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# Some inequalities for the numerical radius and spectral norm for operators in Hilbert $C^*$-modules space

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**Abstract**. This paper introduces a new method for studying the numerical radius of bounded operators on Hilbert $C^*$-modules. Our approach leads to unique discoveries and expands existing theorems for bounded adjointable operators in Hilbert $C^*$-module spaces. Moreover, we find an upper bound for power of the numerical radius of $t^\alpha y s^{1-\alpha}$ under assumption $0 \leq \alpha \leq 1$. In fact, we prove

$$w_c\left(t^\alpha y s^{1-\alpha}\right) \leq |||y|||^r |||\alpha t^r + (1-\alpha)s^r|||$$

for all $0 \leq \alpha \leq 1$ and $r \geq 2$.

***Keywords.*** Numerical radius, inner product space, $C^*$-algebra, $A$-module

## 1 Introduction

The notion of a Hilbert $C^*$-module initiated by Kaplansky [4] as a generalization of a Hilbert space in which the inner product takes its values in a $C^*$-algebra (see also [7, 8, 10, 11]).

Let $\mathfrak{A}$ be a $C^*$-algebra. A pre-Hilbert $\mathfrak{A}$-module or an inner product $\mathfrak{A}$-module is a complex linear space $\mathfrak{E}$ which is a right A-module with compatible scalar multiplication $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in \mathfrak{E}, a \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, together with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E} \longrightarrow \mathfrak{A}$ that satisfies the following properties:

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$;

(ii) $\langle x, ya \rangle = \langle x, y \rangle a$;

(ii) $\langle x, y \rangle = \langle y, x \rangle^*$;

(iv) $\langle x, x \rangle \geq 0$; if $\langle x, x \rangle = 0$, then $x = 0$

for each $x, y, z \in \mathfrak{E}$, $a \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{C}$.

The notion of a left Hilbert $\mathfrak{A}$-module can be defined similarly. Note that the condition (i) is understood as a statement in the $C^*$-algebra $\mathfrak{A}$, where an element $a$ is called positive if it

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can be represented as $bb^*$ for some $b \in \mathfrak{A}$. The conditions (ii) and (iv) imply the inner product to be conjugate-linear in its first variable. Validity of a useful version of the classical Cauchy-Schwartz inequality follows that $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ defines a norm on $\mathfrak{E}$ making it into a normed right $A$-module. An inner product $\mathfrak{A}$-module $\mathfrak{E}$ which is complete with respect to the norm $\|x\|$ is called a Hilbert $\mathfrak{A}$-module or a Hilbert $C^*$-module over the $C^*$-algebra $\mathfrak{A}$. Every $C^*$-algebra $\mathfrak{A}$ is a Hilbert $\mathfrak{A}$-module under the $\mathfrak{A}$-valued inner product $\langle a, b\rangle = a^*b$ $(a, b \in \mathfrak{A})$. Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathfrak{E}$ and $\mathfrak{F}$ are Hilbert $\mathfrak{A}$-modules. We define $\mathscr{L}(\mathfrak{E}, \mathfrak{F})$ to be the set of all maps $t : \mathfrak{E} \longrightarrow \mathfrak{F}$ for which there is a map $t^* : \mathfrak{F} \longrightarrow \mathfrak{E}$ such that $\langle tx, y\rangle = \langle x, t^*y\rangle$, for all $x \in \mathfrak{E}$, $y \in \mathfrak{F}$. It is known that $t$ must be a bounded $\mathfrak{A}$-linear map (that is, $t$ is bounded linear map and $t(xa) = t(x)a$ for all $x \in \mathfrak{E}, a \in \mathfrak{A}$). If $\mathfrak{E} = \mathfrak{F}$, then $\mathscr{L}(\mathfrak{E})$ is a $C^*$-algebra together with the operator norm.

Suppose that $\mathfrak{A}$ is an abelian $C^*$-algebra. Recall that a character $\psi$ on $\mathfrak{A}$ is a non-zero $*$-homomorphism $\psi : \mathfrak{A} \longrightarrow \mathbb{C}$ such that $\|\psi\| = 1$. We denote the set of all characters on $\mathfrak{A}$ by $\varpi(\mathfrak{A})$.

Throughout this paper assume that $\mathfrak{A}$ is abelian $C^*$-algebra.

## 2 Definitions and Complementary results

**Lemma 2.1.** *Let $\mathfrak{E}$ be a Hilbert $\mathfrak{A}$-module. Then for all $x, y \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$, we have*

*(i) (Cauchy-Schwartz inequality)* $|\psi(\langle x, y\rangle)| \le \psi(|x|)\,\psi(|y|)$.

*(ii) (triangle inequality)* $\psi(|x + y|) \le \psi(|x|) + \psi(|y|)$.

*(iii) (Parallelogram Law)* $\psi\left(|x + y|^2\right) + \psi\left(|x - y|^2\right) = 2\left(\psi\left(|x|^2\right) + \psi\left(|y|^2\right)\right)$.

*Proof.* (i) For every $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} 0 \le \psi(\langle x - \lambda y, x - \lambda y\rangle) &= \psi(\langle x, x\rangle) - \psi(\langle x, \lambda y\rangle) - \psi(\langle \lambda y, x\rangle) + \psi(\langle \lambda y, \lambda y\rangle) \\ &= \psi\left(|x|^2\right) - \bar{\lambda}\psi(\langle x, y\rangle) - \lambda\psi(y, x) + |\lambda|^2\psi\left(|y|^2\right) \\ &= \psi\left(|x|^2\right) - 2Re\left(\lambda\psi(\langle y, x\rangle)\right) + |\lambda|^2\psi\left(|y|^2\right). \end{aligned} \tag{2.1}$$

If $\psi(\langle x, y\rangle) = 0$, then the inequality is trivial. Suppose that $\psi(\langle x, y\rangle) \neq 0$, letting $\lambda = \frac{\psi\left(|x|^2\right)}{\psi(\langle y, x\rangle)}$ in (2.1) gives

$$0 \le -\psi\left(|x|^2\right) + \frac{\psi\left(|x|^4\right)\psi\left(|y|^2\right)}{|\psi(\langle x, y\rangle)|^2}.$$

Hence

$$\psi\left(|x|^2\right) \le \frac{\psi\left(|x|^4\right)\psi\left(|y|^2\right)}{|\psi(\langle x, y\rangle)|^2}$$

and this implies that $|\psi(\langle x, y\rangle)|^2 \le \psi\left(|x|^2\right)\psi\left(|y|^2\right)$ and so

$$|\psi(\langle x, y\rangle)| \le \psi(|x|)\,\psi(|y|).$$

(ii) By (i), we have

$$\begin{aligned}\psi\left(|x+y|^2\right) &= \psi\left(\langle x+y,x+y\rangle\right) = \psi\left(|x|^2\right) + 2Re\psi\left(\langle x,y\rangle\right) + \psi\left(|y|^2\right)\\ &\leq \psi\left(|x|^2\right) + 2\psi\left(|x|\right)\psi\left(|y|\right) + \psi\left(|y|^2\right)\\ &= \left(\psi\left(|x|\right) + \psi\left(|y|\right)\right)^2\end{aligned}$$

and so the result.

(iii) We have

$$\begin{aligned}\psi\left(|x+y|^2\right) + \psi\left(|x-y|^2\right) &= \psi\left(|x|^2\right) + 2Re\psi\left(\langle x,y\rangle\right) + \psi\left(|y|^2\right)\\ &+ \psi\left(|x|^2\right) - 2Re\psi\left(\langle x,y\rangle\right) + \psi\left(|y|^2\right)\\ &= 2\left(\psi\left(|x|^2\right) + \psi\left(|y|^2\right)\right).\end{aligned}$$

□

**Definition 1.** Let $t \in \mathscr{L}(\mathfrak{E})$ and $\psi \in \varpi(\mathfrak{A})$. Then

$$|||t||| := \sup\left\{\psi\left(|tx|\right) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1\right\}, \tag{2.2}$$

where $|x| = \langle x,x\rangle^{\frac{1}{2}}$.

It is known from [10] that $|||\cdot|||$ is a norm on $\mathscr{L}(\mathfrak{E})$. And if $\mathfrak{E}$ is a Hilbert space, then $\|t\| = |||t|||$. The following result was investigated in [10].

**Lemma 2.2.** *Let $t \in \mathscr{L}(\mathfrak{E})$. Then*

$$|||t||| = \sup\left\{|\psi\left(\langle x,ty\rangle\right)| : x,y \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = \psi(|y|) = 1\right\}.$$

**Definition 2.** Let $t \in \mathscr{L}(\mathfrak{E})$. Then the spectrum of $t$, denoted by $\sigma(t)$, is defined by

$$\sigma(t) = \left\{\lambda \in \mathbb{C} : t - \lambda 1 \text{ is not invertible}\right\}.$$

And $\lambda \in \mathbb{C}$ is called an eigenvalue of $t$ if there is a non-zero vector $x \in \mathfrak{E}$ such that $tx = \lambda x$. Equivalently, $\lambda$ is an eigenvalue if there is a vector $x \in \mathfrak{E}$ with $\psi\left(|x|\right) = 1$ such that $|||(t-\lambda 1)x||| = 0$.

**Definition 3.** $\lambda \in \mathbb{C}$ is called an approximate point spectrum of $t \in \mathscr{L}(\mathfrak{E})$ if there is a sequence $\{x_n\}$ of vectors in $\mathfrak{E}$ with $\psi\left(|x_n|\right) = 1$ such that $|||(t-\lambda 1)x_n||| \longrightarrow 0$, the set of approximate point spectrum is denoted by $\sigma_a(t)$.

**Definition 4.** If $t \in \mathscr{L}(\mathfrak{E})$, then the spectral radius of $t$ is the number defined by

$$r(t) = \sup\left\{|\lambda| : \lambda \in \sigma(t)\right\}.$$

Clearly, $0 \leq r(t) \leq |||t|||$ and it follows from spectral theorem that $r(t^n) = (r(t))^n$. Moreover, it is well-known that $r(t) = \lim_{n\longrightarrow\infty} |||t^n|||^{\frac{1}{n}}$ (see [8]). Recall that a function $f$ which maps A Hilbert $\mathfrak{A}$-module $\mathfrak{E}$ into $\mathbb{C}$ is called a functional. If $f$ is in $\mathscr{L}(\mathfrak{E},\mathbb{C})$, then $f$ is called a linear functional on $\mathfrak{E}$.

**Lemma 2.3.** *If $f$ is a bounded linear functional on a Hilbert $\mathfrak{A}$-module $\mathfrak{E}$, then there exists a unique $y \in \mathfrak{E}$ such that for all $x \in \mathfrak{E}$, $f(x) = \psi\left(\langle y,x\rangle\right)$. Moreover, $|||f||| = \psi\left(|y|\right)$.*

*Proof.* If $f = 0$, take $y = 0$. Suppose that $f \neq 0$. Then $(f)$ is a proper closed subspace of $\mathfrak{E}$. Hence there exists a $v \neq 0$ in $(f)^{\perp}$.

Let $y = \alpha v$, where $\alpha = \frac{\overline{f(v)}}{\psi(|v|^2)}$. Then $y \perp (f)$ (because $v \perp (f)$) and $f(y) = \psi(\langle y, y\rangle)$ since

$$\begin{aligned} f(y) &= \alpha f(v) = \frac{|f(v)|^2}{\psi(|v|^2)} \text{ and} \\ \psi(\langle y, y\rangle) &= |\alpha|^2 \psi\left(|v|^2\right) = \frac{|f(v)|^2}{\psi(|v|^4)} \psi\left(|v|^2\right) = \frac{|f(v)|^2}{\psi(|v|^2)}. \end{aligned}$$

Now, given $x \in \mathfrak{E}$, then $x$ can be represented as $x = \beta y + z$, where $\beta \in \mathbb{C}$ and $z \in (f)$. From the previous arguments, we have

$$f(x) = f(\beta y) = \beta f(y) = \beta \psi(\langle y, y\rangle) = \psi(\langle y, \beta y + z\rangle) = \psi(\langle y, x\rangle).$$

To show that $y$ is unique, suppose there is $w \in \mathfrak{E}$ such that $f(x) = \psi(\langle w, x\rangle)$ for all $x \in \mathfrak{E}$. Then

$$0 = f(x) - f(x) = \psi(\langle y - w, x\rangle) \text{ for all } x \in \mathfrak{E}.$$

In particular, $\psi(\langle y - w, y - w\rangle) = 0$ and so $y = w$.

Finally, for each $y \in \mathfrak{E}$ the functional $f$ defined on $\mathfrak{E}$ is linear. Moreover

$$|f(x)| = |\psi(y, x)| \leq \psi(|x|)\psi(|y|) \text{ for all } x \in \mathfrak{E}.$$

Thus $f$ is bounded and $|||f||| \leq \psi(|y|)$. Since

$$|||f|||\psi(|y|) \geq |f(y)| = \psi(\langle y, y\rangle) = \psi\left(|y|^2\right)$$

and so $|||f||| \geq \psi(|y|)$ and consequently $|||f||| = \psi(|y|)$. □

**Lemma 2.4.** *[10] If $t \in \mathscr{L}(\mathfrak{E})$, then*

$$|||t||| = \sup\{|\psi(x, tx)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1\}.$$

The following results are very useful in the sequel.

**Proposition 2.1.** *[11] Let $t \in \mathscr{L}(\mathfrak{E})$ and $\psi \in \varpi(\mathfrak{A})$. The following statements are equivalent:*

*(a) $\psi(\langle x, tx\rangle) = 0$ for every $x \in \mathfrak{E}$ with $\psi(|x|) = 1$;*

*(b) $\psi(\langle x, tx\rangle) = 0$ for every $x \in \mathfrak{E}$.*

**Proposition 2.2.** *[11] For every $t \in \mathscr{L}(\mathfrak{E})$, the following assertions hold.*

*(i) $t = 0$ if and only if $\psi(\langle x, tx\rangle) = 0$ for every $x \in \mathfrak{E}$.*

*(ii) $t$ is positive if and only if $\psi(\langle x, tx\rangle)$ is positive for every $x \in \mathfrak{E}$.*

*(iii) $t$ is self-adjoint if and only if $\psi(\langle x, tx\rangle)$ is self-adjoint for every $x \in \mathfrak{E}$.*

*(iv) $t = 0$ if and only if $\psi(\langle x, tx\rangle) = 0$ for every $x \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$.*

*(v) $Re\psi(\langle x, tx\rangle) = \psi(\langle x, Re(t)x\rangle)$ for all $x \in \mathfrak{E}$.*

**Lemma 2.5.** *[10] If $t \in \mathscr{L}(\mathfrak{E})$ is self-adjoint, then*

$$|||t||| = \sup \{ |\psi(\langle x, tx \rangle)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1 \}.$$

**Theorem 2.1.** *Suppose $t \in \mathscr{L}(\mathfrak{E})$ is self-adjoint.*

*(i) Let*

$$\lambda = \inf \{ \psi(\langle x, tx \rangle) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1 \}.$$

*If there exists an $x_0 \in \mathfrak{E}$ such that $\psi(|x_0|) = 1$ and $\lambda = \psi(\langle x_0, tx_0 \rangle)$, then $\lambda$ is an eigenvalue of $t$ with corresponding eigenvector $x_0$.*

*(ii) Let*

$$\mu = \sup \{ \psi(\langle x, tx \rangle) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1 \}.$$

*If there exists an $x_1 \in \mathfrak{E}$ such that $\psi(|x_1|) = 1$ and $\mu = \psi(\langle x_1, tx_1 \rangle)$, then $\mu$ is an eigenvalue of $t$ with corresponding eigenvector $x_1$.*

*Proof.* (i) For every $\alpha \in \mathbb{C}$ and every $y \in \mathfrak{E}$, it follows from the definition of $\lambda$ that

$$\psi(\langle x_0 + \alpha y, t(x_0 + \alpha y) \rangle) \geq \lambda \psi(\langle x_0 + \alpha y, x_0 + \alpha y \rangle).$$

Expanding the inner product and setting $\lambda = \psi(\langle x_0, tx_0 \rangle)$, we get the inequality

$$2Re\alpha\psi(\langle (t - \lambda 1)x_0, y \rangle) + |\alpha|^2 \psi(\langle y, (t - \lambda 1)y \rangle) \geq 0.$$

Taking $\alpha = \overline{r\psi(\langle (t - \lambda 1)x_0, y \rangle)}$, where $r \in \mathbb{R}$, it follows that

$$2r |\psi(\langle (t - \lambda 1)x_0, y \rangle)|^2 + r^2 |\psi(\langle (t - \lambda 1)x_0, y \rangle)|^2 \psi(\langle y, (t - \lambda 1)y \rangle) \geq 0.$$

Since $r$ is arbitrary, it follows that $\psi(\langle (t - \lambda 1)x_0, y \rangle) = 0$ and since $y$ is arbitrary, we have $tx_0 = \lambda x_0$ as required.

(ii) The second statement of the theorem follows from part(i) applied to the self-adjoint $-A$. □

**Definition 5.** An operator $t \in \mathscr{L}(\mathfrak{E}, \mathfrak{F})$ is said to be compact if for each sequence $\{x_n\}$ in $\mathfrak{E}$ with $\psi(|x_n|) = 1$ and $\psi \in \varpi(\mathfrak{A})$, the sequence $\{tx_n\}$ has a subsequence which converges in $\mathfrak{F}$.

**Theorem 2.2.** *If $t \in \mathscr{L}(\mathfrak{E})$ is compact and self-adjoint, then at least one the numbers $|||t|||$ or $-|||t|||$ is an eigenvalue of $t$.*

*Proof.* The result is trivial if $t = 0$. Assume that $t \neq 0$, since

$$|||t||| = \sup \{ |\psi(\langle x, tx \rangle)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1 \}$$

then there exists a sequence $\{x_n\}$ in $\mathfrak{E}$ with $\psi(|x_n|) = 1$ and a real number $\lambda$ such that $|\lambda| = |||t||| \neq 0$ and $\psi(\langle x_n, tx_n \rangle) \longrightarrow \lambda$.
Now

$$\begin{aligned} 0 \leq \psi\left(|tx_n - \lambda x_n|^2\right) &= \psi\left(|tx_n|^2\right) - 2\lambda\psi(x_n, tx_n) + \lambda^2 \\ &\leq 2\lambda^2 - 2\lambda\psi(x_n, tx_n) \longrightarrow 2\lambda^2 - 2\lambda^2 = 0 \end{aligned}$$

and so

$$tx_n - \lambda x_n \longrightarrow 0. \tag{2.3}$$

Since $t$ is compact, there exists a subsequence $\{tx_{n'}\}$ of $\{tx_n\}$ which converges to some $y \in \mathfrak{E}$. Thus (2.3) implies that $x_{n'} \longrightarrow \frac{1}{\lambda}y$ and by the continuity of $t$, $y = \lim_{n' \longrightarrow \infty} tx_{n'} = \frac{1}{\lambda}ty$. Hence $ty = \lambda y$ and $y \neq 0$. Since

$$\psi(|y|) = \lim_{n' \longrightarrow \infty} \psi(|\lambda x_{n'}|) = |\lambda| = |||t|||$$

and so $\lambda$ is an eigenvalue of $t$, as required. □

**Definition 6.** Let $t \in \mathscr{L}(\mathfrak{E})$. Then the numerical range of $t$ is defined by

$$W_c(t) = \{\psi(\langle x, tx\rangle) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \text{ and } \psi(|x|) = 1\}.$$

The next result represent some of the basic properties for the numerical range (see [10]).

**Lemma 2.6.** *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then the following assertions hold.*

*(i)* $W_c(t^*) = \overline{W_c(T)}$, *where* $\overline{W_c(T)}$ *is the conjugate of* $W_c(t)$.

*(ii)* $W_c(T) \subseteq \mathbb{R}$ *if and only if* $t$ *is a self-adjoint.*

*(iii) If* $u$ *is unitary, then* $W_c(u^*tu) = W_c(t)$.

*(iv) If* $\alpha, \beta \in \mathbb{C}$, *then* $W_c(\alpha t + \beta 1) = \alpha W_c(t) + \beta$.

*(v)* $W_c(t+s) \subset W_c(t) + W_c(s)$.

**Definition 7.** Let $t \in \mathscr{L}(\mathfrak{E})$. Then the numerical radius of $t$ is defined by

$$w_c(t) = \sup\{|\psi(\langle x, tx\rangle)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \text{ and } \psi(|x|) = 1\}.$$

It is easy to show that $w_c(\cdot)$ is a norm on $\mathscr{L}(\mathfrak{E})$.

The following is useful in the sequel.

**Lemma 2.7.** *If $\mathfrak{E}$ is a Hilbert $\mathfrak{A}$-module, then for every $\psi \in \varpi(\mathfrak{A})$, $x \in \mathfrak{E}$,*

$$\psi(|\langle x, tx\rangle|) \le \psi\left(|x|^2\right) w_c(t)$$

**Theorem 2.3.** *If $t \in \mathscr{L}(\mathfrak{E})$ is normal, then*

$$|||t||| = r(t) = w_c(t).$$

*Proof.* First we want to show $|||t^n||| = |||t|||^n$. by induction, for $n = 1$ the equality is trivial. Assume that its true for $k$ such that $1 \le k \le n$.

$$\begin{aligned} |||t^n x|||^2 &= \psi(\langle t^n x, t^n x\rangle) = \psi\left(\langle t^* t^n x, t^{n-1} x\rangle\right) \\ &\le |||t^* t^n x||| |||t^{n-1} x||| \le \left|\left|\left|t^{n+1} x\right|\right|\right| \left|\left|\left|t^{n-1}\right|\right|\right| \psi\left(|x|^2\right) \quad (t \text{ is normal}). \end{aligned}$$

and so, $|||t^n|||^2 \le \left|\left|\left|t^{n+1}\right|\right|\right| \left|\left|\left|t^{n-1}\right|\right|\right|$. But $|||t^n||| = |||t|||^n$ for all $k$ such that $1 \le k \le n$ and this implies that $|||t|||^{2n} \le \left|\left|\left|t^{n+1}\right|\right|\right| |||t|||^{n-1}$ and hence $|||t^n||| = |||t|||^n$ for all $n \in \mathbb{N}$.

Now, $r(t) = \lim_{n \longrightarrow \infty} |||t^n|||^{\frac{1}{n}} = |||t|||$. But its known that $r(t) \le w_c(t) \le |||t|||$ and so we have the desired equality. □

**Lemma 2.8.** *If $t \in \mathscr{L}(\mathfrak{E})$ is normal and $\lambda \notin \sigma(t)$, then*

$$|||(t-\lambda 1)^{-1}||| = \frac{1}{d(\lambda, \sigma(t))},$$

*where $d(\lambda, \sigma(t))$ is the distance from $\lambda$ to $\sigma(t)$.*

*Proof.* we have

$$r((t-\lambda 1)^{-1}) = \sup\left\{\frac{1}{|\mu-\lambda|} : \mu \in \sigma(t)\right\} = \frac{1}{\inf\{|\mu-\lambda : \mu \in \sigma(t)\}} = \frac{1}{d(\lambda, \sigma(t))}.$$

So, if $t$ is normal, then $(t-\lambda 1)^{-1}$ is normal for $\lambda \notin \sigma(t)$ and hence

$$|||(t-\lambda 1)^{-1}||| = r((t-\lambda 1)^{-1}) = \frac{1}{d(\lambda, \sigma(t))}.$$

□

**Theorem 2.4.** *If $t \in \mathscr{L}(\mathfrak{E})$ is normal, then $\overline{W_c(t)} = Conv\,\sigma(t)$, where $Conv\,\sigma(t)$ is the convex hull of the spectrum of $t$.*

*Proof.* We need only to show $\overline{W_c(t)} \subset \overline{Conv\,\sigma(t)}$. To see this, it sufficient to show that any closed half-plane which contains $\sigma(t)$ also contain $\overline{W_c(t)}$. By translation and rotation this reduces to shown that $Re\sigma(t) \leq 0$ implies $Re\overline{W_c(t)} \leq 0$.

Let $x \in \mathfrak{E}$ such that $\psi(|x|) = 1$ and $tx = (a+ib)x + y$ with $a, b$ are real and $x$ orthogonal to $y$. Now from Lemma 2.8,we have $|||(t-c)x||| \geq dist(c, \sigma(t)) \geq c$ for all $c > 0$. Indeed, if $c \notin \sigma(t)$, then $|||(t-c)^{-1}x||| |||(t-c)x||| \geq |||(t-c)^{-1}(t-c)x||| = \psi(|x|) = 1$ and so $|||(t-c)x||| \geq \frac{1}{|||(t-c)^{-1}|||} = d(c, \sigma(t)) \geq c$. So that

$$\begin{aligned} c^2 &\leq |||(t-c)x|||^2 = |||(a-c)x + ibx + y|||^2 = |||(a-c)x + ibx|||^2 + \psi\left(|y|^2\right) \\ &= (a-c)^2 + b^2 + \psi\left(|y|^2\right). \end{aligned}$$

Consequently,

$$2ac \leq a^2 + b^2 + \psi\left(|y|^2\right).$$

Since this hold for all $c > 0$. This implies that $Re\psi(x, tx) = a \leq 0$ as required. □

# 3 A numerical radius inequality

In order to prove our desired numerical radius inequality, we need the following lemmas. The first lemma, which is a generalized Schwartz inequality, can be found in [11, Corollary 3.11]

**Lemma 3.1.** *(Geralized-Cauchy Schwartz) For $\psi \in \varpi(\mathfrak{A})$, $\psi(\langle\cdot,\cdot\rangle)$ is a semi-inner product. Suppose that $t \in \mathscr{L}(\mathfrak{E})$ and $\alpha \in [0,1]$, then*

$$|\psi(\langle x, ty\rangle)|^2 \leq \psi\left(\langle x, |t|^{2\alpha}x\rangle\right)\psi\left(\left\langle y, |t^*|^{2(1-\alpha)}y\right\rangle\right), \ \ x, y \in \mathfrak{E}.$$

*If $\alpha = \frac{1}{2}$, then*

$$|\psi(\langle x, ty\rangle)|^2 \leq \psi(\langle x, |t|x\rangle)\psi(\langle y, |t^*|y\rangle), \ \ x, y \in \mathfrak{E}.$$

*Here $|t|$ stands for the positive (semi-definite) operator $(t^*t)^{\frac{1}{2}}$.*

The second lemma contains a special case of a more general norm inequality that is equivalent to some Löwner–Heinz type inequalities. See [6].

**Lemma 3.2.** *If* $t, s \in \mathscr{L}(\mathfrak{E})$ *are positive, then*

$$\left|\left|\left| t^{\frac{1}{2}} s^{\frac{1}{2}} \right|\right|\right| \leq |||ts|||^{\frac{1}{2}}.$$

The third lemma contains a recent norm inequality for sums of positive operators that is sharper than the triangle inequality.

**Lemma 3.3.** *If* $t, s \in \mathscr{L}(\mathfrak{E})$ *are positive, then*

$$|||t+s||| \leq \frac{1}{2}\left( |||t||| + |||s||| + \sqrt{(|||t||| - |||s|||)^2 + 4\left|\left|\left| t^{\frac{1}{2}} s^{\frac{1}{2}} \right|\right|\right|^2} \right). \tag{3.1}$$

Now we are in a position to present our refined numerical radius inequality.

**Theorem 3.1.** *If* $t \in \mathscr{L}(\mathfrak{E})$, *then*

$$w_c(t) \leq \frac{1}{2}\left( |||t||| + \left|\left|\left| t^2 \right|\right|\right|^{\frac{1}{2}} \right). \tag{3.2}$$

*Proof.* By Lemma 3.1 and by the arithmetic-geometric mean inequality, we have for every $x \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$,

$$\begin{aligned} |\psi(\langle x, tx\rangle)| &\leq \psi(\langle x, |t|x\rangle)^{\frac{1}{2}} \psi(\langle x, |t^*|x\rangle)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\left(\psi(\langle x, |t|x\rangle) + \psi(\langle x, |t^*|x\rangle)\right) \\ &= \frac{1}{2}\left(\psi(\langle x, (|t| + |t^*|)x\rangle)\right). \end{aligned}$$

Thus

$$\begin{aligned} w_c(t) &= \sup\{|\psi(\langle x, tx\rangle)| : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1\} \\ &\leq \frac{1}{2}\sup\{(\psi(\langle x, (|t| + |t^*|)x\rangle)) : x \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \psi(|x|) = 1\} \\ &= \frac{1}{2}|||\,|t| + |t^*|\,|||. \end{aligned} \tag{3.3}$$

Applying Lemmas 3.2 and 3.3 to the positive operators $|t|$ and $|t^*|$, and using the facts that $|||\,|t|\,||| = |||\,|t^*|\,||| = |||t|||$ and $|||\,|t||t^*|\,||| = \left|\left|\left| t^2 \right|\right|\right|$, we have

$$|||\,|t| + |t^*|\,||| \leq |||t||| + \left|\left|\left| t^2 \right|\right|\right|^{\frac{1}{2}}. \tag{3.4}$$

The desired inequality (3.2) now follows from (3.3) and (3.4). □

To see that (3.2) is a refinement of the second inequality in [11, Theorem 2.13], one has to recall that $\left|\left|\left| t^2 \right|\right|\right| \leq |||t|||^2$ for every $t \in \mathscr{L}(\mathfrak{E})$.
It has been mentioned in [11, Theorem 2.17] that if $t \in \mathscr{L}(\mathfrak{E})$ is such that $t^2 = 0$, then $w_c(t) = \frac{1}{2}|||t|||$. This can be easily seen as an immediate consequence of the first inequality in [11, Theorem 2.13] and the inequality (3.2).

**Corollary 3.2.** *If $t \in \mathscr{L}(\mathfrak{E})$ is such that $t^2 = 0$, then $w_c(t) = \frac{1}{2}|||t|||$.*

*Proof.* Combining the first inequality [11, Theorem 2.13] and the inequality (3.2), we have

$$\frac{1}{2}|||t||| \leq w_c(t) \leq \frac{1}{2}\left(|||t||| + \left|\left|\left|t^2\right|\right|\right|^{\frac{1}{2}}\right) \tag{3.5}$$

for every $t \in \mathscr{L}(\mathfrak{E})$. Thus, if $t^2 = 0$, then $w_c(t) = \frac{1}{2}|||t|||$ as required. □

The following result is another consequence of the inequality (3.2).

**Corollary 3.3.** *If $t \in \mathscr{L}(\mathfrak{E})$ is such that $w_c(t) = |||t|||$, then $\left|\left|\left|t^2\right|\right|\right| = |||t|||^2$.*

*Proof.* It follows from the inequality (3.2) that

$$2w_c(t) \leq |||t||| + \left|\left|\left|t^2\right|\right|\right|^{\frac{1}{2}}$$

for every $t \in \mathscr{L}(\mathfrak{E})$. Thus, if $w_c(t) = |||t|||$, then $|||t||| \leq \left|\left|\left|t^2\right|\right|\right|^{\frac{1}{2}}$, and hence $|||t|||^2 \leq \left|\left|\left|t^2\right|\right|\right|$. But the reverse inequality is always true. Thus $\left|\left|\left|t^2\right|\right|\right| = |||t|||^2$ as required. □

## 4 Power Inequalities For The Numerical Radius

To prove our generalized numerical radius, we need several well-known lemmas.

**Lemma 4.1.** *[9] Let $a, b \geq 0$, $0 \leq \alpha \leq 1$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then*

*(i) $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}$;*

*(ii) $ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}$;*

*for all $r \geq 1$.*

**Lemma 4.2.** *Let $t, s \in \mathscr{L}(\mathfrak{E})$, and let $f$ and $g$ be non-negative functions on $[0, \infty)$ which are continuous such that $f(\tau)g(\tau) = \tau$ for all $\tau \in [0, \infty)$ Then*

$$|\psi\,(y, tx)| \leq |||f(|t|)x|||\,|||g(|t^*|)y|||,$$

*for all $x, y \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$.*

**Lemma 4.3.** *[11, Hölder-McCarthy inequality in Hilbert $C^*$-Modules] Let $t \in \mathscr{L}(\mathfrak{E})$, $t > 0$, then for every $\psi \in \mathfrak{S}(\mathfrak{A})$*

*(i) $(\psi\,\langle x, tx\rangle_{\mathfrak{A}})^r \leq \|x\|^{2(1-r)}\,\psi\,\langle x, t^r x\rangle_{\mathfrak{A}}$ for $r > 1$ and*

*(ii) $(\psi\,\langle x, tx\rangle_{\mathfrak{A}})^r \geq \|x\|^{2(1-r)}\,\psi\,\langle x, t^r x\rangle_{\mathfrak{A}}$ for $0 < r \leq 1$*

**Theorem 4.1.** *Let $t \in \mathscr{L}(\mathfrak{E})$ be self-adjoint. Then*

$$w_c^2(t) \leq \frac{1}{2}\left(w_c(t^2) + |||t|||^2\right).$$

*Proof.* We recall the following refinement of the Cauchy–Schwartz inequality obtained by Dragomir in [1] with slight modification. It says that

$$\begin{aligned}\psi(|u|)\,\psi(|v|) &\geq |\psi(\langle u,v\rangle)-\psi(\langle u,z\rangle)\,\psi(\langle z,v\rangle)|+|\psi(\langle u,z\rangle)\,\psi(\langle z,v\rangle)| \\ &\geq |\psi(\langle u,v\rangle)|, \end{aligned}\tag{4.1}$$

for all $u,v,z\in\mathfrak{E}$ with $\psi(|z|)=1$. From inequality (4.1), we deduce that

$$|\psi(\langle u,z\rangle)\,\psi(\langle z,v\rangle)|\leq\frac{1}{2}\left(\psi(|u|)\,\psi(|v|)+|\psi(\langle u,v\rangle)|\right).\tag{4.2}$$

In the inequality (4.2), put $z=x$ with $\psi(|x|)=1$, $u=t^*x$ and $v=tx$, we get

$$|\psi(\langle t^*x,x\rangle)\,\psi(\langle x,tx\rangle)|\leq\frac{1}{2}\left(\psi(|t^*x|)\,\psi(|tx|)+|\psi(\langle t^*x,tx\rangle)|\right).$$

Hence

$$|\psi(\langle x,tx\rangle)|^2\leq\frac{1}{2}\left(\psi(|tx|)^2+\psi\left(\langle x,t^2x\rangle\right)\right).\tag{4.3}$$

Taking the supremum over all vectors $x\in\mathfrak{E}$ with $\psi(|x|)=1$, we get the desired result. □

**Theorem 4.2.** *Let $t\in\mathscr{L}(\mathfrak{E})$ and let $f$ and $g$ be as in Lemma 4.2. Then we have*

$$w_c^2(t)\leq\frac{1}{2}\left(|||t|||^2+\left|\left|\left|\frac{1}{p}f^p(|t|^2)+\frac{1}{q}g^q(|t|^2)\right|\right|\right|\right)\tag{4.4}$$

*for all $p\geq q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* Let $x\in\mathfrak{E}$ such that $\psi(|x|)=1$. We have

$$\begin{aligned}\left|\psi\left(x,t^2x\right)\right| &\leq |||f(|t^2|x)|||\,|||g(|(t^*)^2|)||| \quad \text{(by Lemma 4.2)}\\ &= \psi\left(x,f^2(|t^2|)x\right)^{\frac{1}{2}}\psi\left(x,g^2(|(t^*)^2|)x\right)^{\frac{1}{2}}\\ &\leq \frac{1}{p}\psi\left(x,f^2(|t^2|)x\right)^{\frac{p}{2}}+\frac{1}{q}\psi\left(x,g^2(|(t^*)^2|)x\right)^{\frac{q}{2}} \quad \text{(by Lemma 4.1(ii))}\\ &\leq \frac{1}{p}\psi\left(x,f^p(|t^2|)x\right)+\frac{1}{q}\psi\left(x,g^q(|(t^*)^2|)x\right) \quad \text{(by Lemma 4.3)}\\ &= \psi\left(\left\langle x,\left(\frac{1}{p}f^p(|t^2|)+\frac{1}{q}g^q(|(t^*)^2|)\right)x\right\rangle\right).\end{aligned}$$

It follows from the inequality (4.3)that

$$|\psi(\langle x,tx\rangle)|^2\leq\frac{1}{2}\left(\psi(|tx|)^2+\psi\left(\left\langle x,\left(\frac{1}{p}f^p(|t^2|)+\frac{1}{q}g^q(|(t^*)^2|)\right)x\right\rangle\right)\right).$$

Taking the supremum over all vectors $x\in\mathfrak{E}$ with $\psi(|x|)=1$, we get the desired result. □

The following lemma is useful in the sequel.

**Lemma 4.4.** *[11] Let $t\in\mathscr{L}(\mathfrak{E})$ and $\psi\in\varpi(\mathfrak{A})$ then for every $x\in\mathfrak{E}$*

$$Re\psi(\langle x,tx\rangle)=\psi(\langle x,Re(t)x\rangle),$$

*where $Re(t)$ denotes the real part of the operator $t\in\mathscr{L}(\mathfrak{E})$.*

**Theorem 4.3.** *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then*

$$w_c(s^*t) \leq \frac{1}{4}\left|\!\left|\!\left| |t^*|^2 + |s^*|^2 \right|\!\right|\!\right| + \frac{1}{2}w_c(ts^*).$$

*Proof.* First of all, we note that

$$w_c(t) = \sup_{\theta \in \mathbb{R}} \left|\!\left|\!\left| Re\left(e^{i\theta}t\right) \right|\!\right|\!\right|. \tag{4.5}$$

For every vector $x \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$ with $\psi(|x|) = 1$, we have

$$\begin{aligned}
Re\psi\left(\langle x, e^{i\theta}s^*tx\rangle\right) &= Re\psi\left(sx, e^{i\theta}tx\right) \\
&= \frac{1}{4}\left|\!\left|\!\left| \left(e^{i\theta}t + s\right)x \right|\!\right|\!\right|^2 - \frac{1}{4}\left|\!\left|\!\left| \left(e^{i\theta}t + s\right)x \right|\!\right|\!\right|^2 \quad \text{(by Polarization identity)} \\
&\leq \frac{1}{4}\left|\!\left|\!\left| \left(e^{i\theta}t + s\right)x \right|\!\right|\!\right|^2 \leq \frac{1}{4}\left|\!\left|\!\left| e^{i\theta}t + s \right|\!\right|\!\right|^2 \\
&= \frac{1}{4}\left|\!\left|\!\left| \left(e^{-i\theta}t^* + s^*\right) \right|\!\right|\!\right|^2 \quad (\text{since } |\!|\!|y|\!|\!| = |\!|\!|y^*|\!|\!|) \\
&= \frac{1}{4}\left|\!\left|\!\left| \left(e^{-i\theta}t^* + s^*\right)^*\left(e^{-i\theta}t^* + s^*\right) \right|\!\right|\!\right| \quad \left(\text{since } |\!|\!|y|\!|\!|^2 = |\!|\!|y^*y|\!|\!|\right) \\
&= \frac{1}{4}\left|\!\left|\!\left| tt^* + ss^* + e^{i\theta}ts^* + e^{-i\theta}st^* \right|\!\right|\!\right| \\
&\leq \frac{1}{4}|\!|\!|tt^* + ss^*|\!|\!| + \frac{1}{2}\left|\!\left|\!\left| Re(e^{i\theta}ts^*) \right|\!\right|\!\right|
\end{aligned}$$

Taking the supremum over all vectors $x \in \mathfrak{E}$ with $\psi(|x|) = 1$, we obtain

$$w_c(s^*t) \leq \frac{1}{4}\left|\!\left|\!\left| |t^*|^2 + |s^*|^2 \right|\!\right|\!\right| + \frac{1}{2}w_c(ts^*)$$

as required. ∎

The following theorem gives us a new bound for powers of the numerical radius.

**Theorem 4.4.** *Suppose $t, s, y \in \mathscr{L}(\mathfrak{E})$ such that $t, s$ are positive. Then*

$$w_c\left(t^\alpha y s^\alpha\right) \leq |\!|\!|y|\!|\!|^r \left|\!\left|\!\left| \frac{1}{p}t^{pr} + \frac{1}{q}s^{qr} \right|\!\right|\!\right|^\alpha$$

*for all $0 \leq \alpha \leq 1$, $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$.*

*Proof.* For every vector $x \in \mathfrak{E}$ with $\psi(|x| = 1)$, $\psi \in \varpi(\mathfrak{A})$, we have

$$\begin{aligned}
|\psi(\langle x, t^\alpha y s^\alpha x\rangle)|^r &= |\psi(\langle t^\alpha x, y s^\alpha x\rangle)|^r \\
&\leq |\!|\!|y|\!|\!|^r |\!|\!|t^\alpha x|\!|\!|^r |\!|\!|s^\alpha x|\!|\!|^r \\
&\leq |\!|\!|y|\!|\!|^r \psi\left(\left\langle x, t^{2\alpha}x\right\rangle^{\frac{r}{2}}\right)\psi\left(\left\langle x, s^{2\alpha}x\right\rangle^{\frac{r}{2}}\right) \\
&\leq |\!|\!|y|\!|\!|^r \left(\frac{1}{p}\psi\left(\langle x, t^{2\alpha}x\rangle\right)^{\frac{rp}{2}} + \frac{1}{q}\psi\left(\langle x, s^{2\alpha}x\rangle\right)^{\frac{qr}{2}}\right) \quad \text{(by Lemma 4.1(ii))} \\
&\leq |\!|\!|y|\!|\!|^r \left(\frac{1}{p}\psi\left(\langle x, t^{pr}x\rangle\right)^\alpha + \frac{1}{q}\psi\left(\langle x, s^{qr}x\rangle\right)^\alpha\right) \quad \text{(by Lemma 4.3)}
\end{aligned}$$

$$\begin{aligned} &\le \|y\|^r \left(\frac{1}{p}\psi\left(\langle x, t^{pr}x\rangle\right) + \frac{1}{q}\psi\left(\langle x, s^{qr}x\rangle\right)\right)^\alpha \quad \text{(by the concavity of } f(\tau)=\tau^\alpha) \\ &= \|y\|^r \psi\left(\left\langle x, \left(\frac{1}{p}t^{pr} + \frac{1}{q}t^{qr}\right)x\right\rangle\right)^\alpha . \end{aligned}$$

Taking the supremum over all vectors $x \in \mathfrak{E}$ with $\psi(|x|) = 1$, we obtain the desired result. □

Our next result is to find an upper bound for power of the numerical radius of $t^\alpha y s^{1-\alpha}$ under assumption $0 \le \alpha \le 1$.

**Theorem 4.5.** *Suppose $t, s, y \in \mathscr{L}(\mathfrak{E})$ such that $t, s$ are positive. Then*

$$w_c\left(t^\alpha y s^{1-\alpha}\right) \le \|y\|^r \|\alpha t^r + (1-\alpha)s^r\|$$

*for all $0 \le \alpha \le 1$ and $r \ge 2$.*

*Proof.* For every vector $x \in \mathfrak{E}$ with $\psi(|x| = 1)$, $\psi \in \varpi(\mathfrak{A})$, we have

$$\begin{aligned} \left|\psi\left(\left\langle x, t^\alpha y s^{1-\alpha}x\right\rangle\right)\right|^r &= \left|\psi\left(\left\langle t^\alpha x, y s^{1-\alpha}x\right\rangle\right)\right|^r \\ &\le \|y\|^r \|t^\alpha x\|^r \left\|s^{1-\alpha}x\right\|^r \\ &\le \|y\|^r \psi\left(\left\langle x, t^{2\alpha}x\right\rangle\right)^{\frac{r}{2}} \psi\left(\left\langle x, s^{2(1-\alpha)}x\right\rangle\right)^{\frac{r}{2}} \\ &\le \|y\|^r \psi\left(\langle x, t^r x\rangle\right)^\alpha \psi\left(\langle x, s^r x\rangle\right)^{1-\alpha} \quad \text{(by Lemma 4.3)} \\ &\le \|y\|^r \psi\left(\langle x, (\alpha t^r + (1-\alpha)s^r)x\rangle\right) \quad \text{(by Lemma 4.1(i))}. \end{aligned}$$

Hence

$$\left|\psi\left(\left\langle x, t^\alpha y s^{1-\alpha}x\right\rangle\right)\right|^r \le \|y\|^r \psi\left(\langle x, (\alpha t^r + (1-\alpha)s^r)x\rangle\right). \tag{4.6}$$

Taking the supremum over all vectors $x \in \mathfrak{E}$ with $\psi(|x|) = 1$, we obtain the desired result. □

**Remark 1.** Note that our inequality in the previous theorem is a generalization of the second inequality in Theorem 2.13 of [11] when we set $s = t = 1$.

Now assume that $t, s, y \in \mathscr{L}(\mathfrak{E})$. The Heinz mean for matrices are defined by

$$H_\alpha(t, s) = \frac{t^\alpha y s^{1-\alpha} + t^{1-\alpha} y s^\alpha}{2}$$

in which $\alpha \in [0, 1]$ and $t, s \ge 0$, see [7].

The goal of the following result is to find a numerical radius inequality for Heinz means. For this purpose, we use Theorem 4.5 and the convexity of function $f(\tau) = \tau^r$ $(r \ge 1)$.

**Theorem 4.6.** *Suppose $t, s, y \in \mathscr{L}(\mathfrak{E})$ such that $t, s$ are positive. Then*

$$\begin{aligned} w_c^r\left(t^{\frac{1}{2}} y s^{\frac{1}{2}}\right) &\le w_c^r\left(\frac{t^\alpha y s^{1-\alpha} + t^{1-\alpha} y s^\alpha}{2}\right) \\ &\le \|y\|^r w_c\left(\frac{t^r + s^r}{2}\right) \\ &\le \frac{\|y\|^r}{2}\left(\|\alpha t^r + (1-\alpha)s^r\| + \|\alpha s^r + (1-\alpha)t^r\|\right) \end{aligned}$$

*for all $r \ge 2$ and $\alpha \in [0, , 1]$.*

To prove Theorem 4.6, we need the following lemma.

**Lemma 4.5.** *Let $t, s \in \mathscr{L}(\mathfrak{E})$ be invertible self-adjoint operators and $y \in \mathscr{L}(\mathfrak{E})$. Then*

$$w_c(y) \leq w_c\left(\frac{tys^{-1} + t^{-1}ys}{2}\right). \tag{4.7}$$

*Proof.* First of all, we shall show the case $t = s$ and $y$ is self-adjoint. Let $\lambda \in \sigma(y)$. Then

$$\lambda \in \sigma(y) = \sigma(tyt^{-1}) \subseteq \overline{W(tyt^{-1})}.$$

Since $\lambda \in \mathbb{R}$ we have

$$\lambda = Re(\lambda) \in Re\overline{W(tyt^{-1})} = \overline{W\left(Re(tyt^{-1})\right)}.$$

So we obtain

$$w_c(y) = r(y) \leq w_c\left(Re(tyt^{-1})\right) = w_c\left(\frac{tys^{-1} + t^{-1}ys}{2}\right).$$

Next we shall show this lemma for arbitrary $y \in \mathscr{L}(\mathfrak{E})$ and invertible self-adjoint operators $t$ and $s$. Let $\tilde{y} = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}$ and $\tilde{t} = \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix}$. Then $\tilde{y}$ and $\tilde{t}$ are self-adjoint. Hence we have

$$w_c(\tilde{y}) \leq w_c\left(\frac{\tilde{t}\tilde{y}\tilde{t}^{-1} + \tilde{t}^{-1}\tilde{y}\tilde{t}}{2}\right).$$

Here $w_c(\tilde{y}) = w_c(y)$ and

$$\begin{aligned} w_c\left(\frac{\tilde{t}\tilde{y}\tilde{t}^{-1} + \tilde{t}^{-1}\tilde{y}\tilde{t}}{2}\right) &= \frac{1}{2} w_c\left(\begin{pmatrix} 0 & tys^{-1} + t^{-1}ys \\ s^{-1}y^*t + sy^*t^{-1} & 0 \end{pmatrix}\right) \\ &= \frac{1}{2} w_c\left(tys^{-1} + t^{-1}ys\right). \end{aligned}$$

Therefore we obtain the desired inequality. □

*Proof of Theorem 4.6.* We may assume that $t$ and $s$ are invertible. By Lemma 4.5, we have

$$\begin{aligned} w_c^r\left(t^{\frac{1}{2}}ys^{\frac{1}{2}}\right) &\leq w_c^r\left(\frac{t^{\alpha-\frac{1}{2}}t^{\frac{1}{2}}ys^{\frac{1}{2}}s^{\frac{1}{2}-\alpha} + t^{\frac{1}{2}-\alpha}t^{\frac{1}{2}}ys^{\frac{1}{2}}s^{\alpha-\frac{1}{2}}}{2}\right) \\ &= w_c^r\left(\frac{t^{\alpha}ys^{1-\alpha} + t^{1-\alpha}ys^{\alpha}}{2}\right). \end{aligned}$$

On the other hand, by inequality (4.6), for $r \geq 2$ we have

$$\left|\psi\left(\langle x, t^{\alpha}ys^{1-\alpha}x\rangle\right)\right|^r \leq \|\|y\|\|^r \psi\left(\langle x, (\alpha t^r + (1-\alpha)s^r)\, x\rangle\right).$$

Hence we have

$$\begin{aligned} \left|\psi\left(\left\langle x, \left(\frac{t^{\alpha}ys^{1-\alpha} + t^{1-\alpha}ys^{\alpha}}{2}\right)x\right\rangle\right)\right|^r &\leq \left(\frac{\left|\psi\left(\langle x, t^{\alpha}ys^{1-\alpha}x\rangle\right)\right| + \left|\psi\left(\langle x, t^{1-\alpha}ys^{\alpha}x\rangle\right)\right|}{2}\right)^r \\ &\leq \frac{\left|\psi\left(\langle x, t^{\alpha}ys^{1-\alpha}x\rangle\right)\right|^r + \left|\psi\left(\langle x, t^{1-\alpha}ys^{\alpha}x\rangle\right)\right|^r}{2} \\ &\qquad \text{(by the convexity of } f(\tau) = \tau^r\text{)} \end{aligned}$$

$$\begin{aligned} &\leq \frac{|||y|||^r}{2}\left[\psi\left(\langle x,(\alpha t^r+(1-\alpha)s^r)x\rangle\right)+\psi\left(\langle x,((1-\alpha)t^r+\alpha s^r)\rangle\right)\right] \\ &= |||y|||\psi\left(\left\langle x,\frac{t^r+s^r}{2}x\right\rangle\right). \end{aligned}$$

Thus we obtain

$$\begin{aligned} w_c^r\left(\frac{t^\alpha y s^{1-\alpha}+t^{1-\alpha}ys^\alpha}{2}\right) &\leq |||y|||w_c\left(\frac{t^r+s^r}{2}\right) \\ &\leq \frac{|||y|||}{2}\left(w_c\left(\alpha t^r+(1-\alpha)s^r\right)+w_c\left((1-\alpha)t^r+\alpha s^r\right)\right) \\ &= \frac{|||y|||}{2}\left(|||\alpha t^r+(1-\alpha)s^r|||+|||(1-\alpha)t^r+\alpha s^r|||\right). \end{aligned}$$

□

**Theorem 4.7.** *Let $a,b,c,d\in\mathscr{L}(\mathfrak{E})$ and $\mu,\nu\geq 1$. Then*

$$|||b^*a+d^*c|||^2\leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)}\left|\left|\left||a|^{2\mu}+|b|^{2\mu}\right|\right|\right|^{\frac{1}{\mu}}\left|\left|\left||c|^{2\nu}+|d|^{2\nu}\right|\right|\right|^{\frac{1}{\nu}}. \tag{4.8}$$

*Proof.* By the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left|\psi\left(\langle y,(b^*a+d^*c)\,x\rangle\right)\right|^2 &= \left|\psi\left(\langle y,b^*ax\rangle\right)+\psi\left(\langle y,d^*cx\rangle\right)\right|^2 \\ &\leq \left[\left|\psi\left(\langle y,b^*ax\rangle\right)\right|+\left|\psi\left(\langle y,d^*cx\rangle\right)\right|\right]^2 \qquad (4.9) \\ &\leq \left[\psi\left(\langle x,a^*ax\rangle\right)^{\frac{1}{2}}\psi\left(\langle y,b^*by\rangle\right)^{\frac{1}{2}}+\psi\left(\langle x,c^*cx\rangle\right)^{\frac{1}{2}}\psi\left(\langle y,d^*dy\rangle\right)^{\frac{1}{2}}\right]^2 \end{aligned}$$

for all $x,y\in\mathfrak{E}$.

Now, on utilizing the elementary inequality

$$(\kappa_1\kappa_2+\kappa_3\kappa_4)^2\leq\left(\kappa_1^2+\kappa_3^2\right)\left(\kappa_2^2+\kappa_4^2\right),\quad \kappa_i\in\mathbb{R}(i=1,2,3,4).$$

we then conclude that

$$\begin{aligned} &\left[\psi\left(\langle x,a^*ax\rangle\right)^{\frac{1}{2}}\psi\left(\langle y,b^*by\rangle\right)^{\frac{1}{2}}+\psi\left(\langle x,c^*cx\rangle\right)^{\frac{1}{2}}\psi\left(\langle y,d^*dy\rangle\right)^{\frac{1}{2}}\right]^2 \\ &=\left(\psi\left(\langle x,a^*ax\rangle\right)+\psi\left(\langle x,c^*cx\rangle\right)\right)\left(\psi\left(\langle y,b^*by\rangle\right)+\psi\left(\langle y,d^*dy\rangle\right)\right) \qquad (4.10) \end{aligned}$$

for all $x,y\in\mathfrak{E}$.

Utilizing the arithmetic mean - geometric mean inequality and then the convexity of the function $f(\tau)=\tau^\delta,\delta\geq 1$, we have successively,

$$\begin{aligned} &\left(\psi\left(\langle x,a^*ax\rangle\right)+\psi\left(\langle x,c^*cx\rangle\right)\right)\left(\psi\left(\langle y,b^*by\rangle\right)+\psi\left(\langle y,d^*dy\rangle\right)\right) \qquad (4.11) \\ &\leq 4\left(\frac{\psi\left(\langle x,\left((a^*a)^\mu+(c^*c)^\mu\right)x\rangle\right)}{2}\right)^{\frac{1}{\mu}}\left(\frac{\psi\left(\langle y,\left((b^*b)^\nu+(d^*d)^\nu\right)y\rangle\right)}{2}\right)^{\frac{1}{\nu}} \end{aligned}$$

for all $x,y\in\mathfrak{E}$ with $\psi(|x|)=\psi(|y|)=1$ and for all $\mu\geq 1$ and $\nu\geq 1$. Consequently, by (4.9)-(4.11) we have

$$\left|\psi\left(\langle y,(b^*a+d^*c)\,x\rangle\right)\right|^2$$

$$\leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)} \left(\psi\left(\langle x, ((a^*a)^\mu + (c^*c)^\mu)\, x\rangle\right)\right)^{\frac{1}{\mu}} \left(\psi\left(\langle y, ((b^*b)^\nu + (d^*d)^\nu)\, y\rangle\right)\right)^{\frac{1}{\nu}}$$

for all $x, y \in \mathfrak{E}$ with $\psi(|x|) = \psi(|y|) = 1$. Taking the supremum over $x, y \in \mathfrak{E}$ with $\psi(|x|) = \psi(|y|) = 1$ we deduce the desired inequality (4.8). □

**Remark 2.** (i) If $\mu = \nu$, then the inequality (4.8) is equivalent to

$$|||b^*a + d^*c|||^{2\mu} \leq 2^{2\mu-2} |||(a^*a)^\mu + (c^*c)^\mu||| \, |||(b^*b)^\mu + (d^*d)^\mu||| \tag{4.12}$$

(ii) If $b = d = 1$, then inequality (4.8) is equivalent to

$$|||a + c|||^{2\mu} \leq 2^{2\mu-1} |||(a^*a)^\mu + (c^*c)^\mu||| \tag{4.13}$$

for all $\mu \geq 1$.

(iii) If $b = a^*$ and $d = c^*$, then inequality (4.8) is equivalent to

$$\left|\left|\left|a^2 + c^2\right|\right|\right|^2 \leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)} |||(a^*a)^\mu + (c^*c)^\mu|||^{\frac{1}{\mu}} \, |||(b^*b)^\nu + (d^*d)^\nu|||^{\frac{1}{\nu}} \tag{4.14}$$

for all $\mu, \nu \geq 1$.

If we put $d = a$ and $c = b$ in the equality (4.8), we get the following result.

**Corollary 4.8.** *If $a, b \in \mathscr{L}(\mathfrak{E})$. Then*

$$|||b^*a + a^*b|||^2 \leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)} \left|\left|\left| |a|^{2\mu} + |b|^{2\mu} \right|\right|\right|^{\frac{1}{\mu}} \left|\left|\left| |a|^{2\nu} + |b|^{2\nu} \right|\right|\right|^{\frac{1}{\nu}}, \tag{4.15}$$

*for $\mu, \nu \geq 1$. In particular*

$$|||b^*a + a^*b|||^{\mu} \leq 2^{\mu-1} \left|\left|\left| |a|^{2\mu} + |b|^{2\mu} \right|\right|\right| \tag{4.16}$$

*for all $\mu \geq 1$.*

Another particular case that might be of interest is the following one.

**Corollary 4.9.** *For $a, d \in \mathscr{L}(\mathfrak{E})$, we have*

$$|||a + d|||^2 \leq 2^{2-\left(\frac{1}{\mu}+\frac{1}{\nu}\right)} \left|\left|\left| |a|^{2\mu} + 1 \right|\right|\right|^{\frac{1}{\mu}} \left|\left|\left| |d|^{2\nu} + 1 \right|\right|\right|^{\frac{1}{\nu}}, \tag{4.17}$$

*for all $\mu, \nu \geq 1$. In particular*

$$|||a|||^{2\mu} \leq \frac{1}{4} \left|\left|\left| |a|^{2\mu} + 1 \right|\right|\right|^2. \tag{4.18}$$

*for all $\mu \geq 1$.*

*Proof.* The proof of the inequality (4.17) is obvious by the inequality (4.8) on choosing $b = 1, c = 1$ and writing the inequality for $d^*$ instead of $d$. □

**Remark 3.** If $t \in \mathscr{L}(\mathfrak{E})$ and $t = a + ic$, i.e., $a$ and $c$ are its Cartesian decomposition, then we get from (4.13) that

$$|||t|||^{2\mu} \leq 2^{2\mu-1} \left|\left|\left| a^{2\mu} + c^{2\mu} \right|\right|\right|,$$

for all $\mu \geq 1$. Also, since $a = Re(t) = \frac{t+t^*}{2}$ and $c = Im(t) = \frac{t-t^*}{2i}$, then from (4.13) we get the following inequalities as well

$$|||Re(t)|||^{2\mu} \leq \frac{1}{2} \left|\left|\left| |t|^{2\mu} + |t^*|^{2\mu} \right|\right|\right|$$

and

$$|||Im(t)|||^{2\mu} \leq \frac{1}{2} \left|\left|\left| |t|^{2\mu} + |t^*|^{2\mu} \right|\right|\right|$$

for any $\mu \geq 1$.

**Theorem 4.10.** *Let $t = a + ib$ be the Cartesian decomposition of $t \in \mathscr{L}(\mathfrak{E})$. Then for $\mu, \nu \in \mathbb{R}$,*

$$\sup_{\mu^2+\nu^2=1} |||\mu a + \nu b||| = w_c(t). \tag{4.19}$$

*In particular,*

$$\frac{1}{2}|||t + t^*||| \leq w_c(t) \ \ \text{and} \ \ \frac{1}{2}|||t - t^*||| \leq w_c(t). \tag{4.20}$$

*Proof.* First of all, we note that

$$w(t) = \sup_{\theta \in \mathbb{R}} \left|\left|\left| Re(e^{i\theta} t) \right|\right|\right|. \tag{4.21}$$

In fact, $\sup_{\theta\in\mathbb{R}} Re\left(e^{i\theta}\psi\left(\langle x, tx\rangle\right)\right) = |\psi\left(\langle x, tx\rangle\right)|$ yields that

$$\sup_{\theta\in\mathbb{R}} \left|\left|\left| Re(e^{i\theta} t) \right|\right|\right| = \sup_{\theta\in\mathbb{R}} w_c\left(Re(e^{i\theta} t)\right) = w_c(t).$$

On the other hand, let $t = a + ib$ be the Cartesian decomposition of $t$. Then

$$\begin{aligned} Re\left(e^{i\theta} t\right) &= \frac{e^{i\theta} t + e^{-i\theta} t^*}{2} = \frac{1}{2}\left[\left(\cos\theta + i\sin\theta\right)t + \left(\cos\theta - i\sin\theta\right)t^*\right] \\ &= \cos\theta\left(\frac{t+t^*}{2}\right) - \sin\theta\left(\frac{t-t^*}{2i}\right) = \left(\cos\theta\right)a - \left(\sin\theta\right)b \end{aligned} \tag{4.22}$$

Therefore, by putting $\mu = \cos\theta$ and $\nu = -\sin\theta$ in (4.22), we obtain (4.19). Especially, by setting $(\mu, \nu) = (1, 0)$ and $(\mu, \nu) = (0, 1)$, we reach (4.20). □

**Remark 4.** By using (4.20), we get some known inequalities:

(i) $|||t||| = |||a + ib||| \leq |||a||| + |||b||| \leq 2w_c(t)$.

(ii) If $t$ is self adjoint, then $t = a$. Hence we have $|||t||| = |||a||| \leq w_c(t) \leq |||t|||$ and so $w_c(t) = |||t|||$.

(iii) By an easy calculation, we have $\frac{t^*t+tt^*}{2} = a^2 + b^2$. Hence,

$$\frac{1}{4}|||t^*t + tt^*||| = \frac{1}{2}\left|\left|\left| a^2 + b^2 \right|\right|\right| \leq \frac{1}{2}\left(|||a|||^2 + |||b|||^2\right) \leq w_c^2(t). \tag{4.23}$$

(iv) Let $\mu, \nu \in \mathbb{R}$ satisfy $\mu^2 + \nu^2 = 1$. Then for any vector $x \in \mathfrak{E}$ with $\psi(|x|) = 1$, $\psi \in \varpi(\mathfrak{A})$, we have

$$\begin{aligned} \|\|(\mu a + \nu b)x\|\| &= \left\|\left\|\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mu x \\ \nu x \end{bmatrix}\right\|\right\| \leq \left\|\left\|\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right\|\right\| = \left\|\left\|\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}\right\|\right\|^{\frac{1}{2}} \\ &= \left\|\left\|a^2 + b^2\right\|\right\|^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\|\|t^*t + tt^*\|\|^{\frac{1}{2}} \end{aligned}$$

Hence we have

$$w_c^2(t) = \sup_{\mu^2+\nu^2=1} \|\|\mu a + \nu b\|\|^2 \leq \frac{1}{2}\|\|t^*t + tt^*\|\|. \tag{4.24}$$

(v) Combining the inequalities (4.23) and (4.24), we obtain Theorem 3.2 of [11].

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