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Existence and multiplicity solutions for a singular elliptic p(x)-Laplacian equation

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Abstract. This paper deals with the existence and multiplicity of nontrivial weak solutions for the following equation involving variable exponents:

$$\begin{cases} -\triangle_{p(x)} u + \dfrac{|u|^{r-2}u}{|x|^r} = \lambda h(x,u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth enough boundary which is subject to Dirichlet boundary condition. Using a variational method and Krasnoselskii's genus theory, we would show the existence and multiplicity of the solutions. Next, we study closedness of set of eigenfunctions, such that $p(x) \equiv p$.

Keywords. p(x)-Laplacian, variational method, genus theory, Sobolev space.

1 Introduction

In this paper, we study the following problem

$$\begin{cases} -\triangle_{p(x)} u + \dfrac{|u|^{r-2}u}{|x|^r} = \lambda h(x,u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth enough boundary. Let λ be a positive real parameter and p be real continuous function on $\overline{\Omega}$ with $1 < r < p(x) < p^*(x)$, where $p^*(x) = \dfrac{Np(x)}{N-p(x)}$ and $p(x) < N$ for all $x \in \overline{\Omega}$, $\triangle_{p(x)} u = div\left(|\nabla u|^{p(x)-2}\nabla u\right)$ denotes the $p(x)$ -Laplacian operator (for details, see [8, 10]).

We assume throughout this paper that the function h satisfies the following hypotheses:

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(H_1) $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $C_1|t|^{\beta(x)-1} \leq h(x,t) \leq C_2 t^{\alpha(x)-1}$, for all $t \geq 0$ and for all $x \in \overline{\Omega}$, where C_1, C_2 are positive constants and $\alpha, \beta \in C(\overline{\Omega})$ such that $1 < \beta(x) < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

(H_2) h is an odd function according to t, that is $h(x,t) = -h(x,-t)$ for all $t \in \mathbb{R}$ and for $x \in \overline{\Omega}$.

In recent years, the study of $p(x)$-Laplacian problems in the variable exponent Lebesgue Sobolev spaces is an interesting topic. For many problems, authors studied the existence one, two and three solutions. For example, in 2021, Ragusa- Razani- Safari [2], considered the existence of one solution for a $p(x)$-Laplacian problem with Dirichlet boundary condition, by using variational principle. In 2022, Yucedag [22, 23], for Steklov boundary problems and in 2023, Chu- Xie- Zhou [7], for a new $p(x)$-Kirchhoff problem, proved the existence of one solution by using the Mountain pass theorem. In 2022, Heidarkhani- Ghobadi- Avci [14], considered the existence two weak solutions for $p(x)$-equations. In 2014, Allaoui [1], for a Robin problem and in 2021, Aydin-Unal [5], for a Steklov problem, studied the existence three weak solutions by using *Ricceri's* variational principle.

In [3], the authors studied the Kirchhoff type equation:

$$\begin{cases} -M\left(\dfrac{1}{p(x)}\int_\Omega |\nabla u|^{p(x)}dx\right) \triangle_{p(x)}\, u = f(x,u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{1.2}$$

They established the existence and multiplicity of the solutions of the problem (1.2).
The authors in [4], by using the mountain pass theorem, the fountain theorem, the dual fountain theorem and the theory of the variable exponent Sobolev spaces, obtained results on existence and multiplicity of solutions for the following problem:

$$\begin{cases} M\left(\int_\Omega \dfrac{|\nabla u|^{p(x)}}{p(x)}dx\right)\left(-\triangle_{p(x)}\, u\right) = f(x,u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{1.3}$$

In [9], Z. El Allali and S. Taarabti studied the $p(x)$- Kirchhoff equation:

$$-M\left(\frac{1}{p(x)}\int_\Omega |\nabla u|^{p(x)}dx\right) \triangle^2_{p(x)}\, u = f(x,u), in\ \Omega, \tag{1.4}$$

with Neumann boundary conditions, by using the Krasnoselskii's genus theory.
In [18], R. M. Khanghahi and A. Razani showed that the following problem:

$$\begin{cases} -\triangle_{p(x)}\, u + \dfrac{|u|^{s-2}u}{|x|^s} = \lambda f(x,u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{1.5}$$

have two weak solutions in the case when $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying $f(x,t) \leq a_1 + a_2 t^{q(x)-2}$, for all $(x,t) \in \Omega \times \mathbb{R}$, where a_1, a_2 are two positive constants.
Here, we prove at least m pairs of distinct critical points and then infinitely many solutions for equation (1.1) by using variational method and Krasnoselskii's genus theory.

2 Preliminaries

We recall some necessary definitions and propositions concerning the Lebesgue and Sobolev spaces.

Let Ω be a bounded domain of $\mathbb{R}^N$. Set

$$C_+(\bar{\Omega}) := \left\{ s(x) \in C(\bar{\Omega}); s(x) > 1,\ \forall\ x \in \bar{\Omega} \right\}.$$

For any continuous function $s : \Omega \to (1, \infty)$,

$$s^- := \inf_{x\in\Omega} s(x) \quad \text{and} \quad s^+ := \sup_{x\in\Omega} s(x).$$

For $s \in C_+(\bar{\Omega})$, define

$$L^{s(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \ \text{ is a measurable function} : \int_\Omega |u(x)|^{s(x)} dx < +\infty \right\}.$$

Endowed with the norm:

$$\|u\|_{s(x)} := \inf \left\{ \mu > 0 \colon \int_\Omega \left| \frac{u(x)}{\mu} \right|^{s(x)} dx \le 1 \right\}.$$

$L^{s(x)}(\Omega)$ is well known that is a separable reflexive Banach space [6, 12, 16].
The modular of $L^{s(x)}(\Omega)$ is defined by

$$\sigma_{s(x)}(u) := \int_\Omega |u(x)|^{s(x)} dx.$$

Proposition 2.1. *[10, 13]. $\left(L^{s(x)}(\Omega), \|u\|_{s(x)}\right)$ is separable, uniformly convex, reflexive and its conjugate space is $\left(L^{s'(x)}(\Omega), \|u\|_{s'(x)}\right)$, where*

$$\frac{1}{s(x)} + \frac{1}{s'(x)} = 1, \quad \forall x \in \Omega.$$

For all $u \in L^{s(x)}(\Omega),\ w \in L^{s'(x)}(\Omega)$, we have

$$\left| \int_\Omega u\, w\, dx \right| \le \left(\frac{1}{s^-} + \frac{1}{s'^-} \right) \|u\|_{s(x)} \, \|w\|_{s'(x)} \le 2\|u\|_{s(x)} \, \|w\|_{s'(x)}. \tag{2.1}$$

Proposition 2.2. *[11, 16] Suppose that $u, u_n \in L^{s(x)}(\Omega)$, we have*

$$\begin{aligned}
&\|u\|_{s(x)} > 1 \Rightarrow \|u\|_{s(x)}^{s^-} \le \sigma_{s(x)}(u) \le \|u\|_{s(x)}^{s^+};\\
&\|u\|_{s(x)} < 1 \Rightarrow \|u\|_{s(x)}^{s^+} \le \sigma_{s(x)}(u) \le \|u\|_{s(x)}^{s^-};\\
&\|u\|_{s(x)} > 1 \,(\text{respectively}, = 1; < 1) \Leftrightarrow \sigma_{s(x)}(u) > 1 \ (\text{respectively}, \ = 1; < 1)\,;\\
&\|u_n\|_{s(x)} \longrightarrow 0 (\text{respectively}, \longrightarrow +\infty) \Leftrightarrow \sigma_{s(x)}(u_n) \longrightarrow 0 (\text{respectively}, \longrightarrow +\infty);\\
&\lim_{n\to\infty} \|u_n - u\|_{s(x)} = 0 \Longleftrightarrow \lim_{n\to\infty} \sigma_{s(x)}(u_n - u) = 0.
\end{aligned}$$

The Sobolev space $W^{1,s(x)}(\Omega)$ is defined by

$$W^{1,s(x)}(\Omega) := \left\{ u \in L^{s(x)}(\Omega) \colon |\nabla u| \in L^{s(x)}(\Omega) \right\}.$$

It is a separable and reflexive Banach spaces with norm:

$$\|u\|_{1,s(x)} = \|u\|_{s(x)} + \|\nabla u\|_{s(x)}.$$

(For more details, we refer to [8, 16]). Denote by $W_0^{1,s(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the following norm:

$$\|u\| = \inf \left\{ \mu > 0 \colon \int_\Omega \left| \frac{\nabla u(x)}{\mu} \right|^{s(x)} dx \leq 1 \right\}.$$

It is well known that

$$W_0^{1,s(x)}(\Omega) = \left\{ u \in L^{s(x)}(\Omega); u\big|_{\partial\Omega} = 0, \ |\nabla u| \in L^{s(x)}(\Omega) \right\}.$$

For more details, we refer to [6, 11, 15].

Proposition 2.3. *(Sobolev Embedding[10]) For $s, s' \in C_+(\bar{\Omega})$ and $1 < s'(x) < s^*(x)$ for all $x \in \bar{\Omega}$, there is a continuous compact embedding*

$$W_0^{1,s(x)}(\Omega) \hookrightarrow L_{s'(x)}(\Omega),$$

which is continuous and compact. Therefore, there is a constant $c_0 > 0$ such that

$$\|u\|_{s'(x)} \leq c_0 \|u\|.$$

Proposition 2.4. *(Poincare Inequality [19]) There is a constant $c > 0$ such that*

$$\|u\|_{s(x)} \leq c \|\nabla u\|_{s(x)}, \tag{2.2}$$

for all $u \in W_0^{1,s(x)}(\Omega)$.

Remark 1. From Proposition 2.4, $\|\nabla u\|_{s(x)}$ and $\|u\|_{1,s(x)}$ are equivalent norms on $W_0^{1,s(x)}(\Omega)$.

Proposition 2.5. *[10, 13] The functional $\Lambda : W_0^{1,s(x)}(\Omega) \to \mathbb{R}$ defined by $\Lambda = \int_\Omega \frac{1}{s(x)} |\nabla u|^{s(x)} dx$ is convex. The mapping $\Lambda' : W_0^{1,s(x)}(\Omega) \to \left(W_0^{1,s(x)}(\Omega) \right)^*$ is a strictly monotone, bounded homeomorphism and of (S_+) type, if $u_n \to u$ (weakly) as $n \to \infty$ and $\overline{\lim}_{n\to\infty} (\Lambda'(u_n), u_n - u) \leq 0$ implies $u_n \to u$ (strongly).*

Definition 1. [20] Let $1 < r < N$, we have

$$\int_\Omega \frac{|u(x)|^r}{|x|^r} dx \leq \frac{1}{M} \int_\Omega |\nabla u(x)|^r dx, \tag{2.3}$$

for all $u \in W_0^{1,s(x)}(\Omega)$, where $M := \left(\frac{N-r}{r} \right)^r$.

Definition 2. Let U be a real Banach space. Set $R := \{B \subset U - \{0\};\ B \text{ is compact and symmetric}\}$. Let $B \in R$ and we define the genus of B as follows:

$$\gamma(B) := \inf\{m \geq 1;\ \exists g \in C(B, \mathbb{R}^m \backslash \{0\});\ g \text{ is odd}\}.$$

And $\gamma(B) = \infty$, if does not exist such a map g. $\gamma(\emptyset) = 0$ by definition. For more details, we refer to [9].

3 Main results

Let $E : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is a functional defined by

$$E(u) := \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \frac{1}{r} \int_\Omega \frac{|u|^r}{|x|^r} dx, \tag{3.1}$$

where $1 < r < p(x) < \infty$. By [21] and [[16], Theorem 3.1],

- $E \in C^1$.
- For all $u, w \in W_0^{1,p(x)}(\Omega)$,
  $$E'(u)(w) := \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla w + \frac{|u|^{r-2} uw}{|x|^r} \right) dx. \tag{3.2}$$
- $E' : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega) \right)^*$ defined by (3.2) is strictly monotone.
- $E' : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega) \right)^*$ is homeomorphism and a mapping of type (S_+).

Now, let $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function. For all $(x, z) \in W_0^{1,p(x)}(\Omega)$, define

$$H(x, z) := \int_\Omega^Z h(x, t) dt. \tag{3.3}$$

For $u \in W_0^{1,p(x)}(\Omega)$, define $F : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$F(u) := \int_\Omega H(x, u(x)) dx, \tag{3.4}$$

$F \in C^1$ and has compact derivative such that

$$F'(u)(w) := \int_\Omega h(x, u(x)) w(x) dx, \tag{3.5}$$

for u, w in $W_0^{1,p(x)}(\Omega)$ (see [21]).

Definition 3. $u \in W_0^{1,p(x)}(\Omega)$ is called a weak solution of (1.1) if

$$\int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \, \nabla w + \frac{|u|^{r-2} uw}{|x|^r} \right) dx = \lambda \int_\Omega h(x, u) w \, dx,$$

for all $w \in W_0^{1,p(x)}(\Omega)$.

The energy functional associated with problem (1.1) can obtained by

$$\tau(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{1}{r} \int_\Omega \frac{|u|^r}{|x|^r} dx - \lambda \int_\Omega H(x, u) \, dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$. It is well defined, C^1 functional and for all $u, w \in W_0^{1,p(x)}(\Omega)$

$$\langle \tau'(u), w \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla w + \frac{|u|^{r-2} uw}{|x|^r} \right) dx - \lambda \int_\Omega h(x, u) w \, dx.$$

We consider $\Omega \subset \mathbb{R}^N (N > 3)$ as a bounded domain with smooth boundary and $p \in C_+(\Omega)$ such that

$$1 < \beta^- \le \beta(x) \le \beta^+ < \alpha^- \le \alpha(x) \le \alpha^+ < r < p^- \le p(x) \le p^+ \le p^*(x) \tag{3.6}$$

and $p(x) < N$ for all $x \in \bar{\Omega}$.

Definition 4. The functional τ satisfies in the Palais-Smale condition at the level c, $(PS)_c$, if for every sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ satisfying

$$\tau(u_n) \to c \quad and \quad \tau'(u_n) \to 0 \quad as\ n \to \infty,$$

has a convergence subsequence.

Theorem 3.1. *[9]. Let $\tau \in C^1(W_0^{1,p(x)}, \mathbb{R})$ and satisfies the $(PS)_C$ condition. We assume the following conditions:*

i) τ is even and bounded from below;

ii) There exists a $T \in R$ such that $\gamma(T) = m$ and $\sup_{x\in T} \tau(x) < \tau(0)$.

Then problem (1.1) has at least m pairs of distinct critical points and their corresponding critical values are less than $\tau(0)$.

Theorem 3.2. *If (3.6), (H_1) and (H_2) hold, then there are at least m pairs of distinct critical point for (1.1).*

Lemma 3.1. *Under assumptions (3.6), (H_1) and (H_2), τ is coercive on $W_0^{1,p(x)}(\Omega)$ and bounded from below.*

Proof. For any $u \in W_0^{1,p(x)}(\Omega)$, we have

$$\begin{aligned}\tau(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\ dx + \frac{1}{r}\frac{|u|^r}{|x|^r}\ dx - \lambda \int_\Omega H(x,u)\ dx \\ &\geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\ dx - \lambda \frac{C_2}{a^-}\int_\Omega |u|^{\alpha(x)}\ dx.\end{aligned}$$

If $\sigma_{p(x)}(u) = \int_\Omega |u|^{p(x)}\ dx$, by Proposition 2.2 and Proposition 2.4, we have two cases:
i) If $\sigma_{p(x)}(u) > 1$,

$$\tau(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \lambda\frac{C_2}{a^-}\|u\|^{a^+}.$$

According to (3.6), τ is coercive and bounded from below.
ii) If $\sigma_p(u) < 1$,

$$\tau(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda C_2}{a^-}\|u\|^{a^-}.$$

Because of (3.6), so τ is coercive and bounded from below too. □

Lemma 3.2. *If (3.6), (H_1) and (H_2) hold, then $\tau := E - \lambda F$ satisfies the $(PS)_c$ condition.*

Proof. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a $(PS)_c$ sequence. Initially we prove that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Assume by contradiction the contrary. Then, passing eventually to a subsequence, $\|u_n\| \to +\infty$ as $n \to +\infty$. We choose θ, $0 < \theta < \dfrac{1}{p^+}$. By Definition 4, for large enough n,

$$\begin{aligned}C + \|u_n\| &\geq \tau(u_n) - \theta\langle \tau'(u_n), u_n\rangle \\ &= \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\ dx + \frac{1}{r}\int_\Omega \frac{|u_n|^r}{|x|^r}\ dx - \lambda\int_\Omega H(x,u_n)\ dx\end{aligned}$$

$$\begin{aligned}
&-\theta\int_\Omega\left(|\nabla u_n|^{p(x)}+\frac{|u_n|^r}{|x|^r}\right)\,dx-\lambda\int_\Omega h(x,u_n)u_n\,dx\\
&\geq\frac{1}{p^+}\int_\Omega|\nabla u_n|^{p(x)}\,dx-\frac{C_2}{\alpha^-}\int_\Omega|u_n|^{\alpha(x)}\,dx-\theta\int_\Omega|\nabla u_n|^{p(x)}\,dx+\lambda\theta C_2\int_\Omega|u_n|^{\alpha(x)}\,dx\\
&=\left(\frac{1}{p^+}-\theta\right)\int_\Omega|\nabla u_n|^{p(x)}\,dx-\lambda C_2\left(\frac{1}{\alpha^-}-\theta\right)\int_\Omega|\nabla u_n|^{\alpha(x)}\,dx.
\end{aligned}$$

By Proposition 2.3, there is a constant $C_0>0$, such that

$$-\lambda\left(\frac{1}{\alpha^-}-\theta\right)\|u_n\|_\alpha\geq-\lambda C_0\left(\frac{1}{\alpha^-}-\theta\right)\|u_n\|.$$

So

$$C+\|u_n\|\geq\left(\frac{1}{p^+}-\theta\right)\|u_n\|^{p^-}-\lambda C_3\left(\frac{1}{\alpha^-}-\theta\right)\|u_n\|.$$

It follows from (3.6),when we divide the last inequality by $\|u_n\|$ and pass to the limit as $n\to+\infty$, we obtion a contradiction. Thus $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Then, we prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$. It follows from Proposition 2.3 and reflexivity of $W_0^{1,p(x)}(\Omega)$, we may assume that

$$u_n\rightharpoonup u\text{ in }W_0^{1,p(x)}(\Omega),\;u_n\to u\text{ in }L^{s(x)}(\Omega),\;u_n(x)\to u(x),\;a.e.\;in\;\Omega,\tag{3.7}$$

where $1\leq s(x)<p^*(x)$.

From (H_1), (2.1) and (3.7)

$$\begin{aligned}
\left|\int_\Omega h(x,u_n)(u_n-u)\,dx\right|&\leq C_2\left|\int_\Omega|u_n|^{\alpha(x)-2}u_n(u_n-u)\,dx\right|\leq C_2\int_\Omega|u_n|^{\alpha(x)-1}|u_n-u|dx\\
&\leq C_4\left||u_n|^{\alpha(x)-1}\right|_{\frac{\alpha(x)}{\alpha(x)-1}}|u_n-u|_{\alpha(x)}.
\end{aligned}$$

Because $\{u_n\}$ converges strongly to u in $L^{\alpha(x)}(\Omega)$, that is $|u_n-u|_{\alpha(x)}\to0$ as $n\to\infty$, we get

$$\int_\Omega h(x,u_n)(u_n-u)\,dx\to0.\tag{3.8}$$

Similarly, by (2.1) and (2.3), we have

$$\int_\Omega\frac{|u_n|^{r-2}u_n(u_n-u)}{|x|^r}dx\to0.\tag{3.9}$$

From Definition 4,

$$\langle\tau'(u_n),u_n-u\rangle\to0.$$

Thus,

$$\begin{aligned}
\langle\tau'(u_n),u_n-u\rangle=&\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx+\int_\Omega\frac{|u_n|^{r-2}u_n(u_n-u)}{|x|^r}\,dx\\
&-\lambda\int_\Omega h(x,u_n)(u_n-u)\,dx\to0.
\end{aligned}$$

From (3.8) and (3.9), we have

$$\Lambda' = \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \to 0. \tag{3.10}$$

Then by (3.10) and Proposition 2.5, the sequence $\{u_n\}$ converges strongly to u in $W_0^{1,p(x)}(\Omega)$. Therefore, τ satisfies the $(PS)_c$ condition. □

Proof of Theorem 3.2. Set $R_m = \{B \subset R;\ \gamma(B) \geq m\}$ and $d_m = \inf_{B\in R_m} \sup_{u\in B} \tau(u)$, $m = 1, 2, \cdots$, then we have

$$-\infty < d_1 \leq d_2 \leq \cdots \leq d_m \leq d_{m+1} \leq \cdots .$$

We will show that $d_m < 0$ for every $m \in \mathbb{N}$. Because $W_0^{1,p(x)}$ is a separable Banach space, for any $m \in \mathbb{N}$, let X_m be a m-dimensional linear subspace of $W_0^{1,p(x)}$ such that $X_m \subset C_0^\infty(\Omega)$. As the norms on X_m are equivalent, there exists $r_m \in (0, 1)$ such that $u \in X_m$ with $\|u\| \leq r_m$ implies $|u|_{L^\infty} \leq \delta$.
Set $S^m_{r_m} = \{u \in X_m;\ \|u\| = r_m\}$. By the compactness of $S^m_{r_m}$ and condition (H_1), there exists a constant $\rho_m > 0$ such that

$$\int_\Omega H(x,u)\ dx \geq \frac{C_2}{\beta^+}\int_\Omega |u|^{\beta(x)}\ dx \geq \rho_m, \quad \forall u \in S^m_{r_m}.$$

For $u \in S^m_{r_m}$ and $t \in (0, 1)$, we have

$$\begin{aligned}\tau(tu) &= \int_\Omega \frac{1}{p(x)}|t\nabla u|^{p(x)}\ dx + \frac{1}{r}\int_\Omega \frac{|tu|^r}{|x|^r}\ dx - \lambda\int_\Omega H(x,tu)\ dx \\ &< \frac{t^{p^-}}{p^-}\int_\Omega |\nabla u|^{p(x)}\ dx + \frac{t^r}{r}\int_\Omega \frac{|u|^r}{|x|^r}\ dx - \lambda\rho_m.\end{aligned}$$

Then

$$\lim_{t\to 0}\tau(tu) \leq \lim_{t\to 0}\left[\frac{t^{p^-}}{p^-}\int_\Omega |\nabla u|^{p(x)}\ dx + \frac{t^r}{r}\int_\Omega \frac{|u|^r}{|x|^r}dx - \lambda\rho_m\right] = -\lambda\rho_m.$$

Consider $t_m \in (0, 1)$ converging to zero such that

$$\lim_{t_m\to 0}\tau(t_m u) \leq -\lambda\rho_m.$$

Then

$$\tau(t_m u) \leq \frac{\lambda\rho_m}{2} - \lambda\rho_m \leq -\frac{1}{2}\lambda\rho_m.$$

Since $\|t_m u\| = t_m r_m$, so

$$\tau(t_m u) \leq -\frac{1}{2}\lambda\rho_m < 0, \quad \forall u \in S^m_{r_m},$$

and

$$\tau(u) \leq -\frac{1}{2}\lambda\rho_m < 0, \quad \forall u \in S^m_{t_m r_m},$$

so

$$\sup_{u\in S^m_{t_m r_m}} \tau(u) \leq -\frac{1}{2}\lambda\rho_m < 0.$$

It is well known that $\gamma(S^m_{t_m r_m}) = m$, $d_m \leq -\frac{1}{2}\lambda\rho_m < 0$. Since τ is even, so by Theorem 3.1, τ has least m pairs of different critical points. □

Corollary 3.3. *If* (3.6) *holds. Then there are infinitely many solutions for* (1.1).

Proof. Since m is arbitrary, so there are infinitely many critical points of τ. □

Example 1. The function $h(x,u) = u^4 \sin u$, satisfies hypotheses H_1 and H_2 and the following problem satisfies Theorem 3.1.

$$\begin{cases} -\triangle_{p(x)}\, u + \dfrac{u^{r-2}u}{|x|^r} = \lambda u^4 \sin u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases}$$

4 Closedness of the set of eigenfunctions

We study closedness of the set of eigenfunctions of the problem (1.1) in typical conditions. We consider the following problem:

$$\begin{cases} -div\left(|\nabla u|^{p-2}\nabla u\right) + \dfrac{|u|^{p-2}u}{|x|^p} = \lambda|u|^{q-2}u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{4.1}$$

where

$$1 < q \leq p < p^*. \tag{4.2}$$

The pair $(u,\lambda) \in W_0^{1,p} \times \mathbb{R}^+$ is a eigenpair of (4.1) if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u\,\nabla w + \frac{|u|^{p-2}u\,w}{|x|^p}\right) dx = \lambda \int_\Omega |u|^{q-2}u\,w\,dx, \tag{4.3}$$

for all $w \in W_0^{1,p}(\Omega)$. Let

$$\langle Au, w\rangle = \int_\Omega |u|^{q-2}u\,w\,dx, \tag{4.4}$$

and

$$\langle Bu, w\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\,\nabla w\,dx + \int_\Omega \frac{|u|^{p-2}u\,w}{|x|^p}\,dx. \tag{4.5}$$

Then (4.3) becomes $Bu = \lambda Au$.

Theorem 4.1. *The sets of eigenvalues of the problem* (4.1) *are closed.*

Proof. Let $\{(u_n,\mu_n)\}$ be a sequence of eigenpairs of (4.3) such that $\mu_n \to \mu$ for some $\mu \geq 0$. We show that there is u such that $u_n \to u$ and (u,μ) is a eigenpair of (4.1). First we prove that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Assume by contradiction for large enough n, by (4.3), we have:

$$\int_\Omega |\nabla u_n|^p\,dx + \int_\Omega \frac{|u_n|^p}{|x|^p}\,dx = \mu_n \int_\Omega |u_n|^q\,dx.$$

So

$$\|u_n\|^p \leq \mu_n \|u_n\|_q.$$

By Sobolev embedding we have:

$$\|u_n\|^p \leq \mu_n c_0 \|u_n\|^q.$$

It follows from (4.2), when we divide the last inequality by $\|u_n\|^q$ and pass to the limit as $n \to \infty$, we obtion a contradiction. Thus $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. We can assume $\|u_n\| = 1$ and thus $\{u_n\}$ has a weakly convergent subsequence. We may assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. By (4.3), (4.4) and (4.5) we have:

$$\langle B(u_n), u_n - u\rangle = \mu_n \langle A(u_n), u_n - u\rangle. \tag{4.6}$$

By (2.1) we have:

$$\langle A(u_n), u_n - u\rangle = |\int_\Omega |u_n|^{q-2} u_n (u_n - u)\, dx| \leq ||u_n|^{q-1}|_{\frac{q}{q-1}} |u_n - u|_q. \tag{4.7}$$

Because $\{u_n\}$ converges strongly to u in $L^q(\Omega)$, that is $|u_n - u|_q \to 0$, as $n \to \infty$, we get $\langle A(u_n), u_n - u\rangle \to 0$, as $n \to \infty$. From (4.6), $\langle B(u_n), u_n - u\rangle \to 0$, as $n \to \infty$. Then by Proposition (2.5), the sequence $\{u_n\}$ converges strongly to u in $W_0^{1,p}(\Omega)$.
To show that μ is an eigenvalue of (4.3) and u is an associated eigenfunction we need to show for any $w \in W_0^{1,p}(\Omega)$ as $n \to \infty$,

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla w\, dx \to \int_\Omega |\nabla u|^{p-2} \nabla u \nabla w\, dx, \tag{4.8}$$

$$\int_\Omega \frac{|u_n|^{p-2} u_n w}{|x|^p}\, dx \to \int_\Omega \frac{|u|^{p-2} u w}{|x|^p}\, dx \tag{4.9}$$

and

$$\int_\Omega |u_n|^{q-2} u_n w\, dx \to \int_\Omega |u|^{q-2} u w\, dx. \tag{4.10}$$

Let $t_n = |\nabla u_n|^{p-2} \nabla u_n$ and $t = |\nabla u|^{p-2} \nabla u$. Then as $u_n \to u$ in $W_0^{1,p}(\Omega)$, $t_n \to t$, a.e. in Ω and

$$\int_\Omega |t_n|^{\frac{p}{p-1}}\, dx \to \int_\Omega |t|^{\frac{p}{p-1}}\, dx.$$

It follows from Lemma A.1 of [17], that $t_n \to t$ in $L^{\frac{p}{p-1}}(\Omega)$. Thus, by (2.1) and (2.3), we obtain (4.8). Similarly, we have (4.9) and (4.10). □

Conclusion

Here, we proved multiplicity and infinitely of solutions for the problem (1.1) by using variational method and genus theory. We also proved the closedness of the set of eigenfunctions for problem (4.1), such that $p(x) \equiv p$.

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