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HANKEL DETERMINANT FOR CERTAIN CLASS OF ANALYTIC FUNCTION DEFINED BY GEBERALIZED DERIVATIVE OPERATOR

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Abstract. The authors in [1] have recently introduced a new generalised derivatives operator $\mu^{n,m}_{\lambda_1,\lambda_2}$, which generalised many well-known operators studied earlier by many different authors. By making use of the generalised derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$, the authors derive the class of function denoted by $\mathcal{H}^{n,m}_{\lambda_1,\lambda_2}$, which contain normalised analytic univalent functions f defined on the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy

$$\mathrm{Re}\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' > 0, \quad (z \in U).$$

This paper focuses on attaining sharp upper bound for the functional $\left|a_2 a_4 - a_3^2\right|$ for functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to the class $\mathcal{H}^{n,m}_{\lambda_1,\lambda_2}$.

1. Introduction and Definitions

Throughout this paper, we use the following notation

$$\begin{aligned} s &= [(1+\lambda_2)(1+3\lambda_2)]^m \\ r &= [(1+\lambda_1)(1+3\lambda_1)]^{m-1} \\ l &= (1+2\lambda_2)^{2m} \\ w &= (1+2\lambda_1)^{2m-2}. \end{aligned}$$

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_k \quad \text{is complex number} \tag{1.1}$$

which are normalised and analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Consider $\mathcal{S}$ denote the subclass of $\mathcal{A}$ normalised analytic univalent functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \quad \text{is complex number} \tag{1.2}$$

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Let $S^*(\alpha)$, $K(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclasses of $\mathscr{S}$ consisting of functions that are starlike of order α and convex of order α in U, respectively. In particular, the classes $S^*(0) = S^*$ and $K(0) = K$ are the familiar classes of starlike and convex functions in U, respectively.

Let be given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Then the Hadamard product (or convolution) $f * g$ of two functions f, g is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k .$$

Next, we state basic ideas on $(x)_k$, which denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{for } k = 0,\ x \in \mathbb{C}\backslash\{0\}, \\ x(x+1)(x+2)\ldots(x+k-1) & \text{for } k \in \mathbb{N} = \{1,2,3,\ldots\} \text{ and } x \in \mathbb{C}. \end{cases}$$

We need the following definitions throughout our investigations.

Definition 1.1. (Noonan and Thomas [15]). For the function f given by (1.1) for $q \geq 1$ and $k \geq 0$, the q^{th} Hankel determinant of f is defined by

$$H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q+1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+q+2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+q-1} & a_{k+q} & \cdots & a_{k+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example Noor in [16] determined the rate of growth $H_q(k)$ as $k \to \infty$ for functions f given by (1.2) with bounded boundary. Ehrenborg in [6] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [11].

A classical theorem of Fekete and Szegö functional [7] considered the Hankel determinant of $f \in \mathscr{S}$ for $q = 2$ and $n = 1$,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$

They made an early study for the estimates of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ and μ real. The well-known result due to this functional states that if $f \in \mathscr{S}$ then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0, \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

Hummel [9, 8] proved the conjecture of Singh that $\left|a_3 - a_2^2\right| \le \frac{1}{3}$ for the class $\mathscr{C}$ of convex functions. Keogh and Merkes [10] obtained sharp estimates for $\left|a_3 - \mu a_2^2\right|$ when f is close-to-convex, starlike and convex in U. Here, we consider the Hankel determinant of $f \in \mathscr{S}$ for $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

In the present paper, we seek upper bound for the functional $\left|a_2 a_4 - a_3^2\right|$ for functions f belonging to the class $\mathscr{H}^{n,m}_{\lambda_1,\lambda_2}$. The subclass $\mathscr{H}^{n,m}_{\lambda_1,\lambda_2}$ is defined as the following:

Definition 1.2. Let f be given by (1.2). Then f is said to be in the class $\mathscr{H}^{n,m}_{\lambda_1,\lambda_2}$ if it satisfies the inequality

$$\operatorname{Re}\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' > 0, \quad (z \in U), \tag{1.3}$$

where $\mu^{n,m}_{\lambda_1,\lambda_2} f(z)$ denote the generalised derivative operator which was introduced by the authors [1] earlier. The generalised derivative operator is given as the following:

Definition 1.3. For $f \in \mathscr{A}$ the generalised derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ is defined by $\mu^{n,m}_{\lambda_1,\lambda_2} : \mathscr{A} \to \mathscr{A}$

$$\mu^{n,m}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} c(n,k) a_k z^k, \quad (z \in U),$$

where $n, m \in \mathbb{N}_0 = \{0, 1, 2 \dots\}$, $\lambda_2 \ge \lambda_1 \ge 0$ and $c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator includes the Ruscheweyh derivative operator in the cases $\mu^{n,1}_{\lambda_1,0} \equiv \mu^{n,m}_{0,0} \equiv \mu^{n,0}_{0,\lambda_2} \equiv R^n$ [18], the Salagean derivative operator $\mu^{0,m+1}_{1,0} \equiv S^n$ [19], the generalised Ruscheweyh derivative operator $\mu^{n,2}_{\lambda_1,0} \equiv R^n_\lambda$ [4], the generalised Salagean derivative operator introduced by Al-Oboudi $\mu^{0,m+1}_{\lambda_1,0} \equiv S^n_\beta$ [2], and the generalised Al-Shaqsi and Darus derivative operator $\mu^{n,m+1}_{\lambda_1,0} \equiv D^n_{\lambda,\beta}$ [3]. It is easily seen that $\mu^{0,1}_{\lambda_1,0} f(z) = \mu^{0,m}_{0,0} f(z) = \mu^{0,0}_{0,\lambda_2} f(z) = f(z)$ and $\mu^{1,1}_{\lambda_1,0} f(z) = \mu^{1,m}_{0,0} f(z) = \mu^{1,0}_{0,\lambda_2} f(z) = z f'(z)$ and also $\mu^{a-1,0}_{\lambda_1,0} f(z) = \mu^{a-1,m}_{0,0} f(z)$ where $a = 1, 2, 3, \dots$.

The subclass $\mathscr{H}^{0,1}_{\lambda_1,0}$ was studied systematically by MacGregor [14] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

We first state some preliminary lemmas which shall be used in our proof.

2. Preliminary Results

To establish our results, we recall the following:

Let P be the family of all functions p analytic in U for which $Re(p(z)) > 0$ and be given by the power series

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots . \quad (z \in U). \tag{2.1}$$

Lemma 2.1. *(Pommerenke [17]). If $p \in P$. Then the sharp estimate*

$$|c_k| \le 2 \qquad \text{for each } k, \tag{2.2}$$

and

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}.$$

Lemma 2.2. *(Libera and Zlotkiewicz [12, 13]). Let the function $p \in P$ be given by the powers series (2.1). Then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.3}$$

for some $x, |x| \le 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{2.4}$$

for some value of $z, |z| < 1$.

3. Main Result

Our main result is the following:

Theorem 3.3. *Let the function f, given by (1.2) be in the class $\mathcal{H}^{n,m}_{\lambda_1,\lambda_2}$. Then*

$$\left| a_2 a_4 - a_3^2 \right| \le \frac{16\,(1 + 2\lambda_2)^{2m}}{9(n+1)^2(n+2)^2\,(1 + 2\lambda_1)^{2m-2}}.$$

The result obtained is sharp.

Proof. We refer to the method by Libera and Zlotkiewicz [12, 13]. For $f \in \mathcal{H}^{n,m}_{\lambda_1,\lambda_2}$, it follows from (1.3) that $\exists p \in P$ such that

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z) \right)' = p(z) = 1 + c_1 z + c_2 z^2 + \dots, \tag{3.1}$$

for some $(z \in U)$. From (3.1) computation and equating coefficients, we obtain

$$\left. \begin{aligned} a_2 &= \frac{(1 + \lambda_2)^m\, c_1}{2(n+1)\,(1 + \lambda_1)^{m-1}} \\ a_3 &= \frac{2\,(1 + 2\lambda_2)^m\, c_2}{3(n+1)(n+2)\,(1 + 2\lambda_1)^{m-1}} \\ a_4 &= \frac{3\,(1 + 3\lambda_2)^m\, c_3}{2(n+1)(n+2)(n+3)\,(1 + 3\lambda_1)^{m-1}} \end{aligned} \right\}. \tag{3.2}$$

From (3.2), it can be easily established that

$$\left|a_2a_4-a_3^2\right|=\frac{1}{(n+1)^2(n+2)}\left|\frac{3sc_1c_3}{4r(n+3)}-\frac{4lc_2^2}{9w(n+2)}\right|. \tag{3.3}$$

Where s, r, l, w as mentioned before.

Since the function $p(z)$ is the member of the class P simultaneously, we assume without loss of generality that $c_1>0$. For convenience of notation, we take $c_1=c$ ($c\in[0,2]$).

Using (2.3) along with (2.4), we get

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &= \frac{1}{(n+1)^2(n+2)}\\ &\quad\times\left|\frac{3sc^4+6s(4-c^2)c^2x-3sc^2(4-c^2)x^2+6sc(4-c^2)(1-|x|^2)z}{16r(n+3)}\right.\\ &\quad\left.-\frac{lc^4}{9w(n+2)}-\frac{lx^2(4-c^2)^2}{9w(n+2)}-\frac{2lc^2(4-c^2)x}{9w(n+2)}\right|,\\ &= \frac{1}{(n+1)^2(n+2)}\left|\left(\frac{27sw(n+2)-16rl(n+3)}{144rw(n+2)(n+3)}\right)c^4\right.\\ &\quad+\left(\frac{27sw(n+2)-16rl(n+3)}{72rw(n+2)(n+3)}\right)c^2(4-c^2)x\\ &\quad-(4-c^2)x^2\left(\frac{[27sw(n+2)-16rl(n+3)]\,c^2+64rl(n+3)}{144rw(n+2)(n+3)}\right)\\ &\quad\left.+\frac{3sc(4-c^2)(1-|x|^2)z}{8r(n+3)}\right|.\end{aligned}$$

By triangle inequality we have

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &\le \frac{1}{(n+1)^2(n+2)}\left\{\frac{|27sw(n+2)-16rl(n+3)|\,c^4}{144rw(n+2)(n+3)}+\frac{3sc(4-c^2)}{8r(n+3)}\right.\\ &\quad+\frac{c^2(4-c^2)\rho\,|27sw(n+2)-16rl(n+3)|}{72rw(n+2)(n+3)}\\ &\quad\left.+\frac{(4-c^2)\rho^2\,(c-2)\,(27sw(n+2)c-16rl(n+3)(c+2))}{144rw(n+2)(n+3)}\right\},\\ &= F(\rho).\end{aligned} \tag{3.4}$$

With $\rho=|x|\le 1$. We assume that the upper bound for (3.4) attains at the interior point of $\rho\in[0,1]$ and $c\in[0,2]$, then

$$\begin{aligned}F'(\rho) &= \frac{1}{(n+1)^2(n+2)}\left\{\frac{c^2(4-c^2)\,|27sw(n+2)-16rl(n+3)|}{72rw(n+2)(n+3)}\right.\\ &\quad\left.+\frac{(4-c^2)\rho\,(c-2)\,(27sw(n+2)c-16rl(n+3)(c+2))}{72rw(n+2)(n+3)}\right\}.\end{aligned}$$

And with elementary calculus, we can show that $F'(\rho) > 0$ for $\rho > 0$, provided that $c-2<0$ and $(27sw(n+2)c - 16rl(n+3)(c+2)) < 0$.

Now, our goal is to prove the inequality

$$[27sw(n+2)c - 16rl(n+3)(c+2)] < 0. \tag{3.5}$$

Now, (3.5) can be simplified to

$$sw(27n+54)c < rl(16n+48)(c+2). \tag{3.6}$$

So (3.6) is true provided that our two inequalities

$$(27n+54)c < (16n+48)(c+2), \tag{3.7}$$

and

$$sw < rl, \tag{3.8}$$

are satisfied.

First, we need to show the inequality (3.7) holds, so from (3.7) we have

$$11nc + 6c < 32n + 96,$$

and immediately implies that

$$n(32-11c) + 6(16-c) > 0.$$

Thus inequality (3.7) is true.

Next, we want to show the inequality $sw < rl$ holds. This inequality reduces to

$$\left[\frac{(1+\lambda_2)(1+3\lambda_2)(1+2\lambda_1)^2}{(1+\lambda_1)(1+3\lambda_1)(1+2\lambda_2)^2}\right]^m \frac{(1+\lambda_1)(1+3\lambda_1)}{(1+2\lambda_1)^2} < 1. \tag{3.9}$$

From (3.9), we must show that the inequalities

$$\frac{(1+\lambda_1)(1+3\lambda_1)}{(1+2\lambda_1)^2} < 1, \tag{3.10}$$

and

$$\frac{(1+\lambda_2)(1+3\lambda_2)(1+2\lambda_1)^2}{(1+\lambda_1)(1+3\lambda_1)(1+2\lambda_2)^2} < 1, \tag{3.11}$$

are true.

Now, from (3.10) it is easy to see that

$$1 + 4\lambda_1 + 3\lambda_1^2 < 1 + 4\lambda_1 + 4\lambda_1^2$$

and obviously

$$\lambda_1^2 > 0.$$

Hence the proof is done for particular inequality (3.10).

Next we need to prove the inequality (3.11) is true. So, by doing tedious calculations for (3.11), we shall get

$$\left(1+4\lambda_2+3\lambda_2^2\right)\left(1+4\lambda_1+4\lambda_1^2\right) < \left(1+4\lambda_1+3\lambda_1^2\right)\left(1+4\lambda_2+4\lambda_2^2\right),$$

and a straightforward calculation and some simplifications, we can conclude that

$$\lambda_1^2-\lambda_2^2+4\lambda_1^2\lambda_2-4\lambda_2^2\lambda_1 < 0,$$

and therefore

$$(\lambda_1-\lambda_2)(\lambda_1+\lambda_2+4\lambda_1\lambda_2) < 0.$$

Hence the proof for inequality (3.11) is complete.

Now after satisfying the inequalities (3.5) and $c-2<0$ we observed that $F'(\rho)>0$ for $\rho \in [0,1]$, implying that F is an increasing function and thus the upper bound for (3.4) corresponds to $\rho = 1$ and so $maxF(\rho) = F(1)$. This contradicts our assumption of having the maximum value in the interior of $\rho \in [0,1]$.

Now let,

$$\begin{aligned} G(c) = F(1) = \frac{1}{(n+1)^2(n+2)} \Bigg\{ & \frac{|27sw(n+2)-16rl(n+3)|\,c^4}{144rw(n+2)(n+3)} + \frac{3sc(4-c^2)}{8r(n+3)} \\ & + \frac{c^2(4-c^2)\,|27sw(n+2)-16rl(n+3)|}{72rw(n+2)(n+3)} \\ & + \frac{(4-c^2)(c-2)(27sw(n+2)c-16rl(n+3)(c+2))}{144rw(n+2)(n+3)} \Bigg\}. \end{aligned}$$

Assume that $G(c)$ has a maximum value in an interior of $c \in [0,2]$, by elementary calculation we find

$$\begin{aligned} G'(c) = & \frac{c}{36rw(n+1)^2(n+2)^2(n+3)} \left[(4-c^2)\,|27sw(n+2)-16rl(n+3)| \right. \\ & \left. + \ 27sw(n+2)(2-c^2)+16rl(n+3)(c^2-4)\right]. \end{aligned} \tag{3.12}$$

Then $G'(0) = 0$ implies the real critical point $c_\bullet = 0$ or

$$c_* = \sqrt{\frac{64rl(n+3)-54sw(n+2)-4\,|27sw(n+2)-16rl(n+3)|}{16rl(n+3)-27sw(n+2)-|27sw(n+2)-16rl(n+3)|}}.$$

Through some calculations we observe that $c_* > 2$, however c_* is out of the interval $[0,2]$. A calculation showed that the maximum value occurs at $c = 0$ or $c = c_*$ which contradicts our assumption of having the maximum value at the interior point of $c \in [0,2]$. Thus any maximum point of G must be on the boundary of $c \in [0,2]$.

At $c = 0$, we have

$$G(c) = G(0) = \frac{16l}{9w(n+2)},$$

and at $c = 2$, we obtain

$$G(c) = G(2) = \frac{|27sw(n+2) - 16rl(n+3)|}{9rw(n+2)(n+3)}.$$

It is obvious that $G(0) > G(2)$ for the two choices of $|27sw(n+2) - 16rl(n+3)|$. Hence G attains maximum value at $c = 0$. Therefore the upper bound for (3.4) corresponds to $\rho = 1$ and $c = 0$ in which case

$$\left|a_2a_4 - a_3^2\right| \leq \frac{16(1+2\lambda_2)^{2m}}{9(n+1)^2(n+2)^2(1+2\lambda_1)^{2m-2}}.$$

Equality holds for the functions in $\mathscr{H}^{n,m}_{\lambda_1,\lambda_2}$ given by

$$f'(z) = \frac{1+z^2}{1-z^2}.$$

This concludes the proof of our theorem.

Note that this problem has yet to be solved for certain classes introduced in various studies (see for examples [5], [21], [22] and [23]). Note that Hankel problems have also been solved successfully for fractional operator which can be seen in [20].

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