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Some inequalities for the numerical radius and spectral norm for operators in Hilbert C^*-modules space

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Abstract. This paper introduces a new method for studying the numerical radius of bounded operators on Hilbert C^*-modules. Our approach leads to unique discoveries and expands existing theorems for bounded adjointable operators in Hilbert C^*-module spaces. Moreover, we find an upper bound for power of the numerical radius of $t^\alpha y s^{1-\alpha}$ under assumption $0 \leq \alpha \leq 1$. In fact, we prove

$$w_c\left(t^\alpha y s^{1-\alpha}\right) \leq |||y|||^r |||\alpha t^r + (1-\alpha)s^r|||$$

for all $0 \leq \alpha \leq 1$ and $r \geq 2$.

Keywords. Numerical radius, inner product space, C^*-algebra, A-module

1 Introduction

The notion of a Hilbert C^*-module initiated by Kaplansky [4] as a generalization of a Hilbert space in which the inner product takes its values in a C^*-algebra (see also [7, 8, 10, 11]).

Let $\mathfrak{A}$ be a C^*-algebra. A pre-Hilbert $\mathfrak{A}$-module or an inner product $\mathfrak{A}$-module is a complex linear space $\mathfrak{E}$ which is a right A-module with compatible scalar multiplication $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in \mathfrak{E}, a \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, together with an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathfrak{E} \times \mathfrak{E} \longrightarrow \mathfrak{A}$ that satisfies the following properties:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;

(ii) $\langle x, ya\rangle = \langle x, y\rangle a$;

(ii) $\langle x, y\rangle = \langle y, x\rangle^*$;

(iv) $\langle x, x\rangle \geq 0$; if $\langle x, x\rangle = 0$, then $x = 0$

for each $x, y, z \in \mathfrak{E}$, $a \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{C}$.

The notion of a left Hilbert $\mathfrak{A}$-module can be defined similarly. Note that the condition (i) is understood as a statement in the C^*-algebra $\mathfrak{A}$, where an element a is called positive if it

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can be represented as bb^* for some $b \in \mathfrak{A}$. The conditions (ii) and (iv) imply the inner product to be conjugate-linear in its first variable. Validity of a useful version of the classical Cauchy-Schwartz inequality follows that $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ defines a norm on $\mathfrak{E}$ making it into a normed right A-module. An inner product $\mathfrak{A}$-module $\mathfrak{E}$ which is complete with respect to the norm $\|x\|$ is called a Hilbert $\mathfrak{A}$-module or a Hilbert C^*-module over the C^*-algebra $\mathfrak{A}$. Every C^*-algebra $\mathfrak{A}$ is a Hilbert $\mathfrak{A}$-module under the $\mathfrak{A}$-valued inner product $\langle a, b\rangle = a^*b$ $(a, b \in \mathfrak{A})$. Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathfrak{E}$ and $\mathfrak{F}$ are Hilbert $\mathfrak{A}$-modules. We define $\mathscr{L}(\mathfrak{E}, \mathfrak{F})$ to be the set of all maps $t : \mathfrak{E} \longrightarrow \mathfrak{F}$ for which there is a map $t^* : \mathfrak{F} \longrightarrow \mathfrak{E}$ such that $\langle tx, y\rangle = \langle x, t^*y\rangle$, for all $x \in \mathfrak{E}$, $y \in \mathfrak{F}$. It is known that t must be a bounded $\mathfrak{A}$-linear map (that is, t is bounded linear map and $t(xa) = t(x)a$ for all $x \in \mathfrak{E}, a \in \mathfrak{A}$). If $\mathfrak{E} = \mathfrak{F}$, then $\mathscr{L}(\mathfrak{E})$ is a C^*-algebra together with the operator norm.

Suppose that $\mathfrak{A}$ is an abelian C^*-algebra. Recall that a character ψ on $\mathfrak{A}$ is a non-zero $*$-homomorphism $\psi : \mathfrak{A} \longrightarrow \mathbb{C}$ such that $\|\psi\| = 1$. We denote the set of all characters on $\mathfrak{A}$ by $\varpi(\mathfrak{A})$.

Throughout this paper assume that $\mathfrak{A}$ is abelian C^*-algebra.

2 Definitions and Complementary results

Lemma 2.1. *Let $\mathfrak{E}$ be a Hilbert $\mathfrak{A}$-module. Then for all $x, y \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$, we have*

(i) (Cauchy-Schwartz inequality) $|\psi(\langle x, y\rangle)| \le \psi(|x|)\,\psi(|y|)$.

(ii) (triangle inequality) $\psi(|x+y|) \le \psi(|x|) + \psi(|y|)$.

(iii) (Parallelogram Law) $\psi\left(|x+y|^2\right) + \psi\left(|x-y|^2\right) = 2\left(\psi\left(|x|^2\right) + \psi\left(|y|^2\right)\right)$.

Proof. (i) For every $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} 0 \le \psi(\langle x - \lambda y, x - \lambda y\rangle) &= \psi(\langle x, x\rangle) - \psi(\langle x, \lambda y\rangle) - \psi(\langle \lambda y, x\rangle) + \psi(\langle \lambda y, \lambda y\rangle) \\ &= \psi\left(|x|^2\right) - \bar{\lambda}\psi(\langle x, y\rangle) - \lambda\psi(y, x) + |\lambda|^2\psi\left(|y|^2\right) \\ &= \psi\left(|x|^2\right) - 2Re\left(\lambda\psi(\langle y, x\rangle)\right) + |\lambda|^2\psi\left(|y|^2\right). \end{aligned} \tag{2.1}$$

If $\psi(\langle x, y\rangle) = 0$, then the inequality is trivial. Suppose that $\psi(\langle x, y\rangle) \neq 0$, letting $\lambda = \frac{\psi\left(|x|^2\right)}{\psi(\langle y, x\rangle)}$ in (2.1) gives

$$0 \le -\psi\left(|x|^2\right) + \frac{\psi\left(|x|^4\right)\psi\left(|y|^2\right)}{|\psi(\langle x, y\rangle)|^2}.$$

Hence

$$\psi\left(|x|^2\right) \le \frac{\psi\left(|x|^4\right)\psi\left(|y|^2\right)}{|\psi(\langle x, y\rangle)|^2}$$

and this implies that $|\psi(\langle x, y\rangle)|^2 \le \psi\left(|x|^2\right)\psi\left(|y|^2\right)$ and so

$$|\psi(\langle x, y\rangle)| \le \psi(|x|)\,\psi(|y|).$$

(ii) By (i), we have

$$\begin{aligned}\psi\left(|x+y|^2\right) &= \psi\left(\langle x+y, x+y\rangle\right) = \psi\left(|x|^2\right) + 2Re\psi\left(\langle x, y\rangle\right) + \psi\left(|y|^2\right) \\ &\leq \psi\left(|x|^2\right) + 2\psi\left(|x|\right)\psi\left(|y|\right) + \psi\left(|y|^2\right) \\ &= \left(\psi\left(|x|\right) + \psi\left(|y|\right)\right)^2\end{aligned}$$

and so the result.

(iii) We have

$$\begin{aligned}\psi\left(|x+y|^2\right) + \psi\left(|x-y|^2\right) &= \psi\left(|x|^2\right) + 2Re\psi\left(\langle x, y\rangle\right) + \psi\left(|y|^2\right) \\ &+ \psi\left(|x|^2\right) - 2Re\psi\left(\langle x, y\rangle\right) + \psi\left(|y|^2\right) \\ &= 2\left(\psi\left(|x|^2\right) + \psi\left(|y|^2\right)\right).\end{aligned}$$

□

Definition 1. Let $t \in \mathscr{L}(\mathfrak{E})$ and $\psi \in \varpi(\mathfrak{A})$. Then

$$|||t||| := \sup\left\{\psi\left(|tx|\right) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1\right\}, \tag{2.2}$$

where $|x| = \langle x, x\rangle^{\frac{1}{2}}$.

It is known from [10] that $|||\cdot|||$ is a norm on $\mathscr{L}(\mathfrak{E})$. And if $\mathfrak{E}$ is a Hilbert space, then $\|t\| = |||t|||$. The following result was investigated in [10].

Lemma 2.2. *Let $t \in \mathscr{L}(\mathfrak{E})$. Then*

$$|||t||| = \sup\left\{|\psi\left(\langle x, ty\rangle\right)| : x, y \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = \psi(|y|) = 1\right\}.$$

Definition 2. Let $t \in \mathscr{L}(\mathfrak{E})$. Then the spectrum of t, denoted by $\sigma(t)$, is defined by

$$\sigma(t) = \left\{\lambda \in \mathbb{C} : t - \lambda 1 \text{ is not invertible}\right\}.$$

And $\lambda \in \mathbb{C}$ is called an eigenvalue of t if there is a non-zero vector $x \in \mathfrak{E}$ such that $tx = \lambda x$. Equivalently, λ is an eigenvalue if there is a vector $x \in \mathfrak{E}$ with $\psi\left(|x|\right) = 1$ such that $|||(t - \lambda 1)x||| = 0$.

Definition 3. $\lambda \in \mathbb{C}$ is called an approximate point spectrum of $t \in \mathscr{L}(\mathfrak{E})$ if there is a sequence $\{x_n\}$ of vectors in $\mathfrak{E}$ with $\psi\left(|x_n|\right) = 1$ such that $|||(t - \lambda 1)x_n||| \longrightarrow 0$, the set of approximate point spectrum is denoted by $\sigma_a(t)$.

Definition 4. If $t \in \mathscr{L}(\mathfrak{E})$, then the spectral radius of t is the number defined by

$$r(t) = \sup\left\{|\lambda| : \lambda \in \sigma(t)\right\}.$$

Clearly, $0 \leq r(t) \leq |||t|||$ and it follows from spectral theorem that $r(t^n) = (r(t))^n$. Moreover, it is well-known that $r(t) = \lim_{n \longrightarrow \infty} |||t^n|||^{\frac{1}{n}}$ (see [8]). Recall that a function f which maps A Hilbert $\mathfrak{A}$-module $\mathfrak{E}$ into $\mathbb{C}$ is called a functional. If f is in $\mathscr{L}(\mathfrak{E}, \mathbb{C})$, then f is called a linear functional on $\mathfrak{E}$.

Lemma 2.3. *If f is a bounded linear functional on a Hilbert $\mathfrak{A}$-module $\mathfrak{E}$, then there exists a unique $y \in \mathfrak{E}$ such that for all $x \in \mathfrak{E}$, $f(x) = \psi\left(\langle y, x\rangle\right)$. Moreover, $|||f||| = \psi\left(|y|\right)$.*

Proof. If $f=0$, take $y=0$. Suppose that $f\neq 0$. Then (f) is a proper closed subspace of $\mathfrak{E}$. Hence there exists a $v\neq 0$ in $(f)^{\perp}$.

Let $y=\alpha v$, where $\alpha=\frac{\overline{f(v)}}{\psi(|v|^2)}$. Then $y\perp(f)$ (because $v\perp(f)$) and $f(y)=\psi(\langle y,y\rangle)$ since

$$\begin{aligned} f(y) &= \alpha f(v)=\frac{|f(v)|^2}{\psi\left(|v|^2\right)} \text{ and} \\ \psi\left(\langle y,y\rangle\right) &= |\alpha|^2\psi\left(|v|^2\right)=\frac{|f(v)|^2}{\psi\left(|v|^4\right)}\psi\left(|v|^2\right)=\frac{|f(v)|^2}{\psi\left(|v|^2\right)}. \end{aligned}$$

Now, given $x\in\mathfrak{E}$, then x can be represented as $x=\beta y+z$, where $\beta\in\mathbb{C}$ and $z\in(f)$. From the previous arguments, we have

$$f(x)=f(\beta y)=\beta f(y)=\beta\psi\left(\langle y,y\rangle\right)=\psi\left(\langle y,\beta y+z\rangle\right)=\psi\left(\langle y,x\rangle\right).$$

To show that y is unique, suppose there is $w\in\mathfrak{E}$ such that $f(x)=\psi\left(\langle w,x\rangle\right)$ for all $x\in\mathfrak{E}$. Then

$$0=f(x)-f(x)=\psi\left(\langle y-w,x\rangle\right) \text{ for all } x\in\mathfrak{E}.$$

In particular, $\psi\left(\langle y-w,y-w\rangle\right)=0$ and so $y=w$.

Finally, for each $y\in\mathfrak{E}$ the functional f defined on $\mathfrak{E}$ is linear. Moreover

$$|f(x)|=|\psi\left(y,x\right)|\leq\psi\left(|x|\right)\psi\left(|y|\right) \text{ for all } x\in\mathfrak{E}.$$

Thus f is bounded and $|||f|||\leq\psi\left(|y|\right)$. Since

$$|||f|||\psi\left(|y|\right)\geq|f(y)|=\psi\left(\langle y,y\rangle\right)=\psi\left(|y|^2\right)$$

and so $|||f|||\geq\psi\left(|y|\right)$ and consequently $|||f|||=\psi\left(|y|\right)$. □

Lemma 2.4. *[10] If $t\in\mathscr{L}(\mathfrak{E})$, then*

$$|||t|||=\sup\left\{|\psi\left(x,tx\right)|: x\in\mathfrak{E},\psi\in\varpi(\mathfrak{A}),\psi\left(|x|\right)=1\right\}.$$

The following results are very useful in the sequel.

Proposition 2.1. *[11] Let $t\in\mathscr{L}(\mathfrak{E})$ and $\psi\in\varpi(\mathfrak{A})$. The following statements are equivalent:*

(a) $\psi\left(\langle x,tx\rangle\right)=0$ for every $x\in\mathfrak{E}$ with $\psi\left(|x|\right)=1$;

(b) $\psi\left(\langle x,tx\rangle\right)=0$ for every $x\in\mathfrak{E}$.

Proposition 2.2. *[11] For every $t\in\mathscr{L}(\mathfrak{E})$, the following assertions hold.*

(i) $t=0$ if and only if $\psi\left(\langle x,tx\rangle\right)=0$ for every $x\in\mathfrak{E}$.

(ii) t is positive if and only if $\psi\left(\langle x,tx\rangle\right)$ is positive for every $x\in\mathfrak{E}$.

(iii) t is self-adjoint if and only if $\psi\left(\langle x,tx\rangle\right)$ is self-adjoint for every $x\in\mathfrak{E}$.

(iv) $t=0$ if and only if $\psi\left(\langle x,tx\rangle\right)=0$ for every $x\in\mathfrak{E}$ and $\psi\in\varpi(\mathfrak{A})$.

(v) $Re\psi\left(\langle x,tx\rangle\right)=\psi\left(\langle x,Re(t)x\rangle\right)$ for all $x\in\mathfrak{E}$.

Lemma 2.5. *[10] If $t \in \mathscr{L}(\mathfrak{E})$ is self-adjoint, then*

$$|||t||| = \sup\left\{|\psi\left(\langle x, tx\rangle\right)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi\left(|x|\right) = 1\right\}.$$

Theorem 2.1. *Suppose $t \in \mathscr{L}(\mathfrak{E})$ is self-adjoint.*

(i) Let

$$\lambda = \inf\left\{\psi\left(\langle x, tx\rangle\right) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi\left(|x|\right) = 1\right\}.$$

If there exists an $x_0 \in \mathfrak{E}$ such that $\psi\left(|x_0|\right) = 1$ and $\lambda = \psi\left(\langle x_0, tx_0\rangle\right)$, then λ is an eigenvalue of t with corresponding eigenvector x_0.

(ii) Let

$$\mu = \sup\left\{\psi\left(\langle x, tx\rangle\right) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi\left(|x|\right) = 1\right\}.$$

If there exists an $x_1 \in \mathfrak{E}$ such that $\psi\left(|x_1|\right) = 1$ and $\mu = \psi\left(\langle x_1, tx_1\rangle\right)$, then μ is an eigenvalue of t with corresponding eigenvector x_1.

Proof. (i) For every $\alpha \in \mathbb{C}$ and every $y \in \mathfrak{E}$, it follows from the definition of λ that

$$\psi\left(\langle x_0 + \alpha y, t(x_0 + \alpha y)\rangle\right) \geq \lambda\psi\left(\langle x_0 + \alpha y, x_0 + \alpha y\rangle\right).$$

Expanding the inner product and setting $\lambda = \psi\left(\langle x_0, tx_0\rangle\right)$, we get the inequality

$$2Re\alpha\psi\left(\langle(t - \lambda 1)x_0, y\rangle\right) + |\alpha|^2\psi\left(\langle y, (t - \lambda 1)y\rangle\right) \geq 0.$$

Taking $\alpha = \overline{r\psi\left(\langle(t - \lambda 1)x_0, y\rangle\right)}$, where $r \in \mathbb{R}$, it follows that

$$2r\left|\psi\left(\langle(t - \lambda 1)x_0, y\rangle\right)\right|^2 + r^2\left|\psi\left(\langle(t - \lambda 1)x_0, y\rangle\right)\right|^2\psi\left(\langle y, (t - \lambda 1)y\rangle\right) \geq 0.$$

Since r is arbitrary, it follows that $\psi\left(\langle(t - \lambda 1)x_0, y\rangle\right) = 0$ and since y is arbitrary, we have $tx_0 = \lambda x_0$ as required.

(ii) The second statement of the theorem follows from part(i) applied to the self-adjoint $-A$. □

Definition 5. An operator $t \in \mathscr{L}(\mathfrak{E}, \mathfrak{F})$ is said to be compact if for each sequence $\{x_n\}$ in $\mathfrak{E}$ with $\psi\left(|x_n|\right) = 1$ and $\psi \in \varpi(\mathfrak{A})$, the sequence $\{tx_n\}$ has a subsequence which converges in $\mathfrak{F}$.

Theorem 2.2. *If $t \in \mathscr{L}(\mathfrak{E})$ is compact and self-adjoint, then at least one the numbers $|||t|||$ or $-|||t|||$ is an eigenvalue of t.*

Proof. The result is trivial if $t = 0$. Assume that $t \neq 0$, since

$$|||t||| = \sup\left\{|\psi\left(\langle x, tx\rangle\right)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi\left(|x|\right) = 1\right\}$$

then there exists a sequence $\{x_n\}$ in $\mathfrak{E}$ with $\psi\left(|x_n|\right) = 1$ and a real number λ such that $|\lambda| = |||t||| \neq 0$ and $\psi\left(\langle x_n, tx_n\rangle\right) \longrightarrow \lambda$.
Now

$$\begin{aligned} 0 \leq \psi\left(|tx_n - \lambda x_n|^2\right) &= \psi\left(|tx_n|^2\right) - 2\lambda\psi\left(x_n, tx_n\right) + \lambda^2 \\ &\leq 2\lambda^2 - 2\lambda\psi\left(x_n, tx_n\right) \longrightarrow 2\lambda^2 - 2\lambda^2 = 0 \end{aligned}$$

and so

$$tx_n - \lambda x_n \longrightarrow 0. \tag{2.3}$$

Since t is compact, there exists a subsequence $\{tx_{n'}\}$ of $\{tx_n\}$ which converges to some $y \in \mathfrak{E}$. Thus (2.3) implies that $x_{n'} \longrightarrow \frac{1}{\lambda}y$ and by the continuity of t, $y = \lim_{n'\longrightarrow\infty} tx_{n'} = \frac{1}{\lambda}ty$. Hence $ty = \lambda y$ and $y \neq 0$. Since

$$\psi(|y|) = \lim_{n'\longrightarrow\infty} \psi(|\lambda x_{n'}|) = |\lambda| = |||t|||$$

and so λ is an eigenvalue of t, as required. □

Definition 6. Let $t \in \mathscr{L}(\mathfrak{E})$. Then the numerical range of t is defined by

$$W_c(t) = \{\psi(\langle x, tx\rangle) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \text{ and } \psi(|x|) = 1\}.$$

The next result represent some of the basic properties for the numerical range (see [10]).

Lemma 2.6. *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then the following assertions hold.*

(i) $W_c(t^*) = \overline{W_c(T)}$, *where* $\overline{W_c(T)}$ *is the conjugate of* $W_c(t)$.

(ii) $W_c(T) \subseteq \mathbb{R}$ *if and only if t is a self-adjoint.*

(iii) If u is unitary, then $W_c(u^*tu) = W_c(t)$.

(iv) If $\alpha, \beta \in \mathbb{C}$, then $W_c(\alpha t + \beta 1) = \alpha W_c(t) + \beta$.

(v) $W_c(t + s) \subset W_c(t) + W_c(s)$.

Definition 7. Let $t \in \mathscr{L}(\mathfrak{E})$. Then the numerical radius of t is defined by

$$w_c(t) = \sup\{|\psi(\langle x, tx\rangle)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \text{ and } \psi(|x|) = 1\}.$$

It is easy to show that $w_c(\cdot)$ is a norm on $\mathscr{L}(\mathfrak{E})$.

The following is useful in the sequel.

Lemma 2.7. *If $\mathfrak{E}$ is a Hilbert $\mathfrak{A}$-module, then for every $\psi \in \varpi(\mathfrak{A})$, $x \in \mathfrak{E}$,*

$$\psi(|\langle x, tx\rangle|) \leq \psi\left(|x|^2\right) w_c(t)$$

Theorem 2.3. *If $t \in \mathscr{L}(\mathfrak{E})$ is normal, then*

$$|||t||| = r(t) = w_c(t).$$

Proof. First we want to show $|||t^n||| = |||t|||^n$. by induction, for $n = 1$ the equality is trivial. Assume that its true for k such that $1 \leq k \leq n$.

$$\begin{aligned} |||t^n x|||^2 &= \psi(\langle t^n x, t^n x\rangle) = \psi\left(\langle t^* t^n x, t^{n-1} x\rangle\right) \\ &\leq |||t^* t^n x||| |||t^{n-1} x||| \leq |||t^{n+1} x||| |||t^{n-1}||| \psi\left(|x|^2\right) \ (t \text{ is normal}). \end{aligned}$$

and so, $|||t^n|||^2 \leq \left|\left|\left|t^{n+1}\right|\right|\right| \left|\left|\left|t^{n-1}\right|\right|\right|$. But $|||t^n||| = |||t|||^n$ for all k such that $1 \leq k \leq n$ and this implies that $|||t|||^{2n} \leq \left|\left|\left|t^{n+1}\right|\right|\right| |||t|||^{n-1}$ and hence $|||t^n||| = |||t|||^n$ for all $n \in \mathbb{N}$.

Now, $r(t) = \lim_{n\longrightarrow\infty} |||t^n|||^{\frac{1}{n}} = |||t|||$. But its known that $r(t) \leq w_c(t) \leq |||t|||$ and so we have the desired equality. □

Lemma 2.8. *If $t \in \mathscr{L}(\mathfrak{E})$ is normal and $\lambda \notin \sigma(t)$, then*

$$|||(t-\lambda 1)^{-1}||| = \frac{1}{d(\lambda, \sigma(t))},$$

where $d(\lambda, \sigma(t))$ is the distance from λ to $\sigma(t)$.

Proof. we have

$$r((t-\lambda 1)^{-1}) = \sup\left\{\frac{1}{|\mu-\lambda|} : \mu \in \sigma(t)\right\} = \frac{1}{\inf\{|\mu-\lambda : \mu \in \sigma(t)\}} = \frac{1}{d(\lambda, \sigma(t))}.$$

So, if t is normal, then $(t-\lambda 1)^{-1}$ is normal for $\lambda \notin \sigma(t)$ and hence

$$|||(t-\lambda 1)^{-1}||| = r((t-\lambda 1)^{-1}) = \frac{1}{d(\lambda, \sigma(t))}.$$

□

Theorem 2.4. *If $t \in \mathscr{L}(\mathfrak{E})$ is normal, then $\overline{W_c(t)} = Conv\,\sigma(t)$, where $Conv\,\sigma(t)$ is the convex hull of the spectrum of t.*

Proof. We need only to show $\overline{W_c(t)} \subset \overline{Conv\,\sigma(t)}$. To see this, it sufficient to show that any closed half-plane which contains $\sigma(t)$ also contain $\overline{W_c(t)}$. By translation and rotation this reduces to shown that $Re\sigma(t) \leq 0$ implies $Re\overline{W_c(t)} \leq 0$.

Let $x \in \mathfrak{E}$ such that $\psi(|x|) = 1$ and $tx = (a+ib)x + y$ with a, b are real and x orthogonal to y. Now from Lemma 2.8,we have $|||(t-c)x||| \geq dist(c, \sigma(t)) \geq c$ for all $c > 0$. Indeed, if $c \notin \sigma(t)$, then $|||(t-c)^{-1}x||| |||(t-c)x||| \geq |||(t-c)^{-1}(t-c)x||| = \psi(|x|) = 1$ and so $|||(t-c)x||| \geq \frac{1}{|||(t-c)^{-1}|||} = d(c, \sigma(t)) \geq c$. So that

$$\begin{aligned} c^2 &\leq |||(t-c)x|||^2 = |||(a-c)x + ibx + y|||^2 = |||(a-c)x + ibx|||^2 + \psi\left(|y|^2\right) \\ &= (a-c)^2 + b^2 + \psi\left(|y|^2\right). \end{aligned}$$

Consequently,

$$2ac \leq a^2 + b^2 + \psi\left(|y|^2\right).$$

Since this hold for all $c > 0$. This implies that $Re\psi(x, tx) = a \leq 0$ as required. □

3 A numerical radius inequality

In order to prove our desired numerical radius inequality, we need the following lemmas. The first lemma, which is a generalized Schwartz inequality, can be found in [11, Corollary 3.11]

Lemma 3.1. *(Geralized-Cauchy Schwartz) For $\psi \in \varpi(\mathfrak{A})$, $\psi(\langle\cdot,\cdot\rangle)$ is a semi-inner product. Suppose that $t \in \mathscr{L}(\mathfrak{E})$ and $\alpha \in [0,1]$, then*

$$|\psi(\langle x, ty\rangle)|^2 \leq \psi\left(\langle x, |t|^{2\alpha}x\rangle\right)\psi\left(\left\langle y, |t^*|^{2(1-\alpha)}y\right\rangle\right), \quad x, y \in \mathfrak{E}.$$

If $\alpha = \frac{1}{2}$, then

$$|\psi(\langle x, ty\rangle)|^2 \leq \psi(\langle x, |t|x\rangle)\,\psi(\langle y, |t^*|y\rangle), \quad x, y \in \mathfrak{E}.$$

Here $|t|$ stands for the positive (semi-definite) operator $(t^*t)^{\frac{1}{2}}$.

The second lemma contains a special case of a more general norm inequality that is equivalent to some Löwner–Heinz type inequalities. See [6].

Lemma 3.2. *If $t, s \in \mathscr{L}(\mathfrak{E})$ are positive, then*

$$\left|\left|\left| t^{\frac{1}{2}} s^{\frac{1}{2}} \right|\right|\right| \leq |||ts|||^{\frac{1}{2}}.$$

The third lemma contains a recent norm inequality for sums of positive operators that is sharper than the triangle inequality.

Lemma 3.3. *If $t, s \in \mathscr{L}(\mathfrak{E})$ are positive, then*

$$|||t+s||| \leq \frac{1}{2}\left(|||t||| + |||s||| + \sqrt{(|||t||| - |||s|||)^2 + 4\left|\left|\left| t^{\frac{1}{2}} s^{\frac{1}{2}} \right|\right|\right|^2} \right). \tag{3.1}$$

Now we are in a position to present our refined numerical radius inequality.

Theorem 3.1. *If $t \in \mathscr{L}(\mathfrak{E})$, then*

$$w_c(t) \leq \frac{1}{2}\left(|||t||| + \left|\left|\left| t^2 \right|\right|\right|^{\frac{1}{2}} \right). \tag{3.2}$$

Proof. By Lemma 3.1 and by the arithmetic-geometric mean inequality, we have for every $x \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$,

$$\begin{aligned} |\psi(\langle x, tx\rangle)| &\leq \psi(\langle x, |t|x\rangle)^{\frac{1}{2}} \psi(\langle x, |t^*|x\rangle)^{\frac{1}{2}} \\ &\leq \frac{1}{2}(\psi(\langle x, |t|x\rangle) + \psi(\langle x, |t^*|x\rangle)) \\ &= \frac{1}{2}(\psi(\langle x, (|t| + |t^*|)x\rangle)). \end{aligned}$$

Thus

$$\begin{aligned} w_c(t) &= \sup\{|\psi(\langle x, tx\rangle)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1\} \\ &\leq \frac{1}{2}\sup\{(\psi(\langle x, (|t| + |t^*|)x\rangle)) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1\} \\ &= \frac{1}{2}\left|\left|\left| |t| + |t^*| \right|\right|\right|. \end{aligned} \tag{3.3}$$

Applying Lemmas 3.2 and 3.3 to the positive operators $|t|$ and $|t^*|$, and using the facts that $|||\,|t|\,||| = |||\,|t^*|\,||| = |||t|||$ and $|||\,|t||t^*|\,||| = \left|\left|\left| t^2 \right|\right|\right|$, we have

$$\left|\left|\left| |t| + |t^*| \right|\right|\right| \leq |||t||| + \left|\left|\left| t^2 \right|\right|\right|^{\frac{1}{2}}. \tag{3.4}$$

The desired inequality (3.2) now follows from (3.3) and (3.4). □

To see that (3.2) is a refinement of the second inequality in [11, Theorem 2.13], one has to recall that $\left|\left|\left| t^2 \right|\right|\right| \leq |||t|||^2$ for every $t \in \mathscr{L}(\mathfrak{E})$.
It has been mentioned in [11, Theorem 2.17] that if $t \in \mathscr{L}(\mathfrak{E})$ is such that $t^2 = 0$, then $w_c(t) = \frac{1}{2}|||t|||$. This can be easily seen as an immediate consequence of the first inequality in [11, Theorem 2.13] and the inequality (3.2).

Corollary 3.2. *If $t \in \mathscr{L}(\mathfrak{E})$ is such that $t^2 = 0$, then $w_c(t) = \frac{1}{2}|||t|||$.*

Proof. Combining the first inequality [11, Theorem 2.13] and the inequality (3.2), we have

$$\frac{1}{2}|||t||| \leq w_c(t) \leq \frac{1}{2}\left(|||t||| + |||t^2|||^{\frac{1}{2}}\right) \tag{3.5}$$

for every $t \in \mathscr{L}(\mathfrak{E})$. Thus, if $t^2 = 0$, then $w_c(t) = \frac{1}{2}|||t|||$ as required. □

The following result is another consequence of the inequality (3.2).

Corollary 3.3. *If $t \in \mathscr{L}(\mathfrak{E})$ is such that $w_c(t) = |||t|||$, then $|||t^2||| = |||t|||^2$.*

Proof. It follows from the inequality (3.2) that

$$2w_c(t) \leq |||t||| + |||t^2|||^{\frac{1}{2}}$$

for every $t \in \mathscr{L}(\mathfrak{E})$. Thus, if $w_c(t) = |||t|||$, then $|||t||| \leq |||t^2|||^{\frac{1}{2}}$, and hence $|||t|||^2 \leq |||t^2|||$. But the reverse inequality is always true. Thus $|||t^2||| = |||t|||^2$ as required. □

4 Power Inequalities For The Numerical Radius

To prove our generalized numerical radius, we need several well-known lemmas.

Lemma 4.1. *[9] Let $a, b \geq 0$, $0 \leq \alpha \leq 1$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(i) $a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}$;

(ii) $ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}$;

for all $r \geq 1$.

Lemma 4.2. *Let $t, s \in \mathscr{L}(\mathfrak{E})$, and let f and g be non-negative functions on $[0, \infty)$ which are continuous such that $f(\tau)g(\tau) = \tau$ for all $\tau \in [0, \infty)$ Then*

$$|\psi\left(y, tx\right)| \leq |||f(|t|)x||| \, |||g(|t^*|)y|||,$$

for all $x, y \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$.

Lemma 4.3. *[11, Hölder-McCarthy inequality in Hilbert C^*-Modules] Let $t \in \mathscr{L}(\mathfrak{E})$, $t > 0$, then for every $\psi \in \mathfrak{S}(\mathfrak{A})$*

(i) $(\psi\left\langle x, tx\right\rangle_{\mathfrak{A}})^r \leq \|x\|^{2(1-r)}\, \psi\left\langle x, t^r x\right\rangle_{\mathfrak{A}}$ for $r > 1$ and

(ii) $(\psi\left\langle x, tx\right\rangle_{\mathfrak{A}})^r \geq \|x\|^{2(1-r)}\, \psi\left\langle x, t^r x\right\rangle_{\mathfrak{A}}$ for $0 < r \leq 1$

Theorem 4.1. *Let $t \in \mathscr{L}(\mathfrak{E})$ be self-adjoint. Then*

$$w_c^2(t) \leq \frac{1}{2}\left(w_c(t^2) + |||t|||^2\right).$$

Proof. We recall the following refinement of the Cauchy–Schwartz inequality obtained by Dragomir in [1] with slight modification. It says that

$$\begin{aligned}\psi(|u|)\,\psi(|v|) &\geq |\psi(\langle u,v\rangle)-\psi(\langle u,z\rangle)\,\psi(\langle z,v\rangle)|+|\psi(\langle u,z\rangle)\,\psi(\langle z,v\rangle)|\\ &\geq |\psi(\langle u,v\rangle)|,\end{aligned}\tag{4.1}$$

for all $u,v,z\in\mathfrak{E}$ with $\psi(|z|)=1$. From inequality (4.1), we deduce that

$$|\psi(\langle u,z\rangle)\,\psi(\langle z,v\rangle)|\leq\frac{1}{2}\left(\psi(|u|)\,\psi(|v|)+|\psi(\langle u,v\rangle)|\right).\tag{4.2}$$

In the inequality (4.2), put $z=x$ with $\psi(|x|)=1$, $u=t^*x$ and $v=tx$, we get

$$|\psi(\langle t^*x,x\rangle)\,\psi(\langle x,tx\rangle)|\leq\frac{1}{2}\left(\psi(|t^*x|)\,\psi(|tx|)+|\psi(\langle t^*x,tx\rangle)|\right).$$

Hence

$$|\psi(\langle x,tx\rangle)|^2\leq\frac{1}{2}\left(\psi(|tx|)^2+\psi\left(\langle x,t^2x\rangle\right)\right).\tag{4.3}$$

Taking the supremum over all vectors $x\in\mathfrak{E}$ with $\psi(|x|)=1$, we get the desired result. □

Theorem 4.2. *Let $t\in\mathscr{L}(\mathfrak{E})$ and let f and g be as in Lemma 4.2. Then we have*

$$w_c^2(t)\leq\frac{1}{2}\left(\|\!|t|\!\|^2+\left\|\!\left|\frac{1}{p}f^p(|t|^2)+\frac{1}{q}g^q(|t|^2)\right|\!\right\|\right)\tag{4.4}$$

for all $p\geq q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $x\in\mathfrak{E}$ such that $\psi(|x|)=1$. We have

$$\begin{aligned}\left|\psi\left(x,t^2x\right)\right| &\leq \|\!|f(|t^2|x)|\!\|\,\|\!|g(|(t^*)^2|)|\!\| \quad \text{(by Lemma 4.2)}\\ &= \psi\left(x,f^2(|t^2|)x\right)^{\frac{1}{2}}\psi\left(x,g^2(|(t^*)^2|)x\right)^{\frac{1}{2}}\\ &\leq \frac{1}{p}\psi\left(x,f^2(|t^2|)x\right)^{\frac{p}{2}}+\frac{1}{q}\psi\left(x,g^2(|(t^*)^2|)x\right)^{\frac{q}{2}} \quad \text{(by Lemma 4.1(ii))}\\ &\leq \frac{1}{p}\psi\left(x,f^p(|t^2|)x\right)+\frac{1}{q}\psi\left(x,g^q(|(t^*)^2|)x\right) \quad \text{(by Lemma 4.3)}\\ &= \psi\left(\left\langle x,\left(\frac{1}{p}f^p(|t^2|)+\frac{1}{q}g^q(|(t^*)^2|)\right)x\right\rangle\right).\end{aligned}$$

It follows from the inequality (4.3)that

$$|\psi(\langle x,tx\rangle)|^2\leq\frac{1}{2}\left(\psi(|tx|)^2+\psi\left(\left\langle x,\left(\frac{1}{p}f^p(|t^2|)+\frac{1}{q}g^q(|(t^*)^2|)\right)x\right\rangle\right)\right).$$

Taking the supremum over all vectors $x\in\mathfrak{E}$ with $\psi(|x|)=1$, we get the desired result. □

The following lemma is useful in the sequel.

Lemma 4.4. *[11] Let $t\in\mathscr{L}(\mathfrak{E})$ and $\psi\in\varpi(\mathfrak{A})$ then for every $x\in\mathfrak{E}$*

$$Re\psi(\langle x,tx\rangle)=\psi(\langle x,Re(t)x\rangle),$$

where $Re(t)$ denotes the real part of the operator $t\in\mathscr{L}(\mathfrak{E})$.

Theorem 4.3. *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then*

$$w_c(s^*t) \leq \frac{1}{4}\left|\!\left|\!\left| |t^*|^2 + |s^*|^2 \right|\!\right|\!\right| + \frac{1}{2} w_c(ts^*).$$

Proof. First of all, we note that

$$w_c(t) = \sup_{\theta \in \mathbb{R}} \left|\!\left|\!\left| Re\left(e^{i\theta} t\right) \right|\!\right|\!\right|. \tag{4.5}$$

For every vector $x \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$ with $\psi(|x|) = 1$, we have

$$\begin{aligned}
Re\psi\left(\langle x, e^{i\theta} s^* t x\rangle\right) &= Re\psi\left(sx, e^{i\theta} tx\right) \\
&= \frac{1}{4}\left|\!\left|\!\left|\left(e^{i\theta} t + s\right) x\right|\!\right|\!\right|^2 - \frac{1}{4}\left|\!\left|\!\left|\left(e^{i\theta} t + s\right) x\right|\!\right|\!\right|^2 \quad \text{(by Polarization identity)} \\
&\leq \frac{1}{4}\left|\!\left|\!\left|\left(e^{i\theta} t + s\right) x\right|\!\right|\!\right|^2 \leq \frac{1}{4}\left|\!\left|\!\left| e^{i\theta} t + s\right|\!\right|\!\right|^2 \\
&= \frac{1}{4}\left|\!\left|\!\left|\left(e^{-i\theta} t^* + s^*\right)\right|\!\right|\!\right|^2 \quad (\text{since } |\!|\!| y |\!|\!| = |\!|\!| y^* |\!|\!|) \\
&= \frac{1}{4}\left|\!\left|\!\left|\left(e^{-i\theta} t^* + s^*\right)^* \left(e^{-i\theta} t^* + s^*\right)\right|\!\right|\!\right| \quad \left(\text{since } |\!|\!| y |\!|\!|^2 = |\!|\!| y^* y |\!|\!|\right) \\
&= \frac{1}{4}\left|\!\left|\!\left| tt^* + ss^* + e^{i\theta} ts^* + e^{-i\theta} st^*\right|\!\right|\!\right| \\
&\leq \frac{1}{4}|\!|\!| tt^* + ss^* |\!|\!| + \frac{1}{2}\left|\!\left|\!\left| Re(e^{i\theta} ts^*)\right|\!\right|\!\right|
\end{aligned}$$

Taking the supremum over all vectors $x \in \mathfrak{E}$ with $\psi(|x|) = 1$, we obtain

$$w_c(s^*t) \leq \frac{1}{4}\left|\!\left|\!\left| |t^*|^2 + |s^*|^2 \right|\!\right|\!\right| + \frac{1}{2} w_c(ts^*)$$

as required. □

The following theorem gives us a new bound for powers of the numerical radius.

Theorem 4.4. *Suppose $t, s, y \in \mathscr{L}(\mathfrak{E})$ such that t, s are positive. Then*

$$w_c\left(t^\alpha y s^\alpha\right) \leq |\!|\!| y |\!|\!|^r \left|\!\left|\!\left| \frac{1}{p} t^{pr} + \frac{1}{q} s^{qr} \right|\!\right|\!\right|^\alpha$$

for all $0 \leq \alpha \leq 1$, $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$.

Proof. For every vector $x \in \mathfrak{E}$ with $\psi(|x| = 1)$, $\psi \in \varpi(\mathfrak{A})$, we have

$$\begin{aligned}
|\psi(\langle x, t^\alpha y s^\alpha x\rangle)|^r &= |\psi(\langle t^\alpha x, y s^\alpha x\rangle)|^r \\
&\leq |\!|\!| y |\!|\!|^r |\!|\!| t^\alpha x |\!|\!|^r |\!|\!| s^\alpha x |\!|\!|^r \\
&\leq |\!|\!| y |\!|\!|^r \psi\left(\left\langle x, t^{2\alpha} x\right\rangle^{\frac{r}{2}}\right) \psi\left(\left\langle x, s^{2\alpha} x\right\rangle^{\frac{r}{2}}\right) \\
&\leq |\!|\!| y |\!|\!|^r \left(\frac{1}{p}\psi\left(\langle x, t^{2\alpha} x\rangle\right)^{\frac{rp}{2}} + \frac{1}{q}\psi\left(\langle x, s^{2\alpha} x\rangle\right)^{\frac{qr}{2}}\right) \quad \text{(by Lemma 4.1(ii))} \\
&\leq |\!|\!| y |\!|\!|^r \left(\frac{1}{p}\psi\left(\langle x, t^{pr} x\rangle\right)^\alpha + \frac{1}{q}\psi\left(\langle x, s^{qr} x\rangle\right)^\alpha\right) \quad \text{(by Lemma 4.3)}
\end{aligned}$$

$$\begin{aligned}&\leq \|y\|^r\left(\frac{1}{p}\psi\left(\langle x,t^{pr}x\rangle\right)+\frac{1}{q}\psi\left(\langle x,s^{qr}x\rangle\right)\right)^{\alpha} \quad \text{(by the concavity of } f(\tau)=\tau^{\alpha})\\ &= \|y\|^r\psi\left(\left\langle x,\left(\frac{1}{p}t^{pr}+\frac{1}{q}t^{qr}\right)x\right\rangle\right)^{\alpha}.\end{aligned}$$

Taking the supremum over all vectors $x\in\mathfrak{E}$ with $\psi(|x|)=1$, we obtain the desired result. □

Our next result is to find an upper bound for power of the numerical radius of $t^{\alpha}ys^{1-\alpha}$ under assumption $0\leq\alpha\leq1$.

Theorem 4.5. *Suppose $t,s,y\in\mathscr{L}(\mathfrak{E})$ such that t,s are positive. Then*

$$w_c\left(t^{\alpha}ys^{1-\alpha}\right)\leq\|y\|^r\|\alpha t^r+(1-\alpha)s^r\|$$

for all $0\leq\alpha\leq1$ and $r\geq2$.

Proof. For every vector $x\in\mathfrak{E}$ with $\psi(|x|=1)$, $\psi\in\varpi(\mathfrak{A})$, we have

$$\begin{aligned}\left|\psi\left(\left\langle x,t^{\alpha}ys^{1-\alpha}x\right\rangle\right)\right|^r &= \left|\psi\left(\left\langle t^{\alpha}x,ys^{1-\alpha}x\right\rangle\right)\right|^r\\ &\leq \|y\|^r\|t^{\alpha}x\|^r\left\|s^{1-\alpha}x\right\|^r\\ &\leq \|y\|^r\psi\left(\left\langle x,t^{2\alpha}x\right\rangle\right)^{\frac{r}{2}}\psi\left(\left\langle x,s^{2(1-\alpha)}x\right\rangle\right)^{\frac{r}{2}}\\ &\leq \|y\|^r\psi\left(\langle x,t^rx\rangle\right)^{\alpha}\psi\left(\langle x,s^rx\rangle\right)^{1-\alpha} \quad \text{(by Lemma 4.3)}\\ &\leq \|y\|^r\psi\left(\langle x,(\alpha t^r+(1-\alpha)s^r)x\rangle\right) \quad \text{(by Lemma 4.1(i))}.\end{aligned}$$

Hence

$$\left|\psi\left(\left\langle x,t^{\alpha}ys^{1-\alpha}x\right\rangle\right)\right|^r\leq\|y\|^r\psi\left(\langle x,(\alpha t^r+(1-\alpha)s^r)x\rangle\right). \tag{4.6}$$

Taking the supremum over all vectors $x\in\mathfrak{E}$ with $\psi(|x|)=1$, we obtain the desired result. □

Remark 1. Note that our inequality in the previous theorem is a generalization of the second inequality in Theorem 2.13 of [11] when we set $s=t=1$.

Now assume that $t,s,y\in\mathscr{L}(\mathfrak{E})$. The Heinz mean for matrices are defined by

$$H_{\alpha}(t,s)=\frac{t^{\alpha}ys^{1-\alpha}+t^{1-\alpha}ys^{\alpha}}{2}$$

in which $\alpha\in[0,1]$ and $t,s\geq0$, see [7].

The goal of the following result is to find a numerical radius inequality for Heinz means. For this purpose, we use Theorem 4.5 and the convexity of function $f(\tau)=\tau^r$ $(r\geq1)$.

Theorem 4.6. *Suppose $t,s,y\in\mathscr{L}(\mathfrak{E})$ such that t,s are positive. Then*

$$\begin{aligned}w_c^r\left(t^{\frac{1}{2}}ys^{\frac{1}{2}}\right) &\leq w_c^r\left(\frac{t^{\alpha}ys^{1-\alpha}+t^{1-\alpha}ys^{\alpha}}{2}\right)\\ &\leq \|y\|^r w_c\left(\frac{t^r+s^r}{2}\right)\\ &\leq \frac{\|y\|^r}{2}\left(\|\alpha t^r+(1-\alpha)s^r\|+\|\alpha s^r+(1-\alpha)t^r\|\right)\end{aligned}$$

for all $r\geq2$ and $\alpha\in[0,,1]$.

To prove Theorem 4.6, we need the following lemma.

Lemma 4.5. *Let $t, s \in \mathscr{L}(\mathfrak{E})$ be invertible self-adjoint operators and $y \in \mathscr{L}(\mathfrak{E})$. Then*

$$w_c(y) \leq w_c\left(\frac{tys^{-1}+t^{-1}ys}{2}\right). \tag{4.7}$$

Proof. First of all, we shall show the case $t = s$ and y is self-adjoint. Let $\lambda \in \sigma(y)$. Then

$$\lambda \in \sigma(y) = \sigma(tyt^{-1}) \subseteq \overline{W(tyt^{-1})}.$$

Since $\lambda \in \mathbb{R}$ we have

$$\lambda = Re(\lambda) \in Re\overline{W(tyt^{-1})} = \overline{W\left(Re(tyt^{-1})\right)}.$$

So we obtain

$$w_c(y) = r(y) \leq w_c\left(Re(tyt^{-1})\right) = w_c\left(\frac{tys^{-1}+t^{-1}ys}{2}\right).$$

Next we shall show this lemma for arbitrary $y \in \mathscr{L}(\mathfrak{E})$ and invertible self-adjoint operators t and s. Let $\tilde{y} = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}$ and $\tilde{t} = \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix}$. Then $\tilde{y}$ and $\tilde{t}$ are self-adjoint. Hence we have

$$w_c(\tilde{y}) \leq w_c\left(\frac{\tilde{t}\tilde{y}\tilde{t}^{-1}+\tilde{t}^{-1}\tilde{y}\tilde{t}}{2}\right).$$

Here $w_c(\tilde{y}) = w_c(y)$ and

$$\begin{aligned} w_c\left(\frac{\tilde{t}\tilde{y}\tilde{t}^{-1}+\tilde{t}^{-1}\tilde{y}\tilde{t}}{2}\right) &= \frac{1}{2}w_c\left(\begin{pmatrix} 0 & tys^{-1}+t^{-1}ys \\ s^{-1}y^*t+sy^*t^{-1} & 0 \end{pmatrix}\right) \\ &= \frac{1}{2}w_c\left(tys^{-1}+t^{-1}ys\right). \end{aligned}$$

Therefore we obtain the desired inequality. □

Proof of Theorem 4.6. We may assume that t and s are invertible. By Lemma 4.5, we have

$$\begin{aligned} w_c^r\left(t^{\frac{1}{2}}ys^{\frac{1}{2}}\right) &\leq w_c^r\left(\frac{t^{\alpha-\frac{1}{2}}t^{\frac{1}{2}}ys^{\frac{1}{2}}s^{\frac{1}{2}-\alpha}+t^{\frac{1}{2}-\alpha}t^{\frac{1}{2}}ys^{\frac{1}{2}}s^{\alpha-\frac{1}{2}}}{2}\right) \\ &= w_c^r\left(\frac{t^{\alpha}ys^{1-\alpha}+t^{1-\alpha}ys^{\alpha}}{2}\right). \end{aligned}$$

On the other hand, by inequality (4.6), for $r \geq 2$ we have

$$\left|\psi\left(\langle x, t^{\alpha}ys^{1-\alpha}x\rangle\right)\right|^r \leq |||y|||^r\psi\left(\langle x, (\alpha t^r + (1-\alpha)s^r)\,x\rangle\right).$$

Hence we have

$$\begin{aligned} \left|\psi\left(\left\langle x, \left(\frac{t^{\alpha}ys^{1-\alpha}+t^{1-\alpha}ys^{\alpha}}{2}\right)x\right\rangle\right)\right|^r &\leq \left(\frac{\left|\psi\left(\langle x, t^{\alpha}ys^{1-\alpha}x\rangle\right)\right|+\left|\psi\left(\langle x, t^{1-\alpha}ys^{\alpha}x\rangle\right)\right|}{2}\right)^r \\ &\leq \frac{\left|\psi\left(\langle x, t^{\alpha}ys^{1-\alpha}x\rangle\right)\right|^r+\left|\psi\left(\langle x, t^{1-\alpha}ys^{\alpha}x\rangle\right)\right|^r}{2} \\ &\quad \text{(by the convexity of } f(\tau)=\tau^r) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{|||y|||^r}{2}\left[\psi\left(\langle x,(\alpha t^r+(1-\alpha)s^r)x\rangle\right)+\psi\left(\langle x,((1-\alpha)t^r+\alpha s^r)\rangle\right)\right]\\
&= |||y|||\psi\left(\left\langle x,\frac{t^r+s^r}{2}x\right\rangle\right).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
w_c^r\left(\frac{t^\alpha y s^{1-\alpha}+t^{1-\alpha}ys^\alpha}{2}\right) &\le |||y|||w_c\left(\frac{t^r+s^r}{2}\right)\\
&\le \frac{|||y|||}{2}\left(w_c\left(\alpha t^r+(1-\alpha)s^r\right)+w_c\left((1-\alpha)t^r+\alpha s^r\right)\right)\\
&= \frac{|||y|||}{2}\left(|||\alpha t^r+(1-\alpha)s^r|||+|||(1-\alpha)t^r+\alpha s^r|||\right).
\end{aligned}
$$

□

Theorem 4.7. *Let $a,b,c,d\in\mathscr{L}(\mathfrak{E})$ and $\mu,\nu\ge 1$. Then*

$$
|||b^*a+d^*c|||^2\le 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)}\left|\left|\left||a|^{2\mu}+|b|^{2\mu}\right|\right|\right|^{\frac{1}{\mu}}\left|\left|\left||c|^{2\nu}+|d|^{2\nu}\right|\right|\right|^{\frac{1}{\nu}}. \tag{4.8}
$$

Proof. By the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
\left|\psi\left(\langle y,(b^*a+d^*c)\,x\rangle\right)\right|^2 &= \left|\psi\left(\langle y,b^*ax\rangle\right)+\psi\left(\langle y,d^*cx\rangle\right)\right|^2\\
&\le \left[\left|\psi\left(\langle y,b^*ax\rangle\right)\right|+\left|\psi\left(\langle y,d^*cx\rangle\right)\right|\right]^2 \qquad (4.9)\\
&\le \left[\psi\left(\langle x,a^*ax\rangle\right)^{\frac12}\psi\left(\langle y,b^*by\rangle\right)^{\frac12}+\psi\left(\langle x,c^*cx\rangle\right)^{\frac12}\psi\left(\langle y,d^*dy\rangle\right)^{\frac12}\right]^2
\end{aligned}
$$

for all $x,y\in\mathfrak{E}$.

Now, on utilizing the elementary inequality

$$
(\kappa_1\kappa_2+\kappa_3\kappa_4)^2\le\left(\kappa_1^2+\kappa_3^2\right)\left(\kappa_2^2+\kappa_4^2\right),\quad \kappa_i\in\mathbb{R}(i=1,2,3,4).
$$

we then conclude that

$$
\begin{aligned}
&\left[\psi\left(\langle x,a^*ax\rangle\right)^{\frac12}\psi\left(\langle y,b^*by\rangle\right)^{\frac12}+\psi\left(\langle x,c^*cx\rangle\right)^{\frac12}\psi\left(\langle y,d^*dy\rangle\right)^{\frac12}\right]^2\\
&=\left(\psi\left(\langle x,a^*ax\rangle\right)+\psi\left(\langle x,c^*cx\rangle\right)\right)\left(\psi\left(\langle y,b^*by\rangle\right)+\psi\left(\langle y,d^*dy\rangle\right)\right) \qquad (4.10)
\end{aligned}
$$

for all $x,y\in\mathfrak{E}$.

Utilizing the arithmetic mean - geometric mean inequality and then the convexity of the function $f(\tau)=\tau^\delta,\delta\ge1$, we have successively,

$$
\begin{aligned}
&\left(\psi\left(\langle x,a^*ax\rangle\right)+\psi\left(\langle x,c^*cx\rangle\right)\right)\left(\psi\left(\langle y,b^*by\rangle\right)+\psi\left(\langle y,d^*dy\rangle\right)\right) \qquad (4.11)\\
&\le 4\left(\frac{\psi\left(\langle x,\left((a^*a)^\mu+(c^*c)^\mu\right)x\rangle\right)}{2}\right)^{\frac{1}{\mu}}\left(\frac{\psi\left(\langle y,\left((b^*b)^\nu+(d^*d)^\nu\right)y\rangle\right)}{2}\right)^{\frac{1}{\nu}}
\end{aligned}
$$

for all $x,y\in\mathfrak{E}$ with $\psi(|x|)=\psi(|y|)=1$ and for all $\mu\ge1$ and $\nu\ge1$. Consequently, by (4.9)-(4.11) we have

$$
\left|\psi\left(\langle y,(b^*a+d^*c)\,x\rangle\right)\right|^2
$$

$$\leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)} \left(\psi\left(\langle x, \left((a^*a)^\mu + (c^*c)^\mu\right) x\rangle\right)\right)^{\frac{1}{\mu}} \left(\psi\left(\langle y, \left((b^*b)^\nu + (d^*d)^\nu\right) y\rangle\right)\right)^{\frac{1}{\nu}}$$

for all $x, y \in \mathfrak{E}$ with $\psi(|x|) = \psi(|y|) = 1$. Taking the supremum over $x, y \in \mathfrak{E}$ with $\psi(|x|) = \psi(|y|) = 1$ we deduce the desired inequality (4.8). □

Remark 2. (i) If $\mu = \nu$, then the inequality (4.8) is equivalent to

$$|||b^*a + d^*c|||^{2\mu} \leq 2^{2\mu-2} |||(a^*a)^\mu + (c^*c)^\mu||| |||(b^*b)^\mu + (d^*d)^\mu||| \tag{4.12}$$

(ii) If $b = d = 1$, then inequality (4.8) is equivalent to

$$|||a + c|||^{2\mu} \leq 2^{2\mu-1} |||(a^*a)^\mu + (c^*c)^\mu||| \tag{4.13}$$

for all $\mu \geq 1$.

(iii) If $b = a^*$ and $d = c^*$, then inequality (4.8) is equivalent to

$$\left|\left|\left|a^2 + c^2\right|\right|\right|^2 \leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)} |||(a^*a)^\mu + (c^*c)^\mu|||^{\frac{1}{\mu}} |||(b^*b)^\nu + (d^*d)^\nu|||^{\frac{1}{\nu}} \tag{4.14}$$

for all $\mu, \nu \geq 1$.

If we put $d = a$ and $c = b$ in the equality (4.8), we get the following result.

Corollary 4.8. *If $a, b \in \mathscr{L}(\mathfrak{E})$. Then*

$$|||b^*a + a^*b|||^2 \leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)} \left|\left|\left||a|^{2\mu} + |b|^{2\mu}\right|\right|\right|^{\frac{1}{\mu}} \left|\left|\left||a|^{2\nu} + |b|^{2\nu}\right|\right|\right|^{\frac{1}{\nu}}, \tag{4.15}$$

for $\mu, \nu \geq 1$. In particular

$$|||b^*a + a^*b|||^\mu \leq 2^{\mu-1} \left|\left|\left||a|^{2\mu} + |b|^{2\mu}\right|\right|\right| \tag{4.16}$$

for all $\mu \geq 1$.

Another particular case that might be of interest is the following one.

Corollary 4.9. *For $a, d \in \mathscr{L}(\mathfrak{E})$, we have*

$$|||a + d|||^2 \leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)} \left|\left|\left||a|^{2\mu} + 1\right|\right|\right|^{\frac{1}{\mu}} \left|\left|\left||d|^{2\nu} + 1\right|\right|\right|^{\frac{1}{\nu}}, \tag{4.17}$$

for all $\mu, \nu \geq 1$. In particular

$$|||a|||^{2\mu} \leq \frac{1}{4} \left|\left|\left||a|^{2\mu} + 1\right|\right|\right|^2. \tag{4.18}$$

for all $\mu \geq 1$.

Proof. The proof of the inequality (4.17) is obvious by the inequality (4.8) on choosing $b = 1, c = 1$ and writing the inequality for d^* instead of d. □

Remark 3. If $t \in \mathscr{L}(\mathfrak{E})$ and $t = a + ic$, i.e., a and c are its Cartesian decomposition, then we get from (4.13) that

$$|||t|||^{2\mu} \leq 2^{2\mu-1} \left|\left|\left| a^{2\mu} + c^{2\mu} \right|\right|\right|,$$

for all $\mu \geq 1$. Also, since $a = Re(t) = \frac{t+t^*}{2}$ and $c = Im(t) = \frac{t-t^*}{2i}$, then from (4.13) we get the following inequalities as well

$$|||Re(t)|||^{2\mu} \leq \frac{1}{2} \left|\left|\left| |t|^{2\mu} + |t^*|^{2\mu} \right|\right|\right|$$

and

$$|||Im(t)|||^{2\mu} \leq \frac{1}{2} \left|\left|\left| |t|^{2\mu} + |t^*|^{2\mu} \right|\right|\right|$$

for any $\mu \geq 1$.

Theorem 4.10. *Let $t = a + ib$ be the Cartesian decomposition of $t \in \mathscr{L}(\mathfrak{E})$. Then for $\mu, \nu \in \mathbb{R}$,*

$$\sup_{\mu^2+\nu^2=1} |||\mu a + \nu b||| = w_c(t). \tag{4.19}$$

In particular,

$$\frac{1}{2}|||t + t^*||| \leq w_c(t) \;\; \textit{and} \;\; \frac{1}{2}|||t - t^*||| \leq w_c(t). \tag{4.20}$$

Proof. First of all, we note that

$$w(t) = \sup_{\theta \in \mathbb{R}} \left|\left|\left| Re(e^{i\theta} t) \right|\right|\right|. \tag{4.21}$$

In fact, $\sup_{\theta \in \mathbb{R}} Re\left(e^{i\theta} \psi\left(\langle x, tx \rangle\right)\right) = |\psi\left(\langle x, tx \rangle\right)|$ yields that

$$\sup_{\theta \in \mathbb{R}} \left|\left|\left| Re(e^{i\theta} t) \right|\right|\right| = \sup_{\theta \in \mathbb{R}} w_c\left(Re(e^{i\theta} t)\right) = w_c(t).$$

On the other hand, let $t = a + ib$ be the Cartesian decomposition of t. Then

$$\begin{aligned} Re\left(e^{i\theta} t\right) &= \frac{e^{i\theta} t + e^{-i\theta} t^*}{2} = \frac{1}{2}\left[(\cos\theta + i \sin\theta)\, t + (\cos\theta - i\sin\theta)\, t^*\right] \\ &= \cos\theta \left(\frac{t + t^*}{2}\right) - \sin\theta \left(\frac{t - t^*}{2i}\right) = (\cos\theta)\, a - (\sin\theta)\, b \end{aligned} \tag{4.22}$$

Therefore, by putting $\mu = \cos\theta$ and $\nu = -\sin\theta$ in (4.22), we obtain (4.19). Especially, by setting $(\mu, \nu) = (1, 0)$ and $(\mu, \nu) = (0, 1)$, we reach (4.20). □

Remark 4. By using (4.20), we get some known inequalities:

(i) $|||t||| = |||a + ib||| \leq |||a||| + |||b||| \leq 2w_c(t)$.

(ii) If t is self adjoint, then $t = a$. Hence we have $|||t||| = |||a||| \leq w_c(t) \leq |||t|||$ and so $w_c(t) = |||t|||$.

(iii) By an easy calculation, we have $\frac{t^*t+tt^*}{2} = a^2 + b^2$. Hence,

$$\frac{1}{4}|||t^*t + tt^*||| = \frac{1}{2}\left|\left|\left| a^2 + b^2 \right|\right|\right| \leq \frac{1}{2}\left(|||a|||^2 + |||b|||^2\right) \leq w_c^2(t). \tag{4.23}$$

(iv) Let $\mu, \nu \in \mathbb{R}$ satisfy $\mu^2 + \nu^2 = 1$. Then for any vector $x \in \mathfrak{E}$ with $\psi(|x|) = 1$, $\psi \in \varpi(\mathfrak{A})$, we have

$$\begin{aligned} |||(\mu a + \nu b)\, x||| &= \left|\left|\left| \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu x \\ \nu x \end{bmatrix} \right|\right|\right| \leq \left|\left|\left| \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right|\right|\right| = \left|\left|\left| \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} \right|\right|\right|^{\frac{1}{2}} \\ &= \left|\left|\left| a^2 + b^2 \right|\right|\right|^{\frac{1}{2}} = \frac{1}{\sqrt{2}} |||t^*t + tt^*|||^{\frac{1}{2}} \end{aligned}$$

Hence we have

$$w_c^2(t) = \sup_{\mu^2+\nu^2=1} |||\mu a + \nu b|||^2 \leq \frac{1}{2} |||t^*t + tt^*|||. \tag{4.24}$$

(v) Combining the inequalities (4.23) and (4.24), we obtain Theorem 3.2 of [11].

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