begin{array}{c} \pm\frac{\sqrt{2(1-m)}}{12\epsilon}\mathrm{sc}\left(\frac{i}{6\epsilon}\left(x-\frac{(m+6\sqrt{1-m}+1)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) \\ \pm\frac{\sqrt{2}}{12\epsilon}\mathrm{cs}\left(\frac{i}{6\epsilon}\left(x-\frac{(m+6\sqrt{1-m}+1)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) \end{array}\right\}e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \quad (3.27)$$

$$u_{(3,7,8)} = \left\{\begin{array}{c} \pm\frac{\sqrt{2(1-m)}}{12\epsilon}\mathrm{sc}\left(\frac{i}{6\epsilon}\left(x-\frac{(m-6\sqrt{1-m}+1)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) \\ \mp\frac{\sqrt{2}}{12\epsilon}\mathrm{cs}\left(\frac{i}{6\epsilon}\left(x-\frac{(m-6\sqrt{1-m}+1)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) \end{array}\right\}e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}. \quad (3.28)$$

In particular, if $m \to 0$, then we get the trigonometric function solutions (periodic solutions) of Eq.(3.1)

$$u_{(3,1,2)} = \pm\frac{\sqrt{2}}{12\epsilon}\tan\left(\frac{i}{6\epsilon}\left(x-\frac{t^\alpha}{36\epsilon\alpha}\right)\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \quad (3.29)$$

$$u_{(3,3,4)} = \pm\frac{\sqrt{2}}{12\epsilon}\cot\left(\frac{i}{6\epsilon}\left(x-\frac{t^\alpha}{36\epsilon\alpha}\right)\right)e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \quad (3.30)$$

$$u_{(3,5,6)} = \left\{\begin{array}{c} \pm\frac{\sqrt{2}}{12\epsilon}\tan\left(\frac{i}{6\epsilon}\left(x-\frac{7t^\alpha}{36\epsilon\alpha}\right)\right) \\ \pm\frac{\sqrt{2}}{12\epsilon}\cot\left(\frac{i}{6\epsilon}\left(x-\frac{7t^\alpha}{36\epsilon\alpha}\right)\right) \end{array}\right\}e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \quad (3.31)$$

$$u_{(3,7,8)} = \left\{\begin{array}{c} \pm\frac{\sqrt{2}}{12\epsilon}\tan\left(\frac{i}{6\epsilon}\left(x+\frac{5t^\alpha}{36\epsilon\alpha}\right)\right) \\ \mp\frac{\sqrt{2}}{12\epsilon}\cot\left(\frac{i}{6\epsilon}\left(x+\frac{5t^\alpha}{36\epsilon\alpha}\right)\right) \end{array}\right\}e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}. \quad (3.32)$$

4. **Using $4^{\mathbf{th}}$ approach:** following the steps indicated above we get the following system of algebraic equations

$$
\begin{aligned}
&24b_1^3k\epsilon - 12b_1\lambda^3\epsilon = 0,\\
&6a_0b_1^2 - 36a_0b_1^2\lambda\epsilon = 0,\\
&-18a_0b_1^2m^4 + 108a_0b_1^2\lambda m\epsilon - 72a_0a_1b_1\lambda\epsilon + 12a_0a_1b_1\\
&-a_0\lambda^2 - 2a_0\mu + 2a_0\lambda^3\epsilon - 12a_0^3\lambda\epsilon + 2a_0^3 = 0,\\
&-108a_0b_1^2\lambda m^2\epsilon + 18a_0b_1^2m^2 + 144a_0a_1b_1\lambda m\epsilon - 24a_0a_1b_1m + 2a_0\lambda^2m + 4a_0\mu m\\
&-4a_0\lambda^3m\epsilon + 24a_0^3\lambda m\epsilon - 4a_0^3m - 36a_0a_1^2\lambda\epsilon + 6a_0a_1^2 = 0,\\
&36a_0b_1^2\lambda m^3\epsilon - 6a_0b_1^2m^3 - 72a_0a_1b_1\lambda m^2\epsilon + 12a_0a_1b_1m^2 - a_0\lambda^2m^2 - 2a_0\mu m^2 + 2a_0\lambda^3m^2\epsilon\\
&-12a_0^3\lambda m^2\epsilon + 2a_0^3m^2 + 36a_0a_1^2\lambda m\epsilon - 6a_0a_1^2m = 0,\\
&12b_1k^2\lambda\epsilon - 2b_1k^2 - 12b_1^3\lambda\epsilon + 2b_1^3 = 0,\\
&-48a_1b_1^2km\epsilon - 48a_0^2b_1km\epsilon - 24a_1^2b_1k\epsilon - 2a_1k\lambda + 2a_1k\omega + 6a_1k\lambda^2\epsilon - 24a_0^2a_1k\epsilon + 4a_1\lambda^3m\epsilon\\
&-2a_1\lambda^3\epsilon + 72b_1^3km^2\epsilon - 4b_1k\lambda m + 4b_1km\omega + 12b_1k\lambda^2m\epsilon - 28b_1\lambda^3m^2\epsilon - 4b_1\lambda^3m\epsilon = 0,\\
&24a_1b_1^2k\epsilon + 24a_0^2b_1k\epsilon + 2b_1k\lambda - 72b_1^3km\epsilon - 2b_1k\omega - 6b_1k\lambda^2\epsilon + 32b_1\lambda^3m\epsilon + 2b_1\lambda^3\epsilon = 0,\\
&24a_1b_1^2km^2\epsilon + 24a_0^2b_1km^2\epsilon + 2a_1k\lambda m - 2a_1km\omega - 6a_1k\lambda^2m\epsilon - 24a_1^3k\epsilon + 8a_1\lambda^3m^2\epsilon\\
&-10a_1\lambda^3m\epsilon - 24b_1^3km^3\epsilon + 2b_1k\lambda m^2 - 2b_1km^2\omega - 6b_1k\lambda^2m^2\epsilon + 8b_1\lambda^3m^3\epsilon + 2b_1\lambda^3m^2\epsilon = 0,\\
&-36a_1b_1^2\lambda\epsilon - 36a_0^2b_1\lambda\epsilon + 6a_1b_1^2 + 6a_0^2b_1 - 24b_1k^2\lambda m\epsilon + 4b_1k^2m - 6b_1k^2\lambda\epsilon - b_1\lambda^2 + b_1k^2\\
&-2b_1\mu + 36b_1^3\lambda m\epsilon - 6b_1^3m + 2b_1\lambda^3\epsilon = 0,\\
&72a_1b_1^2\lambda m\epsilon + 72a_0^2b_1\lambda m\epsilon - 12a_1b_1^2m - 12a_0^2b_1m - 36a_1^2b_1\lambda\epsilon + 6a_1^2b_1 + 12a_1k^2\lambda m\epsilon - 2a_1k^2m\\
&-6a_1k^2\lambda\epsilon - a_1\lambda^2 + a_1k^2 - 2a_1\mu + 2a_1\lambda^3\epsilon - 36a_0^2a_1\lambda\epsilon + 6a_0^2a_1 + 12b_1k^2\lambda m^2\epsilon - 2b_1k^2m^2\\
&+12b_1k^2\lambda m\epsilon - 2b_1k^2m - 36b_1^3\lambda m^2\epsilon + 6b_1^3m^2 + 2b_1\lambda^2m + 4b_1\mu m - 4b_1\lambda^3m\epsilon = 0,\\
&-36a_1b_1^2\lambda m^2\epsilon - 36a_0^2b_1\lambda m^2\epsilon + 6a_1b_1^2m^2 + 6a_0^2b_1m^2 + 36a_1^2b_1\lambda m\epsilon - 6a_1^2b_1m - 6a_1k^2\lambda m\epsilon\\
&+a_1k^2m + a_1\lambda^2m + 2a_1\mu m - 2a_1\lambda^3m\epsilon + 36a_0^2a_1\lambda m\epsilon - 6a_0^2a_1m - 12a_1^3\lambda\epsilon + 2a_1^3\\
&-6b_1k^2\lambda m^2\epsilon + b_1k^2m^2 + 12b_1^3\lambda m^3\epsilon - 2b_1^3m^3 - b_1\lambda^2m^2 - 2b_1\mu m^2 + 2b_1\lambda^3m^2\epsilon = 0,\\
&48a_0b_1^2k\epsilon = 0,\\
&-96a_0b_1^2km\epsilon = 0,\\
&48a_0b_1^2km^2\epsilon - 48a_0a_1^2k\epsilon = 0.
\end{aligned}
\tag{3.33}
$$

Solving the system with the help of ***Mathematica*** , we get

$$
\begin{aligned}
&\left\{a_0 \to 0, a_1 \to \pm\frac{\sqrt{m-1}\sqrt{2m}}{12\epsilon}, b_1 \to 0, k \to \frac{1}{6\epsilon}, \omega \to \frac{4-2m}{36\epsilon}, \lambda \to \frac{1}{6\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\},\\
&\left\{a_0 \to 0, a_1 \to 0, b_1 \to \pm\frac{\sqrt{2}}{12\epsilon}, k \to \frac{1}{6\epsilon}, \omega \to \frac{4-2m}{36\epsilon}, \lambda \to \frac{1}{6\epsilon}, \mu \to -\frac{1}{108\epsilon^2}\right\}
\end{aligned}
\tag{3.34}
$$

$$\begin{Bmatrix} a_0 \to 0, a_1 \to \pm\frac{m-\sqrt{2m(5m-4)}}{24\epsilon}, b_1 \to \mp\frac{\sqrt{2}}{12\epsilon}, k \to \frac{1}{6\epsilon}, \\ \omega \to \frac{-5m+3\sqrt{m(5m-4)}+4}{36\epsilon}, \lambda \to \frac{1}{6\epsilon}, \mu \to -\frac{1}{108\epsilon^2} \end{Bmatrix}, \quad \begin{Bmatrix} a_0 \to 0, a_1 \to \pm\frac{m+\sqrt{2m(5m-4)}}{24\epsilon}, b_1 \to \mp\frac{\sqrt{2}}{12\epsilon}, k \to \frac{1}{6\epsilon}, \\ \omega \to \frac{-5m-3\sqrt{m(5m-4)}+4}{36\epsilon}, \lambda \to \frac{1}{6\epsilon}, \mu \to -\frac{1}{108\epsilon^2} \end{Bmatrix}. \tag{3.35}$$

So the following elliptic traveling wave solutions (double periodic solutions) of EQ. (3.1) are obtained

$$u_{(4,1,2)} = \pm\frac{\sqrt{m-1}\sqrt{2m}}{12\epsilon}\mathrm{sd}\left(\frac{i}{6\epsilon}\left(x - \frac{(4-2m)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \tag{3.36}$$

$$u_{(4,3,4)} = \pm\frac{\sqrt{2}}{12\epsilon}\mathrm{ds}\left(\frac{i}{6\epsilon}\left(x - \frac{(4-2m)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \tag{3.37}$$

$$u_{(4,5,6)} = \begin{Bmatrix} \pm\frac{m-\sqrt{m(5m-4)}}{12\sqrt{2}\epsilon}\mathrm{sd}\left(\frac{i}{6\epsilon}\left(x - \frac{(-5m+3\sqrt{m(5m-4)}+4)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) \\ \mp\frac{\sqrt{2}}{12\epsilon}\mathrm{ds}\left(\frac{i}{6\epsilon}\left(x - \frac{(-5m+3\sqrt{m(5m-4)}+4)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) \end{Bmatrix} e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}, \tag{3.38}$$

$$u_{(4,7,8)} = \begin{Bmatrix} \pm\frac{m+\sqrt{m(5m-4)}}{12\sqrt{2}\epsilon}\mathrm{sd}\left(\frac{i}{6\epsilon}\left(x - \frac{(-5m-3\sqrt{m(5m-4)}+4)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) \\ \mp\frac{\sqrt{2}}{12\epsilon}\mathrm{ds}\left(\frac{i}{6\epsilon}\left(x - \frac{(-5m-3\sqrt{m(5m-4)}+4)t^\alpha}{36\epsilon\alpha}\right)\middle| m\right) \end{Bmatrix} e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{108\epsilon^2\alpha}\right)}. \tag{3.39}$$

In particular, if $m \to 0$, then we get the trigonometric function solutions (periodic solutions) of Eq.(3.1)

$$u_{(4,3,4)} = \pm\frac{\sqrt{2}}{12\epsilon}\csc\left(\frac{i}{6\epsilon}\left(x - \frac{t^\alpha}{9\alpha\epsilon}\right)\right) e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{\alpha(108\epsilon^2)}\right)}, \tag{3.40}$$

and, if $m \to 1$, then we get the hyperbolic function solutions (solitary wave solutions) of Eq.(3.1)

$$u_{(4,3,4)} = \pm\frac{\sqrt{2}}{12\epsilon}\mathrm{csch}\left(\frac{i}{6\epsilon}\left(x - \frac{t^\alpha}{18\alpha\epsilon}\right)\right) e^{i\left(\frac{x}{6\epsilon}-\frac{t^\alpha}{\alpha(108\epsilon^2)}\right)} \tag{3.41}$$

$$u_{(4,7,8)} = \pm\frac{\sqrt{2}}{12\epsilon}\sinh\left(\frac{i}{6\epsilon}\left(x + \frac{t^\alpha}{9\alpha\epsilon}\right)\right) \mp \frac{\sqrt{2}}{12\epsilon}\mathrm{csch}\left(\frac{i}{6\epsilon}\left(x + \frac{t^\alpha}{9\alpha\epsilon}\right)\right). \tag{3.42}$$

Remark 2. We have verified all solutions obtained by substituting them in the equation under study (3.1) and found to be correct.

4 Conclusion

In the present work, the extended Jacobi elliptic function expansion method has been proposed. By implementing procedure of the method on the Conformable Time-fractional Sasa-Satsuma equation, we have shown that the solutions obtained by the extended technique is more and various than the classical Jacobi elliptic function expansion methods. Therefore, the method plays an important role to seek more doubly periodic solutions, solitary wave solutions and trigonometric (periodic) solutions of other nonlinear evolution equations in mathematical physics. We have also studied the accuracy, the geometrical construction and the behavior of some particular solutions by plotting their surfaces and contour plots as shown in figures below. Since the solutions are complex functions, by choosing values for the parameters (m, ϵ and α), we plot graphics of modulus, imaginary part and real part of these special solutions.

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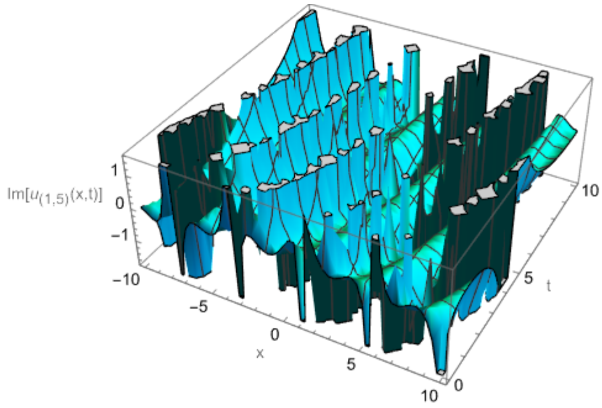
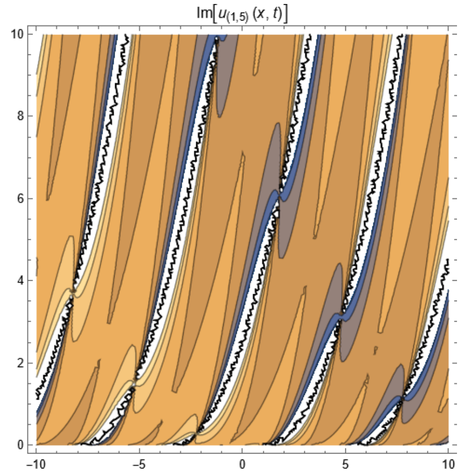


(a) $Im[u_{(1,5)}(x,t)]\,(m = 0.5,\ \epsilon = 0.1,\ \alpha = 0.5)$



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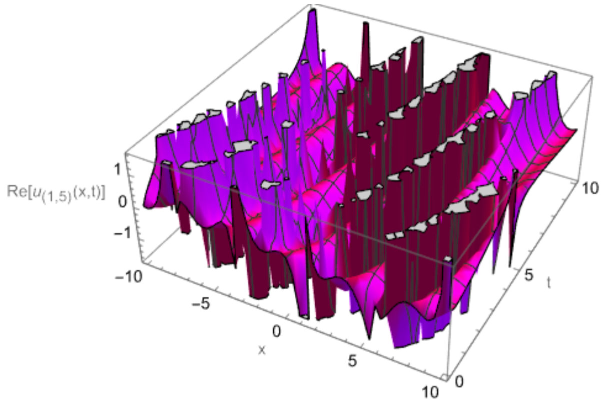

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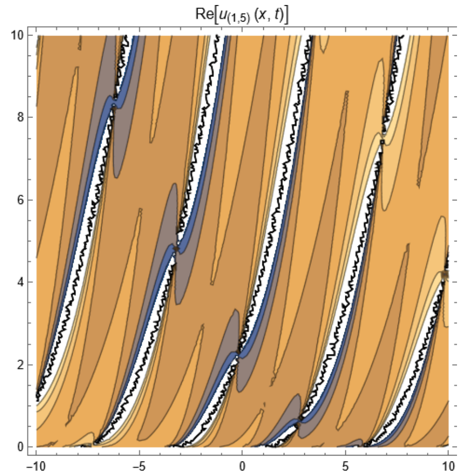
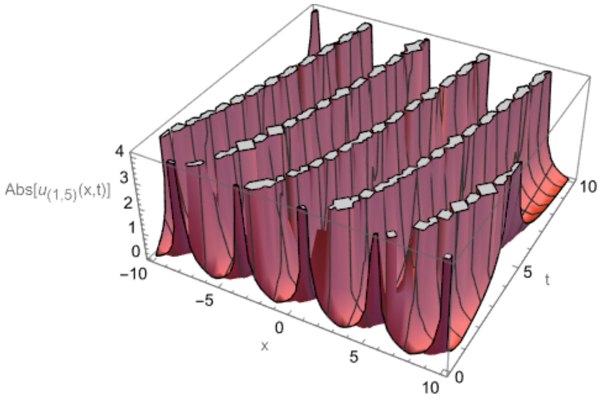
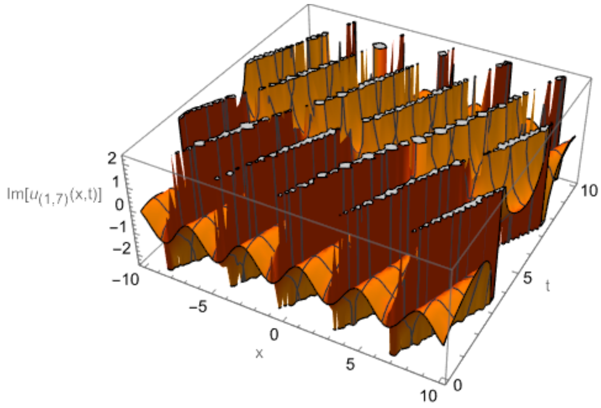


(d) $Re[u_{(1,5)}(x,t)]\,(m = 0.5,\ \epsilon = 0.1,\ \alpha = 0.5)$

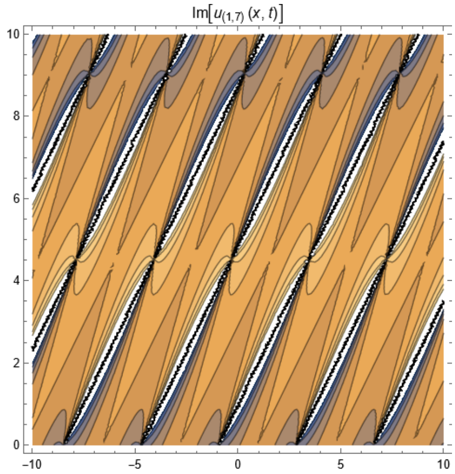


(e) $Abs[u_{(1,5)}(x,t)]\ (m = 0.5,\ \epsilon = 0.1,\ \alpha = 0.5)$

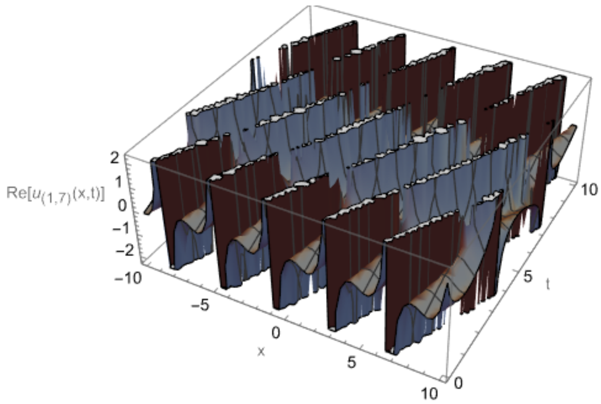
Figure 1: 3DPlot and ContourPlot of the exact solutions of Eq. (3.1) given by (3.7), for $(x,t) \in [-10,10] \times [0,10]$.



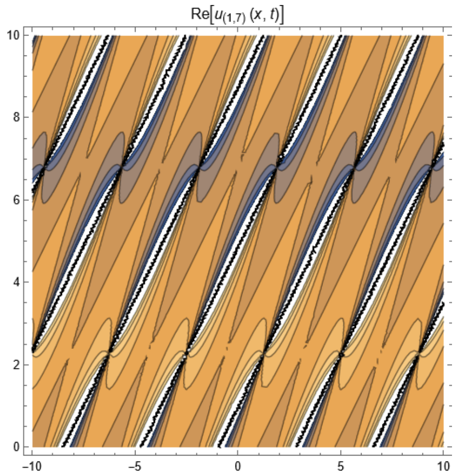
(a) $Im[u_{(1,7)}(x,t)]\,(m=1,\,\epsilon=0.1,\,\alpha=1)$



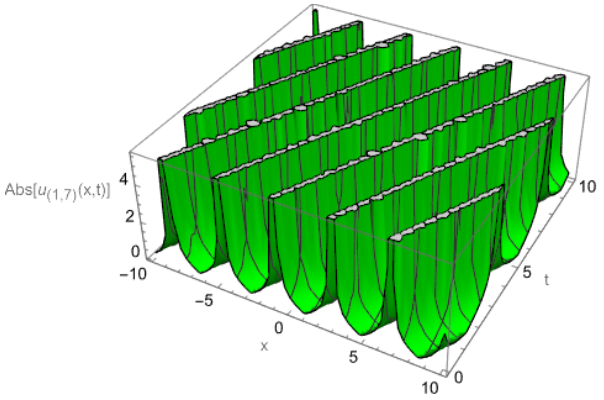
(b) $Im[u_{(1,7)}(x,t)]\,(m=1,\,\epsilon=0.1,\,\alpha=1)$



(c) $Re[u_{(1,7)}(x,t)]\,(m=1,\,\epsilon=0.1,\,\alpha=1)$

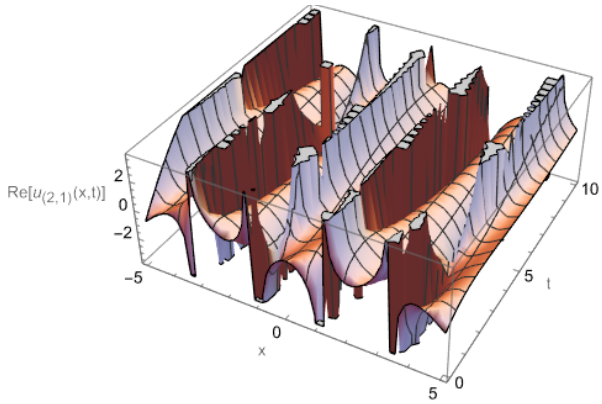


(d) $Re[u_{(1,7)}(x,t)]\,(m=1,\,\epsilon=0.1,\,\alpha=1)$

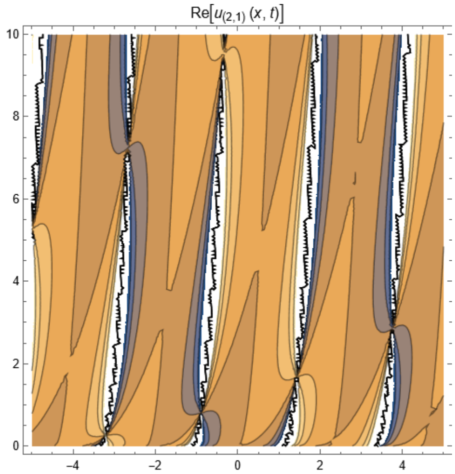


(e) $Abs[u_{(1,7)}(x,t)]\,(m=1,\,\epsilon=0.1,\,\alpha=1)$

Figure 2: 3DPlot and ContourPlot of the exact solutions of Eq. (3.1) given by (3.8), for $(x,t)\in[-10,10]\times[0,10]$.



(a) $Re[u_{(2,1)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.5$)



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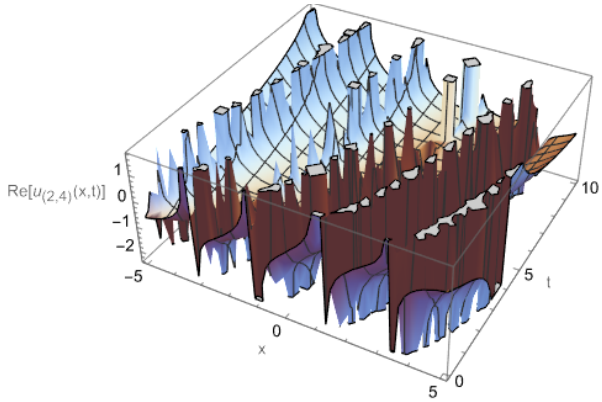
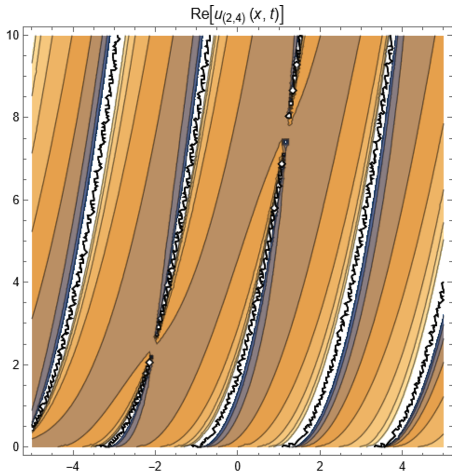
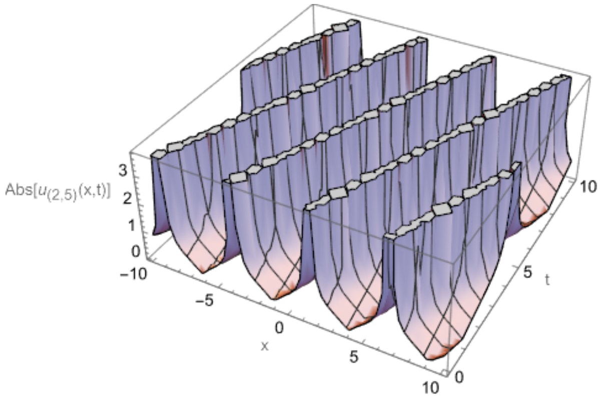


(c) $Re[u_{(2,4)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.5$)

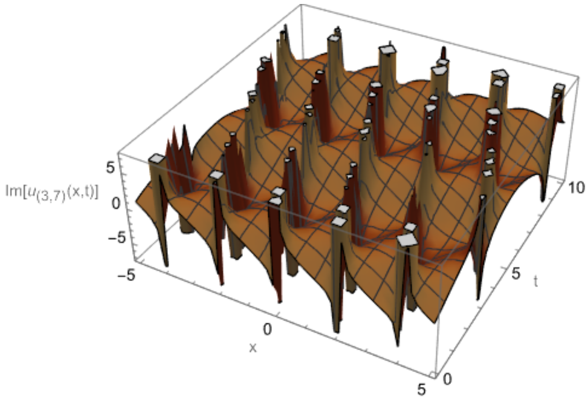


(d) $Re[u_{(2,4)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.5$)

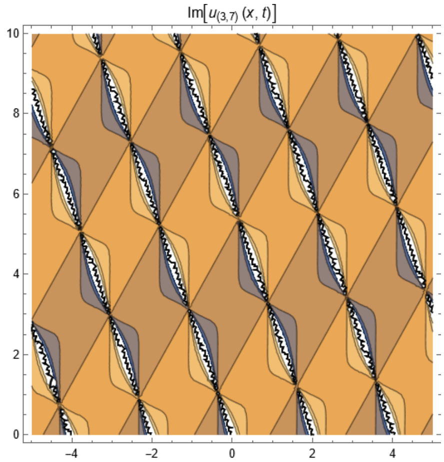


(e) $Abs[u_{(2,5)}(x,t)]$ ($m = 0.25$, $\epsilon = 0.1$, $\alpha = 1$)

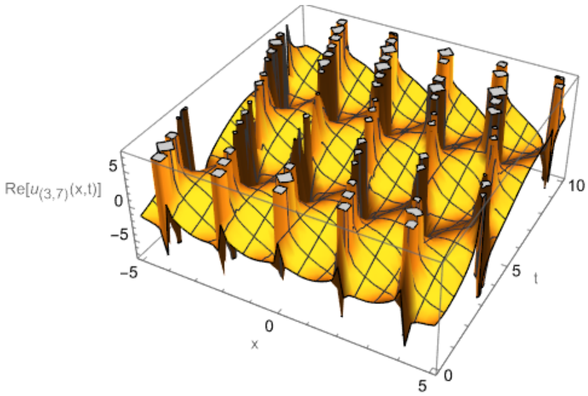
Figure 3: 3DPlot and ContourPlot of the exact solutions of Eq. (3.1) given by (3.15) , (3.16) and (3.17) respectively, for $(x,t) \in [-5,5] \times [0,10]$ and $(x,t) \in [-10,10] \times [0,10]$ respectively.



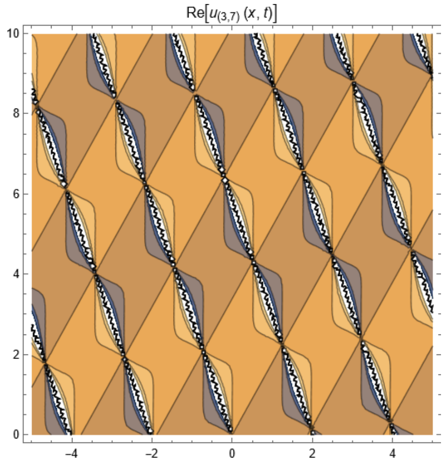
(a) $Im[u_{(3,7)}(x,t)]$ ($m = 0.75$, $\epsilon = 0.1$, $\alpha = 1$)



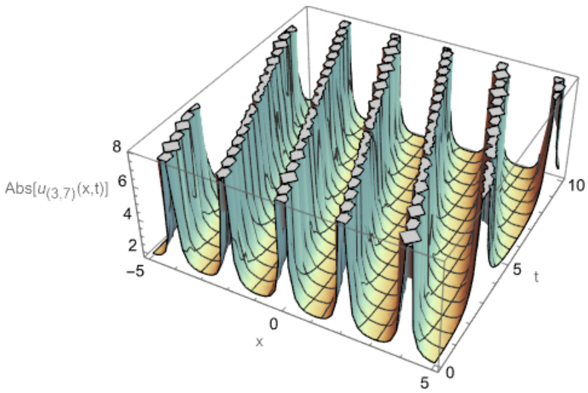
(b) $Im[u_{(3,7)}(x,t)]$ ($m = 0.75$, $\epsilon = 0.1$, $\alpha = 1$)



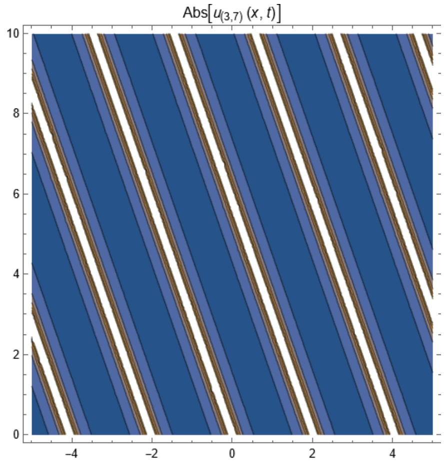
(c) $Re[u_{(3,7)}(x,t)]$ ($m = 0.75$, $\epsilon = 0.1$, $\alpha = 1$)



(d) $Re[u_{(3,7)}(x,t)]$ ($m = 0.75$, $\epsilon = 0.1$, $\alpha = 1$)

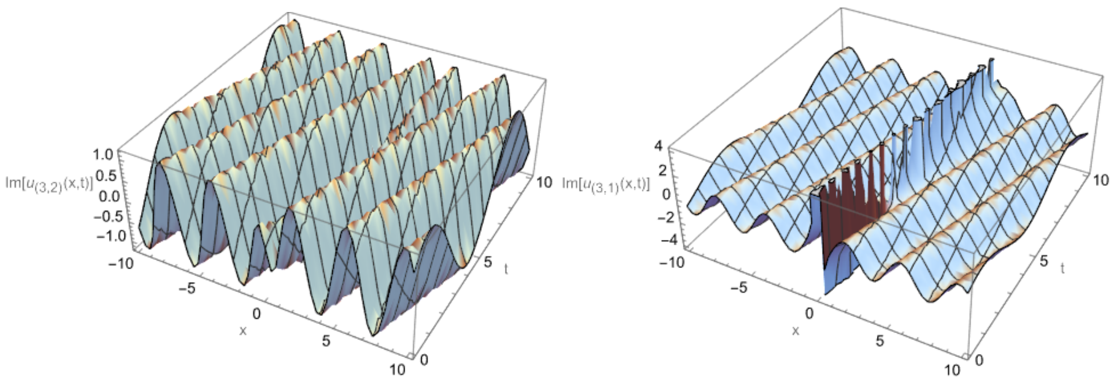


(e) $Abs[u_{(3,7)}(x,t)]$ ($m = 0.75$, $\epsilon = 0.1$, $\alpha = 1$)



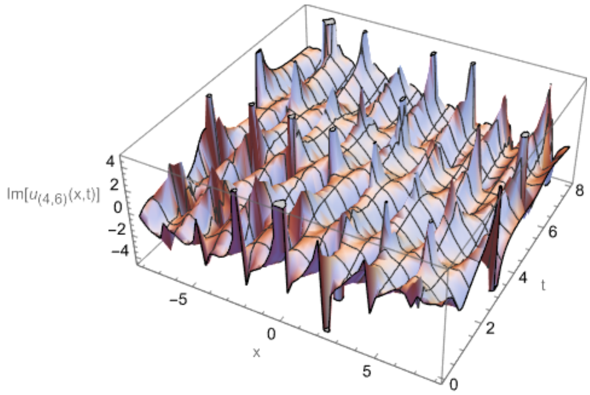
(f) $Abs[u_{(3,7)}(x,t)]$ ($m = 0.75$, $\epsilon = 0.1$, $\alpha = 1$)

Figure 4: 3DPlot of the exact solutions of Eq. (3.1) given by (3.28) respectively, for $(x,t) \in [-5, 10] \times [0, 10]$.

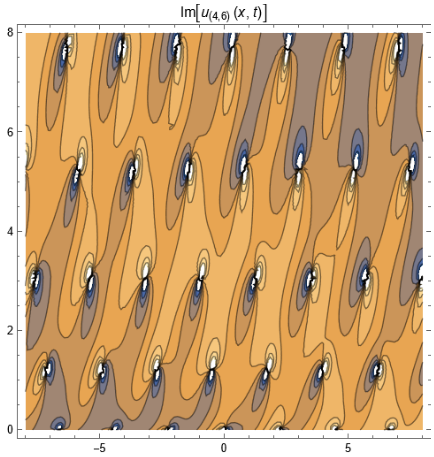


(a) $Im[u_{(3,1)}(x,t)]$ $(m=0,\, \epsilon=0.1,\, \alpha=1)$ (b) $Im[u_{(3,3)}(x,t)]$ $(m=0,\, \epsilon=0.1,\, \alpha=1)$

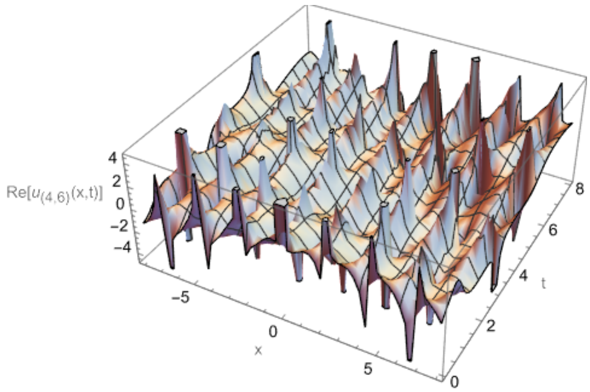
Figure 5: 3DPlot of the exact solutions of Eq. (3.1) given by (3.29) and (3.30) respectively, for $(x,t) \in [-10,10] \times [0,10]$.



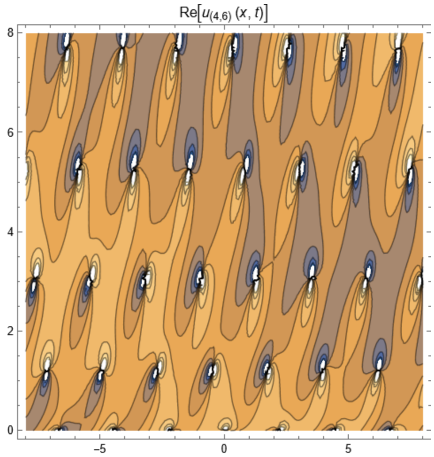
(a) $Im[u_{(4,6)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.75$)



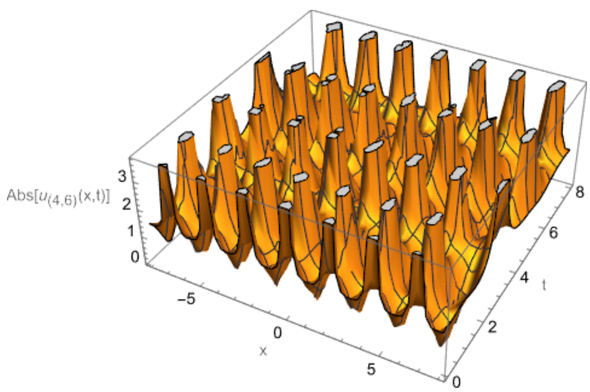
(b) $Im[u_{(4,6)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.75$)



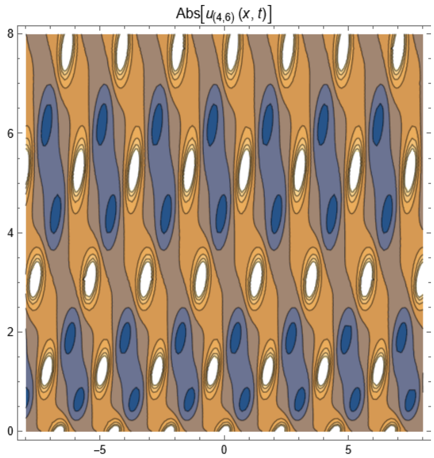
(c) $Re[u_{(4,6)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.75$)



(d) $Re[u_{(4,6)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.75$)

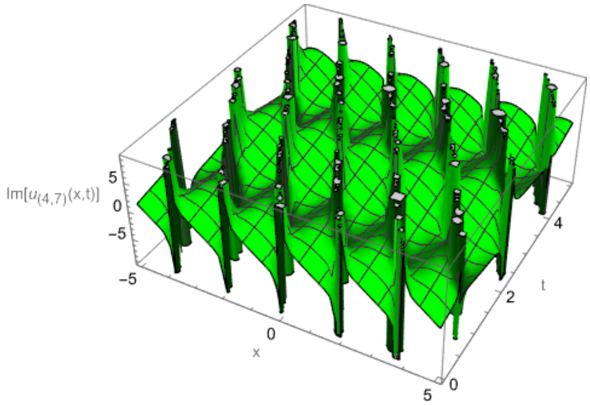


(e) $Abs[u_{(4,6)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.75$)

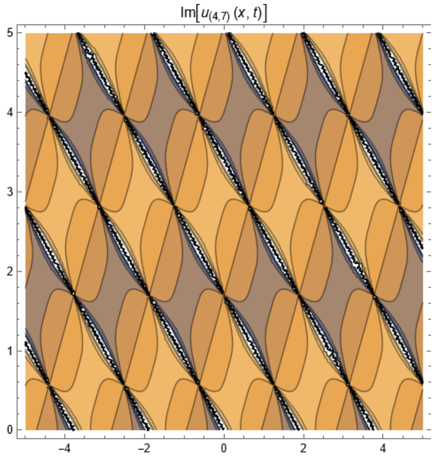


(f) $Abs[u_{(4,6)}(x,t)]$ ($m = 0.5$, $\epsilon = 0.1$, $\alpha = 0.75$)

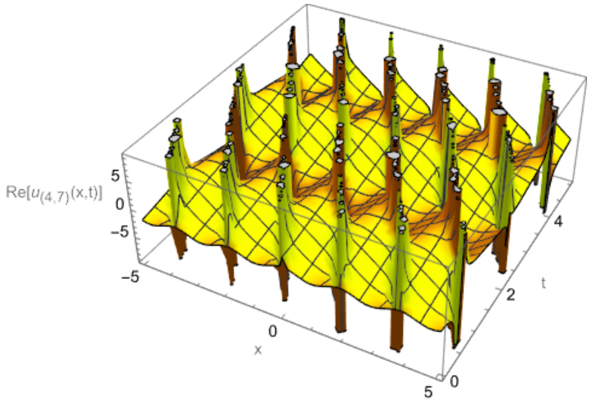
Figure 6: 3DPlot of the exact solutions of Eq. (3.1) given by (3.38) , for $(x,t) \in [-8,8] \times [0,8]$.



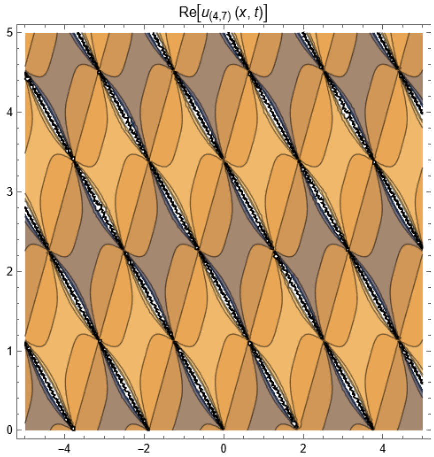
(a) $Im[u_{(4,7)}(x,t)]$ $(m=1,\ \epsilon=0.1,\ \alpha=1)$



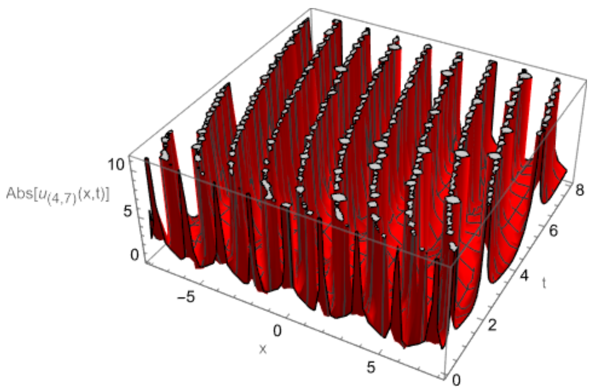
(b) $Im[u_{(4,7)}(x,t)]$ $(m=1,\ \epsilon=0.1,\ \alpha=1)$



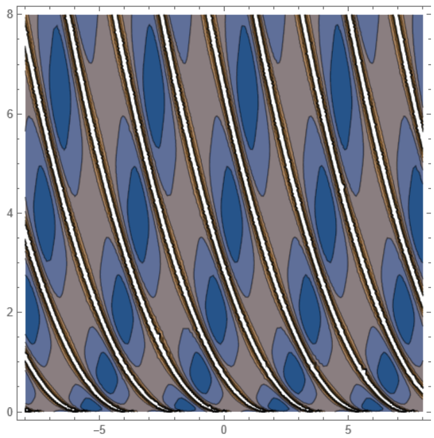
(c) $Re[u_{(4,7)}(x,t)]$ $(m=1,\ \epsilon=0.1,\ \alpha=1)$



(d) $Re[u_{(4,7)}(x,t)]$ $(m=1,\ \epsilon=0.1,\ \alpha=1)$



(e) $Abs[u_{(4,7)}(x,t)]$ $(m=1,\ \epsilon=0.1,\ \alpha=0.5)$



(f) $Abs[u_{(4,7)}(x,t)]$ $(m=1,\ \epsilon=0.1,\ \alpha=0.5)$

Figure 7: 3DPlot of the exact solutions of Eq. (3.1) given by (3.42) respectively, for $(x,t)\in[-5,5]\times[0,5]$ and $(x,t)\in[-8,8]\times[0,8]$ respectvely.