HANKEL DETERMINANT FOR CERTAIN CLASS OF ANALYTIC FUNCTION DEFINED BY GEBERALIZED DERIVATIVE OPERATOR

MA’MOUN HARA YZEH AL-ABBADI AND MASLINA DARUS

Abstract. The authors in [1] have recently introduced a new generalised derivatives operator $\mu_{n,m}^{\lambda_1,\lambda_2}$, which generalised many well-known operators studied earlier by many different authors. By making use of the generalised derivative operator $\mu_{n,m}^{\lambda_1,\lambda_2}$, the authors derive the class of function denoted by $\mathcal{H}_{n,m}^{\lambda_1,\lambda_2}$, which contain normalised analytic univalent functions $f$ defined on the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and satisfy

$$\text{Re} \left( \mu_{n,m}^{\lambda_1,\lambda_2} f(z) \right) > 0, \quad (z \in U).$$

This paper focuses on attaining sharp upper bound for the functional $|a_2a_4 - a_3^2|$ for functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to the class $\mathcal{H}_{n,m}^{\lambda_1,\lambda_2}$.

1. Introduction and Definitions

Throughout this paper, we use the following notation

$$s = \left[ (1 + \lambda_2) (1 + 3\lambda_2) \right]^m$$
$$r = \left[ (1 + \lambda_1) (1 + 3\lambda_1) \right]^{m-1}$$
$$l = (1 + 2\lambda_2)^{2m}$$
$$w = (1 + 2\lambda_1)^{2m-2}.$$

Let $\mathcal{A}$ denote the class of functions $f$ of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k \quad \text{is complex number} \quad (1.1)$$

which are normalised and analytic in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ on the complex plane $\mathbb{C}$. Consider $\mathcal{S}$ denote the subclass of $\mathcal{A}$ normalised analytic univalent functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad a_k \quad \text{is complex number} \quad (1.2)$$

Corresponding author: Maslina Darus.

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Let $S^*(\alpha), K(\alpha)$ \((0 \leq \alpha < 1)\) denote the subclasses of \(S\) consisting of functions that are starlike of order \(\alpha\) and convex of order \(\alpha\) in \(U\), respectively. In particular, the classes $S^*(0) = S^*$ and $K(0) = K$ are the familiar classes of starlike and convex functions in \(U\), respectively.

Let be given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Then the Hadamard product (or convolution) $f \ast g$ of two functions $f, g$ is defined by

$$f(z) \ast g(z) = (f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$ 

Next, we state basic ideas on $(x)_k$, which denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \frac{\Gamma(x + k)}{\Gamma(x)} = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C}\setminus\{0\}, \\ x(x+1)(x+2)\ldots(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \ldots\} \text{ and } x \in \mathbb{C}. \end{cases}$$ 

We need the following definitions throughout our investigations.

**Definition 1.1.** (Noonan and Thomas [15]). For the function $f$ given by (1.1) for $q \geq 1$ and $k \geq 0$, the $q^{th}$ Hankel determinant of $f$ is defined by

$$H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \ldots & a_{k+q+1} \\ a_{k+1} & a_{k+2} & \ldots & a_{k+q+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+q-1} & a_{k+q} & \ldots & a_{k+2q-2} \end{vmatrix}.$$ 

This determinant has also been considered by several authors. For example Noor in [16] determined the rate of growth $H_q(k)$ as $k \to \infty$ for functions $f$ given by (1.2) with bounded boundary. Ehrenborg in [6] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [11].

A classical theorem of Fekete and Szegö functional [7] considered the Hankel determinant of $f \in S$ for $q = 2$ and $n = 1$,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$ 

They made an early study for the estimates of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ and $\mu$ real. The well-known result due to this functional states that if $f \in S$ then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0, \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$
Hummel [9, 8] proved the conjecture of Singh that \(|a_3 - a_2^2| \leq \frac{1}{3}\) for the class \(C\) of convex functions. Keogh and Merkes [10] obtained sharp estimates for \(|a_3 - \mu a_2^2|\) when \(f\) is close-to-convex, starlike and convex in \(U\). Here, we consider the Hankel determinant of \(f \in \mathcal{S}\) for \(q = 2\) and \(n = 2\),

\[
H_2(2) = \begin{vmatrix}
a_2 & a_3 \\
a_3 & a_4
\end{vmatrix}.
\]

In the present paper, we seek upper bound for the functional \(|a_2 a_4 - a_3^2|\) for functions \(f\) belonging to the class \(\mathcal{K}_{\lambda_1, \lambda_2}^m\). The subclass \(\mathcal{K}_{\lambda_1, \lambda_2}^m\) is defined as the following:

**Definition 1.2.** Let \(f\) be given by (1.2). Then \(f\) is said to be in the class \(\mathcal{K}_{\lambda_1, \lambda_2}^m\) if it satisfies the inequality

\[
\text{Re} \left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' > 0, \quad (z \in U),
\]

where \(\mu_{\lambda_1, \lambda_2}^{n,m} f(z)\) denote the generalised derivative operator which was introduced by the authors [1] earlier. The generalised derivative operator is given as the following:

**Definition 1.3.** For \(f \in \mathcal{A}\) the generalised derivative operator \(\mu_{\lambda_1, \lambda_2}^{n,m}\) is defined by \(\mu_{\lambda_1, \lambda_2}^{n,m} : \mathcal{A} \to \mathcal{A}\)

\[
\mu_{\lambda_1, \lambda_2}^{n,m} f(z) = z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1(k-1))^{m-1}}{(1 + \lambda_2(k-1))^m} c(n,k) a_k z^k, \quad (z \in U),
\]

where \(n, m \in \mathbb{N}_0 = \{0, 1, 2, ... \}\), \(\lambda_2 \geq \lambda_1 \geq 0\) and \(c(n,k) = \frac{(n+k-1)!}{n! (1)! (k-1)!}\).

Special cases of this operator includes the Ruscheweyh derivative operator in the cases \(\mu_{\lambda_1, 0}^{n,1} \equiv \mu_{0,0}^{n,1} \equiv \mu_{0,\lambda_2}^{n,0} \equiv R^n [18]\), the Salagean derivative operator \(\mu_{1,0}^{0,m+1} \equiv S^n [19]\), the generalised Ruscheweyh derivative operator \(\mu_{\lambda_1,0}^{n,2} \equiv R^n_\lambda [4]\), the generalised Salagean derivative operator introduced by Al-Oboudi \(\mu_{\lambda_1,0}^{0,m+1} \equiv S^n_\beta [2]\), and the generalised Al-Shaqsi and Darus derivative operator \(\mu_{\lambda_1,0}^{n,m+1} \equiv D^n_\lambda \beta [3]\). It is easily seen that \(\mu_{\lambda_1,0}^{0,1} f(z) = \mu_{0,0}^{0,1} f(z) = \mu_{0,\lambda_2}^{0,1} f(z) = f(z)\) and \(\mu_{\lambda_1,0}^{1,1} f(z) = \mu_{0,0}^{1,0} f(z) = \mu_{0,\lambda_2}^{1,0} f(z) = z f'(z)\) and also \(\mu_{\lambda_1,0}^{a-1,0} f(z) = \mu_{0,0}^{a-1,m} f(z)\) where \(a = 1, 2, 3, ... \).

The subclass \(\mathcal{K}_{\lambda_1,1}^0\) was studied systematically by MacGregor [14] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

We first state some preliminary lemmas which shall be used in our proof.

### 2. Preliminary Results

To establish our results, we recall the following:

Let \(P\) be the family of all functions \(p\) analytic in \(U\) for which \(Re(p(z)) > 0\) and be given by the power series

\[
p(z) = 1 + c_1 z + c_2 z^2 + ... \quad (z \in U).
\] (2.1)
Lemma 2.1. (Pommerenke [17]). If $p \in P$. Then the sharp estimate

$$|c_k| \leq 2 \quad \text{for each } k,$$

(2.2)

and

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

Lemma 2.2. (Libera and Zlotkiewicz [12, 13]). Let the function $p \in P$ be given by the powers series (2.1). Then

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

(2.3)

for some $x, |x| \leq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

(2.4)

for some value of $z, |z| < 1$.

3. Main Result

Our main result is the following:

**Theorem 3.3.** Let the function $f$, given by (1.2) be in the class $\mathcal{H}^{n,m}_{\lambda_1, \lambda_2}$. Then

$$|a_2a_4 - a_3^2| \leq \frac{16 (1 + 2\lambda_2)^{2m}}{9(n + 1)^2(n + 2)^2 (1 + 2\lambda_1)^{2m-2}}.$$

The result obtained is sharp.

**Proof.** We refer to the method by Libera and Zlotkiewicz [12, 13]. For $f \in \mathcal{H}^{n,m}_{\lambda_1, \lambda_2}$, it follows from (1.3) that $\exists p \in P$ such that

$$\left( \mu^{n,m}_{\lambda_1, \lambda_2} f(z) \right)' = p(z) = 1 + c_1z + c_2z^2 + \ldots,$$

(3.1)

for some $(z \in U)$. From (3.1) computation and equating coefficients, we obtain

$$a_2 = \frac{(1 + \lambda_2)^m c_1}{2(n + 1)(1 + \lambda_1)^{m-1}},$$

$$a_3 = \frac{2(1 + 2\lambda_2)^m c_2}{3(n + 1)(n + 2)(1 + 2\lambda_1)^{m-1}},$$

$$a_4 = \frac{3(1 + 3\lambda_2)^m c_3}{2(n + 1)(n + 2)(n + 3)(1 + 3\lambda_1)^{m-1}}.$$

(3.2)
From (3.2), it can be easily established that

$$|a_2a_4 - a_3^2| = \frac{1}{(n+1)^2(n+2)} \left[ \frac{3sc_1c_3}{4r(n+3)} - \frac{4lc_2^2}{9w(n+2)} \right].$$

(3.3)

Where $s, r, l, w$ as mentioned before.

Since the function $p(z)$ is the member of the class $P$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ ($c \in [0, 2]$).

Using (2.3) along with (2.4), we get

$$|a_2a_4 - a_3^2| = \frac{1}{(n+1)^2(n+2)} \left[ \frac{3sc^4 + 6s(4 - c^2)c^2x - 3sc^2(4 - c^2)x^2 + 6sc(4 - c^2)(1 - |x|^2)z}{16r(n+3)} - \frac{lc^4}{9w(n+2)} - \frac{l x^2(4 - c^2)^2}{9w(n+2)} - \frac{2lc^2(4 - c^2)x}{9w(n+2)} \right],$$

$$= \frac{1}{(n+1)^2(n+2)} \left[ \frac{27sw(n+2) - 16rl(n+3)}{144rw(n+2)(n+3)} \right] c^4$$

$$+ \frac{27sw(n+2) - 16rl(n+3)}{72rw(n+2)(n+3)} c^2(4 - c^2)x$$

$$- (4 - c^2)x^2 \frac{[27sw(n+2) - 16rl(n+3)] c^2 + 64rl(n+3)}{144rw(n+2)(n+3)}$$

$$+ \frac{3sc(4 - c^2)(1 - |x|^2)z}{8r(n+3)}. $$

By triangle inequality we have

$$|a_2a_4 - a_3^2| \leq \frac{1}{(n+1)^2(n+2)} \left\{ \frac{|27sw(n+2) - 16rl(n+3)| c^4}{144rw(n+2)(n+3)} + \frac{3sc(4 - c^2)}{8r(n+3)}$$

$$+ \frac{c^2(4 - c^2)\rho |27sw(n+2) - 16rl(n+3)|}{72rw(n+2)(n+3)}$$

$$+ \frac{(4 - c^2)\rho (c - 2) (27sw(n+2)c - 16rl(n+3)(c + 2))}{144rw(n+2)(n+3)} \right\},$$

(3.4)

With $\rho = |x| \leq 1$. We assume that the upper bound for (3.4) attains at the interior point of $\rho \in [0, 1]$ and $c \in [0, 2]$, then

$$F' (\rho) = \frac{1}{(n+1)^2(n+2)} \left\{ \frac{c^2(4 - c^2) |27sw(n+2) - 16rl(n+3)|}{72rw(n+2)(n+3)}$$

$$+ \frac{(4 - c^2)\rho (c - 2) (27sw(n+2)c - 16rl(n+3)(c + 2))}{72rw(n+2)(n+3)} \right\}. $$
And with elementary calculus, we can show that $F'(\rho) > 0$ for $\rho > 0$, provided that $c - 2 < 0$ and $(27sw(n + 2)c - 16rl(n + 3)(c + 2)) < 0$.

Now, our goal is to prove the inequality

$$[27sw(n + 2)c - 16rl(n + 3)(c + 2)] < 0.$$  (3.5)

Now, (3.5) can be simplified to

$$sw(27n + 54)c < rl(16n + 48)(c + 2).$$  (3.6)

So (3.6) is true provided that our two inequalities

$$(27n + 54)c < (16n + 48)(c + 2),$$  (3.7)

and

$$sw < rl,$$  (3.8)

are satisfied.

First, we need to show the inequality (3.7) holds, so from (3.7) we have

$$11nc + 6c < 32n + 96,$$

and immediately implies that

$$n(32 - 11c) + 6(16 - c) > 0.$$

Thus inequality (3.7) is true.

Next, we want to show the inequality $sw < rl$ holds. This inequality reduces to

$$\left[\frac{(1 + \lambda_2)(1 + 3\lambda_2)(1 + 2\lambda_1)^2}{(1 + \lambda_1)(1 + 3\lambda_1)(1 + 2\lambda_2)^2}\right]^m \frac{(1 + \lambda_1)(1 + 3\lambda_1)}{(1 + 2\lambda_1)^2} < 1.$$  (3.9)

From (3.9), we must show that the inequalities

$$\frac{(1 + \lambda_1)(1 + 3\lambda_1)}{(1 + 2\lambda_1)^2} < 1,$$  (3.10)

and

$$\frac{(1 + \lambda_2)(1 + 3\lambda_2)(1 + 2\lambda_1)^2}{(1 + \lambda_1)(1 + 3\lambda_1)(1 + 2\lambda_2)^2} < 1,$$  (3.11)

are true.

Now, from (3.10) it is easy to see that

$$1 + 4\lambda_1 + 3\lambda_1^2 < 1 + 4\lambda_1 + 4\lambda_1^2.$$
and obviously
\[ \lambda_1^2 > 0. \]

Hence the proof is done for particular inequality (3.10).

Next we need to prove the inequality (3.11) is true. So, by doing tedious calculations for (3.11), we shall get
\[ (1 + 4\lambda_2 + 3\lambda_2^2)(1 + 4\lambda_1 + 4\lambda_1^2) < (1 + 4\lambda_1 + 3\lambda_1^2)(1 + 4\lambda_2 + 4\lambda_2^2), \]
and a straightforward calculation and some simplifications, we can conclude that
\[ \lambda_1^2 - \lambda_2^2 + 4\lambda_1^2\lambda_2 - 4\lambda_2^2\lambda_1 < 0, \]
and therefore
\[ (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + 4\lambda_1\lambda_2) < 0. \]
Hence the proof for inequality (3.11) is complete.

Now after satisfying the inequalities (3.5) and \( c - 2 < 0 \) we observed that \( F'(\rho) > 0 \) for \( \rho \in [0, 1] \), implying that \( F \) is an increasing function and thus the upper bound for (3.4) corresponds to \( \rho = 1 \) and so \( \text{max}F(\rho) = F(1) \). This contradicts our assumption of having the maximum value in the interior of \( \rho \in [0, 1] \).

Now let,
\[
G(c) = F(1) = \frac{1}{(n + 1)^2(n + 2)^2} \left\{ \frac{|27sw(n + 2) - 16rl(n + 3)|c^4}{144rwn(n + 2)(n + 3)} + \frac{3sc(4 - c^2)}{8rl(n + 3)} \
+ \frac{c^2(4 - c^2)|27sw(n + 2) - 16rl(n + 3)|}{72rwn(n + 2)(n + 3)} \
+ \frac{(4 - c^2)(c - 2)(27sw(n + 2)c - 16rl(n + 3)(c + 2))}{144rwn(n + 2)(n + 3)} \right\}.
\]
Assume that \( G(c) \) has a maximum value in an interior of \( c \in [0, 2] \), by elementary calculation we find
\[
G'(c) = \frac{c}{36rwn(n + 1)^2(n + 2)^2(n + 3)} \left[ (4 - c^2)|27sw(n + 2) - 16rl(n + 3)| \
+ 27sw(n + 2)(2 - c^2) + 16rl(n + 3)(c^2 - 4) \right].
\]
Then \( G'(0) = 0 \) implies the real critical point \( c_* = 0 \) or
\[
c_* = \sqrt{\frac{64rl(n + 3) - 54sw(n + 2) - 4|27sw(n + 2) - 16rl(n + 3)|}{16rl(n + 3) - 27sw(n + 2) - |27sw(n + 2) - 16rl(n + 3)|}}.
\]
Through some calculations we observe that \( c_* > 2 \), however \( c_* \) is out of the interval \([0, 2]\). A calculation showed that the maximum value occurs at \( c = 0 \) or \( c = c_* \) which contradicts our assumption of having the maximum value at the interior point of \( c \in [0, 2] \). Thus any maximum point of \( G \) must be on the boundary of \( c \in [0, 2] \).

At \( c = 0 \), we have
\[
G(c) = G(0) = \frac{16l}{9w(n + 2)},
\]
and at \( c = 2 \), we obtain
\[
G(c) = G(2) = \frac{|27sw(n + 2) - 16rl(n + 3)|}{9rw(n + 2)(n + 3)}.
\]

It is obvious that \( G(0) > G(2) \) for the two choices of \( |27sw(n + 2) - 16rl(n + 3)| \). Hence \( G \) attains maximum value at \( c = 0 \). Therefore the upper bound for (3.4) corresponds to \( \rho = 1 \) and \( c = 0 \) in which case
\[
|a_2a_4 - a_3^2| \leq \frac{16(1 + 2\lambda_2)^{2m}}{9(n + 1)^2(n + 2)^2(1 + 2\lambda_1)^{2m-2}}.
\]

Equality holds for the functions in \( \mathcal{H}^{n,m}_{\lambda_1,\lambda_2} \) given by
\[
f'(z) = \frac{1 + z^2}{1 - z^2}.
\]

This concludes the proof of our theorem.

Note that this problem has yet to be solved for certain classes introduced in various studies (see for examples [5], [21], [22] and [23]). Note that Hankel problems have also been solved successfully for fractional operator which can be seen in [20].

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References


School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia.

E-mail: mamoun_nn@yahoo.com

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia.

E-mail: maslina@ukm.my