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Common solutions of fixed point and generalized equilibrium problems using asymptotically nonexpansive mapping

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Abstract. In this paper, a common solution of fixed point and generalized equilibrium problems using an asymptotically nonexpansive mapping is determined via iterative approach. Further an application and numerical example of the main result are given.

Keywords. Variational inequality, fixed point, asymptotically nonexpansive mappings, generalized equilibrium problems.

1 Introduction

Let H be a real Hilbert space and $\langle .,. \rangle$ and $\|.\|$ denote inner product and induced norm respectively. Let D $(\neq \phi) \subset H$ be closed and convex and $\{q_n\}$ be a sequence in H, then $q_n \to q$ and $q_n \rightharpoonup q$ denote the strong and weak convergence of sequence $\{q_n\}$ to a point $q \in H$ respectively. For every $q \in H$,

$$\|q - P_D q\| \leq \|q - r\|, \forall r \in D.$$

Then $P_D : H \to D$ is known as metric projection.
A mapping $T : D \to D$ is called asymptotically nonexpansive [8] if there exists a sequence $\{l_n\}$ with $\lim_{n\to\infty} l_n = 1$ and

$$\|T^n q - T^n r\| \leq l_n \|q - r\|, \forall q, r \in D.$$

T is known as l-Lipschitzian ($l \geq 0$) if

$$\|T^n q - T^n r\| \leq l \|q - r\|, \forall q, r \in D, \forall n \in N.$$

If $l_n = 1, \forall n \in N$, then T becomes nonexpansive mapping.
The concept of asymptotically nonexpansive mappings is the generalization of the concept of nonexpansive mappings and it was introduced by Geobel and Kirk [8].
In the fixed point problem (FPP), $q \in D$ is to be determined such that

$$Tq = q. \tag{1.1}$$

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Now,

$$F(T) = \{q \in D : Tq = q\}.$$

Consider a bifunction $G : D \times D \to R$ and a nonlinear mapping $B : D \to H$.
To find $d \in D$ such that

$$G(d, d_1) + \langle Bd, d_1 - d\rangle \geq 0, \forall d_1 \in D, \tag{1.2}$$

is known as generalized equilibrium problem.
EP(G,B) denotes the solution set of problem (1.2).
If $B \equiv 0$ in (1.2), then the reduced form of problem (1.2) of determining $d \in D$ such that

$$G(d, d_1) \geq 0, \forall d_1 \in D, \tag{1.3}$$

is known as equilibrium problem. EP(G) denotes the solution set of problem (1.3).
If $G \equiv 0$ in (1.2), then the reduced form of problem (1.2) of determining $d \in D$ such that

$$\langle Bd, d_1 - d\rangle \geq 0, \forall d_1 \in D, \tag{1.4}$$

is known as classical variational inequality problem. VI(D,B) denotes the solution set of problem (1.4).
If $G(d, d_1) = \langle Bd, d_1 - d\rangle$, for all $d_1, d \in D$, then $d \in EP(G)$ if and only if $d \in VI(D, B)$.
The problem (1.2) is general and it includes many problems like variational inequality, optimization, Nash equilibrium problems etc. as special cases. In [13, 14], the researcher established fixed point iteration results via nonexpansive mappings. Also, the new iterative algorithms are introduced by [15, 19, 21].
[12] "Let $G : D \times D \to R$ be a bifunction which satisfies the following conditions:

(G1) $G(p, p) = 0$ for all $p \in D$;

(G2) G is monotone, i.e., $G(p, q) + G(q, p) \leq 0$ for all $p, q \in D$;

(G3) for each $p, q, r \in D$;
$$\lim_{t \to 0} sup G(tr + (1 - t)p, q) \leq G(p, q);$$

(G4) for each $p \in D$, $q \mapsto G(p, q)$ is convex and lower semicontinuous."

Definition 1. [12] " A mapping $B : D \to H$ is called β-inverse strongly monotone if there exists $\beta \geq 0$ such that $\langle Bp - Bq, p - q\rangle \geq \beta\|Bp - Bq\|^2$, $\forall p, q \in D$."

Takahashi and Takahashi [16] proved the following result:

Theorem 1.1. *[16] "Let D be a nonempty closed convex subset of a real Hilbert space H and let $G : D \times D \to R$ be a bifunction satisfying (G1), (G2), (G3) and (G4). Let $B : D \to H$ be a β-inverse strongly monotone mapping and $T : D \to D$ be nonexpansive mapping such that $F(T) \bigcap EP(G, B) \neq \phi$. Let $u \in D$, $p_1 \in D$ and let $\{r_n\} \subset D$ and $\{p_n\} \subset D$ be sequence defined by*

$$\begin{cases} G(r_n, q) + \langle Bp_n, q - r_n\rangle + \frac{1}{\mu_n}\langle q - r_n, r_n - p_n\rangle \geq 0, \ \forall q \in D, \\ p_{n+1} = \beta_n p_n + (1 - \beta_n)T[\alpha_n u + (1 - \alpha_n)r_n], \ \forall n \in N. \end{cases} \tag{1.5}$$

Let $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{\mu_n\} \subset [0, 2\beta]$ satisfies

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $lim_{n\to\infty}(\mu_n - \mu_{n+1}) = 0$,

(iii) $0 < c \leq \beta_n \leq d < 1$, $0 < a \leq \mu_n \leq b < 2\beta$,

then $\{p_n\}$ *converges strongly to* $r = P_{F(T)\bigcap EP(G,B)}(u)$.*"*

In this paper, the result of Takahashi and Takahashi is further extended for asymptotically nonexpansive mappings. The common solution of FPP and EP is obtained by iterative approach.

2 Preliminaries

Some definitions and lemmas are as follows:

Definition 2. [12] "A mapping $g : D \to D$ is known as contraction mapping if there exists $\eta \in [0, 1)$ such that

$$\|g(p) - g(q)\| \leq \|p - q\|, \ \forall p, q \in D."$$

Definition 3. [4] "Let D be a closed convex subset of a Hilbert space H. A mapping $T : D \to D$ is called asymptotically regular at p if and only if $\lim_{n\to\infty} \|T^n p - T^{n+1}p\| = 0$."

Lemma 2.1. *([3]): "Let D be a nonempty closed convex subset of a real Hilbert space H. Let* $G : D \times D \to R$ *be a bifunction satisfying (G1),(G2), (G3) and (G4). Let* $t > 0$ *and* $p \in H$*. Then, there exists* $r \in D$ *such that*

$$G(r, q) + \frac{1}{t}\langle q - r, r - p\rangle \geq 0, \ \forall q \in D."$$

Lemma 2.2. *([7]): "Let D be a nonempty closed convex subset of a real Hilbert space H and let* $G : D \times D \to R$ *be a bifunction satisfying (G1),(G2), (G3) and (G4). Then, for any* $t > 0$ *and* $p \in H$*, there exists* $r \in D$ *such that*

$$G(r, q) + \frac{1}{t}\langle q - r, r - p\rangle \geq 0, \ \forall q \in D.$$

Also, if

$$T_t p = \{r \in D : G(r, q) + \frac{1}{t}\langle q - r, r - p\rangle \geq 0, \ \forall q \in D\},$$

then

(i) T_t *is single valued,*

(ii) T_t *is firmly nonexpansive, i.e.,*

$$\|T_t p - T_t q\|^2 \leq \langle T_t p - T_t q, p - q\rangle, \ \forall p, q \in H,$$

(iii) $F(T_t) = EP(G)$,

(iv) $EP(G)$ *is closed and convex."*

[12] "Let $G : D \times D \to R$ be a bifunction which satisfy (G1), (G2), (G3) and (G4). Let $B : D \to H$ be a β-inverse strong monotone mapping. Then, by lemma 2.2, for each $t > 0$ and $p \in H$, there exists $s \in D$ such that $T_t(p) = \{s\}$,
where $T_t p = \{r \in D : G(r, q) + \frac{1}{t}\langle q - r, r - p\rangle, \ q \in D\} = \{s\}$."

Lemma 2.3. *([17]): "Let $\{r_n\}$ be a sequence of nonnegative real numbers satisfying*

$$r_{n+1} \leq (1-t_n)r_n + t_n\alpha_n + \beta_n, \ \forall n \geq 0,$$

where $\{t_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions:

(i) $\{t_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $\lim_{n\to\infty} sup\alpha_n \leq 0$,

(iii) $\beta_n \geq 0$, $\sum_{n=1}^{\infty} \beta_n \leq \infty$.

Then, $\lim_{n\to\infty} r_n = 0$."

Lemma 2.4. *([6]): "Let $G : D \times D \to R$ be a bifunction satisfying the conditions (G1) and (G2). Let T_{t_1} and T_{t_2} be defined as in lemma 2.2 with $t_1, t_2 > 0$. For any $p, q \in H$, then*

$$\|T_{t_1}q - T_{t_2}p\| \leq \|q-p\| + |\frac{t_1 - t_2}{t_1}|\|T_{t_1}q - q\|."$$

Lemma 2.5. *([1]): "Let H be a real Hilbert space. Then, for any $u, v \in H$, we have*

$$\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle."$$

Lemma 2.6. *([5]): "Let T be an asymptotically nonexpansive mapping which is defined on a closed and convex subset D of the real Hilbert space H. Then, I-T is demiclosed at 0 i.e., if $\{u_n\}$ is in D, $u_n \rightharpoonup u$ and $u_n - Tu_n \to 0$, then $u \in F(T)$."*

Lemma 2.7. *([18]): "Let $\{r_n\}$ be a sequence of nonnegative real numbers such that*

$$r_{n+1} \leq (1-t_n)r_n + t_n\alpha_n, \ \forall n \geq 0,$$

where $\{t_n\}$ is a sequence in (0,1) and $\{\alpha_n\}$ is a sequence in R such that

(i) $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $\lim_{n\to\infty} sup\alpha_n \leq 0$.

Then, $\lim_{n\to\infty} r_n = 0$."

3 Main Result

The main result is

Theorem 3.1. *Let $D\ (\neq \phi) \subset H$ be closed and convex and let $G : D \times D \to R$ be a bifunction satisfying (G1), (G2), (G3) and (G4), $g : D \to D$ be ρ-contraction, $B : D \to H$ be a β-inverse stronge monotone and $T : D \to D$ be asymptotically nonexpansive as well as asymptotically regular mapping such that $F(T) \bigcap EP(G,B) \neq \phi$. Let $\{\xi_n\} \subset [0,1]$ and $\{\mu_n\} \subset [0,2\beta]$ satisfying*

(i) $\lim_{n\to\infty} \xi_n = 0$, $\sum_{n=1}^{\infty} \xi_n = \infty$,

(ii) $0 \leq a_1 \leq \mu_n \leq b_1 \leq 2\beta$,

(iii) $lim_{n\to\infty}(\mu_n - \mu_{n+1}) = 0,$

(iv) $\lim_{n\to\infty} \frac{l_n - 1}{\xi_n} = 0.$

For $p_1 \in D$*, if* $\{p_n\}$ *be a sequence defined as*

$$\begin{cases} r_n = T_{\mu_n}(p_n - \mu_n B p_n), \\ q_n = T_{\mu_n}(r_n - \mu_n B r_n), \\ p_{n+1} = \xi_n g(p_n) + (1-\xi_n)T^n q_n, \ n = 1, 2, 3, ..., \end{cases} \tag{3.1}$$

then $\{p_n\}$ *strongly converges to* $r = P_{F(T)\bigcap EP(G,B)} g(r)$.

Proof. Step 1. $\{p_n\}$ is bounded.
Let $d \in F(T)\bigcap EP(G,B)$, then

$$\begin{aligned} \|p_{n+1} - d\| &= \|\xi_n g(p_n) + (1-\xi_n)T^n q_n - d\| \\ &= \|\xi_n g(p_n) - \xi_n g(d) + \xi_n g(d) - \xi_n d + \xi_n d + (1-\xi_n)T^n q_n - d\| \\ &= \|\xi_n [g(p_n) - g(d)] + \xi_n [g(d) - d] + (1-\xi_n)T^n q_n + (\xi_n - 1)d\| \\ &\le \xi_n \|g(p_n) - g(d)\| + \xi_n \|g(d) - d\| + (1-\xi_n)\|T^n q_n - d\| \end{aligned}$$

$$\le \xi_n \rho \|p_{n+1} - d\| + \xi_n \|g(d) - d\| + (1-\xi_n) l_n \|q_n - d\|. \tag{3.2}$$

Now,

$$\begin{aligned} \|q_n - d\|^2 &= \|T_{\mu_n}(r_n - \mu_n B r_n) - d\|^2 \\ &= \|T_{\mu_n}(r_n - \mu_n B r_n) - T_{\mu_n}(d - \mu_n B d)\|^2 \\ &\le \|(r_n - \mu_n B r_n) - (d - \mu_n B d)\|^2 \\ &= \|(r_n - d) - \mu_n (B r_n - B d)\|^2 \\ &= \|r_n - d\|^2 - 2\mu_n \langle r_n - d, B r_n - B d\rangle + \mu_n^2 \|B r_n - B d\|^2 \\ &\le \|r_n - d\|^2 - 2\mu_n \beta \|B r_n - B d\|^2 + \mu_n^2 \|B r_n - B d\|^2 \\ &\le \|r_n - d\|^2 + \mu_n(\mu_n - 2\beta)\|B r_n - B d\|^2 \\ &\le \|r_n - d\|^2 \end{aligned}$$

$$\therefore \|q_n - d\| \le \|r_n - d\|. \tag{3.3}$$

Now,

$$\|r_n - d\|^2 = \|T_{\mu_n}(p_n - \mu_n B p_n) - d\|^2.$$

Similarly,

$$\|r_n - d\| \le \|p_n - d\|. \tag{3.4}$$

So, from (3.2), using (3.3) and (3.4),

$$\begin{aligned} \|p_{n+1} - d\| &\le \xi_n \rho \|p_n - d\| + \xi_n \|g(d) - d\| + (1-\xi_n) l_n \|p_n - d\| \\ &= (1-\xi_n)\|p_n - d\| + \xi_n \rho \|p_n - d\| + \xi_n \|g(d) - d\| \end{aligned}$$

$$
\begin{aligned}
&+(1-\xi_n)l_n\|p_n-d\|-(1-\xi_n)\|p_n-d\| \\
&=[1-\xi_n(1-\rho)]\|p_n-d\|+\xi_n\|g(d)-d\|+(1-\xi_n)(l_n-1)\|p_n-d\| \\
&\leq[1-\xi_n(1-\rho)]\|p_n-d\|+\xi_n\|g(d)-d\|+\xi_n\delta\|p_n-d\| \\
&=[1-\xi_n(1-\rho-\delta)]\|p_n-d\|+\xi_n\|g(d)-d\| \\
&\leq max\{\|p_n-d\|,\frac{1}{1-\rho-\delta}\|g(d)-d\|\}.
\end{aligned}
$$

So, $\{p_n\}$ is bounded and hence $\{Bp_n\}$, $\{g(p_n)\}$, $\{T^n r_n\}$ and $\{T^n q_n\}$ are bounded.
Step 2.

$$\lim_{n\to\infty}\|p_{n+1}-p_n\|=0$$

$$
\begin{aligned}
\|p_{n+1}-p_n\| &= \|\xi_n g(p_n)+(1-\xi_n)T^n q_n-\xi_{n-1}g(p_{n-1})-(1-\xi_{n-1})T^{n-1}q_{n-1}\| \\
&=\|\xi_n g(p_n)-\xi_n g(p_{n-1})+\xi_n g(p_{n-1})+(1-\xi_n)T^n q_n \\
&-(1-\xi_n)T^{n-1}q_{n-1}+(1-\xi_n)T^{n-1}q_{n-1}-\chi_{n-1}g(p_{n-1})-(1-\xi_{n-1})T^{n-1}q_{n-1}\| \\
&=\|\xi_n[g(p_n)-g(p_{n-1})]+(\xi_n-\xi_{n-1})g(p_{n-1})+(1-\xi_n)(T^n q_n-T^{n-1}q_{n-1}) \\
&+(1-\xi_n)T^{n-1}q_{n-1}-(1-\xi_{n-1})T^{n-1}q_{n-1}\| \\
&=\|\xi_n[g(p_n)-g(p_{n-1})]+(\xi_n-\xi_{n-1})g(p_{n-1})+(1-\xi_n)(T^n q_n-T^{n-1}q_{n-1}) \\
&+(\xi_{n-1}-\xi_n)T^{n-1}q_{n-1}\| \\
&=\|\xi_n[g(p_n)-g(p_{n-1})]+(\xi_n-\xi_{n-1})(g(p_{n-1})-T^{n-1}q_{n-1}) \\
&+(1-\xi_n)(T^n q_n-T^{n-1}q_{n-1})\| \\
&=\|\xi_n[g(p_n)-g(p_{n-1})]+(\xi_n-\xi_{n-1})(g(p_{n-1})-T^{n-1}q_{n-1}) \\
&+(1-\xi_n)(T^n q_n-T^n q_{n-1}+T^n q_{n-1}-T^{n-1}q_{n-1})\| \\
&\leq\xi_n\|g(p_n)-g(p_{n-1})\|+|\xi_n-\xi_{n-1}|.||g(p_{n-1})-T^{n-1}q_{n-1}|| \\
&+(1-\xi_n)||T^n q_n-T^n q_{n-1}+T^n q_{n-1}-T^{n-1}q_{n-1}\| \\
&\leq\xi_n\|g(p_n)-g(p_{n-1})\|+|\xi_n-\xi_{n-1}|[\|g(p_{n-1})\|+\|T^{n-1}q_{n-1}\|] \\
&+(1-\xi_n)\|T^n q_n-T^n q_{n-1}\|+(1-\xi_n)\|T^n q_{n-1}-T^{n-1}q_{n-1}\| \\
&\leq\xi_n\rho\|p_n-p_{n-1}\|+|\xi_n-\xi_{n-1}|K+(1-\xi_n)l_n\|q_n-q_{n-1}\| \\
&+(1-\xi_n)\|T^n q_{n-1}-T^{n-1}q_{n-1}\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\xi_n\rho\|p_n-p_{n-1}\|+|\xi_n-\xi_{n-1}|K+(1-\xi_n)(l_n-1)\|q_n-q_{n-1}\| \\
+(1-\xi_n)\|q_n-q_{n-1}\|+(1-\xi_n)\|T^n q_{n-1}-T^{n-1}q_{n-1}\|.
\end{aligned}
\tag{3.5}
$$

Now,
$q_n=T_{\mu_n}(r_n-\mu_n Br_n)$ and $q_{n+1}=T_{\mu_{n+1}}(r_{n+1}-\mu_{n+1}Br_{n+1})$

$$G(q_n,q)+\langle Br_n,q-q_n\rangle+\frac{1}{\mu_n}\langle q-q_n,q_n-r_n\rangle\geq 0,\ \forall q\in D \tag{3.6}$$

$$G(q_{n+1},q)+\langle Br_{n+1},q-q_{n+1}\rangle+\frac{1}{\mu_{n+1}}\langle q-q_{n+1},q_{n+1}-r_{n+1}\rangle\geq 0,\ \forall q\in D \tag{3.7}$$

Replacing $q=q_{n+1}$ in (3.6) and $q=q_n$ in (3.7) and getting

$$G(q_n,q_{n+1})+\langle Br_n,q_{n+1}-q_n\rangle+\frac{1}{\mu_n}\langle q_{n+1}-q_n,q_n-r_n\rangle\geq 0 \tag{3.8}$$

$$G(q_{n+1}, q_n) + \langle Br_{n+1}, q_n - q_{n+1}\rangle + \frac{1}{\mu_{n+1}}\langle q_n - q_{n+1}, q_{n+1} - r_{n+1}\rangle \geq 0 \tag{3.9}$$

Adding (3.8) and (3.9),

$$\begin{aligned} &G(q_n, q_{n+1}) + G(q_{n+1}, q_n) + \langle Br_n, q_{n+1} - q_n\rangle + \langle Br_{n+1}, q_n - q_{n+1}\rangle \\ &+\frac{1}{\mu_n}\langle q_{n+1} - q_n, q_n - r_n\rangle + \frac{1}{\mu_{n+1}}\langle q_n - q_{n+1}, q_{n+1} - r_{n+1}\rangle \geq 0. \end{aligned}$$

Using (G2),

$$G(q_n, q_{n+1}) + G(q_{n+1}, q_n) \leq 0$$

$$\begin{aligned} \therefore \langle Br_n, q_{n+1} - q_n\rangle + \langle Br_{n+1}, q_n - q_{n+1}\rangle + \frac{1}{\mu_n}\langle q_{n+1} - q_n, q_n - r_n\rangle& \\ + \frac{1}{\mu_{n+1}}\langle q_n - q_{n+1}, q_{n+1} - r_{n+1}\rangle \geq 0& \\ \langle Br_{n+1}, q_n - q_{n+1}\rangle - \langle Br_n, q_n - q_{n+1}\rangle + \langle q_{n+1} - q_n, \frac{1}{\mu_n}(q_n - r_n)\rangle& \\ - \langle q_{n+1} - q_n, \frac{1}{\mu_{n+1}}(q_{n+1} - r_{n+1})\rangle \geq 0& \\ \langle Br_{n+1} - Br_n, q_n - q_{n+1}\rangle + \langle q_{n+1} - q_n, \frac{1}{\mu_n}(q_n - r_n) - \frac{1}{\mu_{n+1}}(q_{n+1} - r_{n+1})\rangle \geq 0& \\ \langle Br_{n+1} - Br_n, q_n - q_{n+1}\rangle + \langle q_n - q_{n+1}, \frac{1}{\mu_{n+1}}(q_{n+1} - r_{n+1}) - \frac{1}{\mu_n}(r_n - q_n)\rangle \geq 0.& \end{aligned}$$

Now,

$$\langle q_n - q_{n+1}, \mu_n(Br_{n+1} - Br_n) + \frac{\mu_n}{\mu_{n+1}}(q_{n+1} - r_{n+1}) - (q_n - r_n)\rangle$$

$$\begin{aligned} &= \langle q_n - q_{n+1}, \mu_n(Br_{n+1} - Br_n)\rangle + \langle q_n - q_{n+1}, \frac{\mu_n}{\mu_{n+1}}(q_{n+1} - r_{n+1}) - (q_n - r_n)\rangle \\ &= \mu_n\langle q_n - q_{n+1}, Br_{n+1} - Br_n\rangle + \mu_n\langle q_n - q_{n+1}, \frac{1}{\mu_{n+1}}(q_{n+1} - r_{n+1}) - \frac{1}{\mu_n}(q_n - r_n)\rangle \\ &= \mu_n[\langle q_n - q_{n+1}, Br_{n+1} - Br_n\rangle + \langle q_n - q_{n+1}, \frac{1}{\mu_{n+1}}(q_{n+1} - r_{n+1}) - \frac{1}{\mu_n}(q_n - r_n)\rangle] \geq 0 \\ \therefore 0 &\leq \langle q_n - q_{n+1}, \mu_n(Br_{n+1} - Br_n) + \frac{\mu_n}{\mu_{n+1}}(q_{n+1} - r_{n+1}) - (q_n - r_n)\rangle \\ &= \langle q_{n+1} - q_n, -\mu_n(Br_{n+1} - Br_n) - \frac{\mu_n}{\mu_{n+1}}(q_{n+1} - r_{n+1}) + (q_n - r_n)\rangle \\ &= \langle q_{n+1} - q_n, q_n - q_{n+1} + q_{n+1} - \frac{\mu_n}{\mu_{n+1}}q_{n+1} + \frac{\mu_n}{\mu_{n+1}}r_{n+1} - r_{n+1} \\ &+ r_{n+1} + \mu_n Br_{n+1} + \mu_n Br_n - r_n\rangle \\ &= \langle q_{n+1} - q_n, q_n - q_{n+1} + (1 - \frac{\mu_n}{\mu_{n+1}})q_{n+1} + (\frac{\mu_n}{\mu_{n+1}} - 1)r_{n+1} \\ &+ (r_{n+1} - \mu_n Br_{n+1}) - (r_n - \mu_n Br_n)\rangle \\ &= \langle q_{n+1} - q_n, q_n - q_{n+1} + (1 - \frac{\mu_n}{\mu_{n+1}})(q_{n+1} - r_{n+1}) + (r_{n+1} - r_n) - \mu_n(Br_{n+1} - Br_n)\rangle \end{aligned}$$

$$
\begin{aligned}
\therefore \|q_{n+1}-q_n\|^2 \le \|q_{n+1}-q_n\|\{|1-\frac{\mu_n}{\mu_{n+1}}|\|q_{n+1}-r_{n+1}\| + \|r_{n+1}-r_n\|\} \\
\Rightarrow \|q_{n+1}-q_n\| \le |1-\frac{\mu_n}{\mu_{n+1}}|\|q_{n+1}-r_{n+1}\| + \|r_{n+1}-r_n\| \\
= \frac{|\mu_{n+1}-\mu_n|}{\mu_{n+1}}\|q_{n+1}-r_{n+1}\| + \|r_{n+1}-r_n\|
\end{aligned}
$$

$$
\|q_{n+1}-q_n\| \le \|r_{n+1}-r_n\| + \frac{1}{a_1}|\mu_{n+1}-\mu_n|M_1, \tag{3.10}
$$

where $M_1 = max\{\|q_{n+1}-r_{n+1}\|\}$
Now,
$r_n = T_{\mu_n}(q_n - \mu_n Bq_n)$ and $r_{n+1} = T_{\mu_{n+1}}(q_{n+1} - \mu_{n+1}Bq_{n+1})$
Similarly,

$$
\|r_{n+1}-r_n\| \le \|p_{n+1}-p_n\| + \frac{1}{a_1}|\mu_{n+1}-\mu_n|M_2, \tag{3.11}
$$

where $M_2 = max\{\|r_{n+1}-p_{n+1}\|\}$
So, from (3.10) and (3.11),

$$
\|q_{n+1}-q_n\| \le \|p_{n+1}-p_n\| + \frac{1}{a_1}|\mu_{n+1}-\mu_n|M, \tag{3.12}
$$

where $M = M_1 + M_2$.

$$
\therefore\ \|q_n-q_{n-1}\| \le \|p_n-p_{n-1}\| + \frac{1}{a_1}|\mu_n-\mu_{n-1}|M \tag{3.13}
$$

$$
\begin{aligned}
\|T^n q_n - T^{n-1}q_{n-1}\| &= \|T^n q_n - T^n q_{n-1} + T^n q_{n-1} - T^{n-1}q_{n-1}\| \\
&\le \|T^n q_n - T^n q_{n-1}\| + \|T^n q_{n-1} - T^{n-1}q_{n-1}\|
\end{aligned}
$$

$$
\le l_n\|q_n-q_{n-1}\| + \|T^n q_{n-1} - T^{n-1}q_{n-1}\|. \tag{3.14}
$$

From (3.5),

$$
\begin{aligned}
\|p_{n+1}-p_n\| &\le \xi_n\rho\|p_n-p_{n-1}\| + |\xi_n-\xi_{n-1}|K + (1-\xi_n)(l_n-1)\|q_n-q_{n-1}\| + (1-\xi_n)\|q_n-q_{n-1}\| \\
&+ (1-\xi_n)\|T^n q_{n-1} - T^{n-1}q_{n-1}\| \\
&\le \xi_n\rho\|p_n-p_{n-1}\| + |\xi_n-\xi_{n-1}|K + (l_n-1)\|q_n-q_{n-1}\| + (1-\xi_n)\|q_n-q_{n-1}\| \\
&+ \|T^n q_{n-1} - T^{n-1}q_{n-1}\| \\
&\le \xi_n\rho\|p_n-p_{n-1}\| + |\xi_n-\xi_{n-1}|K + (l_n-1)[\|p_n-p_{n-1}\| + \frac{1}{a_1}|\mu_n-\mu_{n-1}|M] \\
&+ (1-\xi_n)[\|p_n-p_{n-1}\| + \frac{1}{a_1}|\mu_n-\mu_{n-1}|M] + \|T^n q_{n-1} - T^{n-1}q_{n-1}\| \\
&= \xi_n\rho\|p_n-p_{n-1}\| + |\xi_n-\xi_{n-1}|K + (l_n-1)\|p_n-p_{n-1}\| + \frac{(l_n-1)}{a_1}|\mu_n-\mu_{n-1}|M \\
&+ (1-\xi_n)\|p_n-p_{n-1}\| + \frac{(1-\xi_n)}{a_1}|\mu_n-\mu_{n-1}|M + \|T^n q_{n-1} - T^{n-1}q_{n-1}\| \\
&\le \xi_n\rho\|p_n-p_{n-1}\| + |\xi_n-\xi_{n-1}|K + \xi_n\delta\|p_n-p_{n-1}\| + \frac{\xi_n\delta}{a_1}|\mu_n-\mu_{n-1}|M
\end{aligned}
$$

$$
\begin{aligned}
&+(1-\xi_n)\|p_n-p_{n-1}\|+\frac{(1-\xi_n)}{a_1}|\mu_n-\mu_{n-1}|M+\|T^nq_{n-1}-T^{n-1}q_{n-1}\|\\
&=[1-\xi_n(1-\rho-\delta)]\|p_n-p_{n-1}\|+\frac{(\xi_n\delta+1-\xi_n)}{a_1}|\mu_n-\mu_{n-1}|M\\
&+|\xi_n-\xi_{n-1}|K+\|T^nq_{n-1}-T^{n-1}q_{n-1}\|\\
&=[1-\xi_n(1-\rho-\delta)]\|p_n-p_{n-1}\|+\xi_n(1-\rho-\delta)\frac{|\mu_n-\mu_{n-1}|}{a_1}M\\
&+\frac{(-2\xi_n+2\xi_n\delta+\xi_n\rho+1)}{a_1}|\mu_n-\mu_{n-1}|M+|\xi_n-\xi_{n-1}|K+\|T^nq_{n-1}-T^{n-1}q_{n-1}\|\\
&\le[1-\xi_n(1-\rho-\delta)]\|p_n-p_{n-1}\|+\xi_n(1-\rho-\delta)\frac{|\mu_n-\mu_{n-1}|}{a_1}M\\
&+[1+\xi_n(2\delta+\rho)]\frac{|\mu_n-\mu_{n-1}|}{a_1}M+|\xi_n-\xi_{n-1}|K+\|T^nq_{n-1}-T^{n-1}q_{n-1}\|,
\end{aligned}
$$

where $K=sup\{\|g(p_n)\|+\|T^nq_n\|\}$.
Taking $s_n=\xi_n(1-\rho-\delta)$, $\eta_n=\frac{|\mu_n-\mu_{n-1}|}{a_1}M$ and
$\gamma_n=[1+\xi_n(2\delta+\rho)]\frac{|\mu_n-\mu_{n-1}|}{a_1}M+|\xi_n-\xi_{n-1}|K+\|T^nq_{n-1}-T^{n-1}q_{n-1}\|$.
Then,
$\|p_{n+1}-p_n\|\le(1-s_n)\|p_n-p_{n-1}\|+s_n\eta_n+\gamma_n$,
using lemma (2.4),

$$\lim_{n\to\infty}\|p_{n+1}-p_n\|=0. \tag{3.15}$$

Using (3.15) and $\lim_{n\to\infty}(\mu_n-\mu_{n+1})=0$, then from (3.11) and (3.12),

$$\lim_{n\to\infty}\|r_{n+1}-r_n\|=0 \tag{3.16}$$

and

$$\lim_{n\to\infty}\|q_{n+1}-q_n\|=0. \tag{3.17}$$

Step 3. $\lim_{n\to\infty}\|Tp_n-p_n\|=0$.
Now,

$$
\begin{aligned}
\|q_n-d\|^2&=\|T_{\mu_n}(I-\mu_nB)r_n-T_{\mu_n}(I-\mu_nB)d\|^2\\
&\le\|(I-\mu_nB)r_n-(I-\mu_nB)d\|^2\\
&=\|(r_n-d)-\mu_n(Br_n-Bd)\|^2\\
&=\|r_n-d\|^2-2\mu_n\langle r_n-d,Br_n-Bd\rangle+\mu_n^2\|Br_n-Bd\|^2\\
&\le\|r_n-d\|^2-2\mu_n\beta\|Br_n-Bd\|^2+\mu_n^2\|Br_n-Bd\|^2\\
&=\|q_n-d\|^2+\mu_n(\mu_n-2\beta)\|Br_n-Bd\|^2
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
\|p_{n+1}-d\|^2&=\|\xi_ng(p_n)+(1-\xi_n)T^nq_n-d\|^2\\
&=\|\xi_ng(p_n)-\xi_nd+\xi_nd+(1-\xi_n)T^nq_n-d\|^2\\
&=\|\xi_n[g(p_n)-d]+(1-\xi_n)T^nq_n-(\xi_n-1)d\|^2\\
&=\|\xi_n[g(p_n)-d]+(1-\xi_n)[T^nq_n-d]\|^2\\
&\le\xi_n\|g(p_n)-d\|^2+(1-\xi_n)\|T^nq_n-d\|^2
\end{aligned}
$$

$$\leq \xi_n \|p_{n+1} - d\|^2 + (1-\xi_n) l_n^2 \|q_n - d\|^2 \tag{3.19}$$

Putting value of $\|q_n - d\|^2$,

$$\leq \xi_n \|g(p_n) - d\|^2 + (1-\xi_n) l_n^2 [\|r_n - d\|^2 + \mu_n(\mu_n - 2\beta)\|Br_n - Bd\|^2]$$

$$\leq \xi_n \|g(p_n) - d\|^2 + (1-\xi_n) l_n^2 \|r_n - d\|^2 + \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2. \tag{3.20}$$

Putting value of $\|r_n - d\|^2$,

$$\begin{aligned}
&\leq \xi_n \|g(p_n) - d\|^2 + (1-\xi_n) l_n^2 [\|p_n - d\|^2 + \mu_n(\mu_n - 2\beta)\|Bp_n - Bd\|^2] \\
&+ \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2 \\
&= \xi_n \|g(p_n) - d\|^2 + (1-\xi_n) l_n^2 \|p_n - d\|^2 + (1-\xi_n) l_n^2 \mu_n(\mu_n - 2\beta)\|Bp_n - Bd\|^2 \\
&+ \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2 \\
&= \xi_n \|g(p_n) - d\|^2 + (1-\xi_n)(l_n^2 - 1)\|p_n - d\|^2 + (1-\xi_n) l_n^2 \mu_n(\mu_n - 2\beta)\|Bp_n - Bd\|^2 \\
&+ \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2 + (1-\xi_n)\|p_n - d\|^2 \\
&\leq \beta_n \|g(p_n) - d\|^2 + (1-\xi_n)(l_n^2 - 1)\|p_n - d\|^2 + \|p_n - d\|^2 \\
&+ (1-\xi_n) l_n^2 \mu_n(\mu_n - 2\beta)\|Bp_n - Bd\|^2 + \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2
\end{aligned}$$

$$\begin{aligned}
\therefore (1-\xi_n) l_n^2 \mu_n(2\beta - \mu_n)\|Bp_n - Bd\|^2 + \mu_n(2\beta - \mu_n) l_n^2 \|Br_n - Bd\|^2 &\leq \xi_n \|g(p_n) - d\|^2 \\
+(1-\xi_n)(l_n^2 - 1)\|p_n - d\|^2 + \|p_n - d\|^2 - \|p_{n+1} - d\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \xi_n = 0$, $\lim_{n\to\infty} l_n = 1$, $\{g(p_n)\}$ and $\{p_n\}$ are bounded, so

$$\begin{aligned}
\lim_{n\to\infty} \mu_n(2\beta - \mu_n)\|Bp_n - Bd\|^2 + \lim_{n\to\infty} \mu_n(2\beta - \mu_n)\|Br_n - Bd\|^2 \\
\leq \lim_{n\to\infty} [\|p_n - d\|^2 - \|p_{n+1} - d\|^2].
\end{aligned}$$

Since $\{p_n\}$ and $\{q_n\}$ are bounded, so

$$\lim_{n\to\infty} [\|p_n - d\|^2 - \|p_{n+1} - d\|^2] = 0.$$

$\therefore \lim_{n\to\infty} \|Bp_n - Bd\| = 0$ and $\lim_{n\to\infty} \|Br_n - Bd\| = 0$.

$$\begin{aligned}
\|q_n - d\|^2 &= \|T_{\mu_n}(r_n - \mu_n Br_n) - T_{\mu_n}(d - \mu_n Bd)\|^2 \\
&\leq \langle q_n - d, (r_n - \mu_n Br_n) - (d - \mu_n Bd)\rangle \\
&= \frac{1}{2}(\|(r_n - \mu_n Br_n) - (d - \mu_n Bd)\|^2 + \|q_n - d\|^2) \\
&- \frac{1}{2}(\|(r_n - \mu_n Br_n) - (d - \mu_n Bd) - (q_n - d)\|^2) \\
&= \frac{1}{2}(\|(I - \mu_n B)(r_n - d)\|^2 + \|q_n - d\|^2) - \frac{1}{2}(\|(r_n - q_n) - \mu_n(Br_n - Bd)\|^2) \\
&\leq \frac{1}{2}(\|r_n - d\|^2 + \|q_n - d\|^2) - \frac{1}{2}(\|(r_n - q_n) - \mu_n(Br_n - Bd)\|^2) \\
&= \frac{1}{2}(\|r_n - d\|^2 + \|q_n - d\|^2) - \frac{1}{2}[\|r_n - q_n\|^2 + \|\mu_n(Br_n - Bd)\|^2 \\
&- 2\langle r_n - q_n, \mu_n(Br_n - Bd)\rangle]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|r_n - d\|^2 + \frac{1}{2}\|q_n - d\|^2 - \frac{1}{2}\|r_n - q_n\|^2 - \frac{1}{2}\mu_n^2\|Br_n - Bd\|^2 \\
&+ \mu_n\langle r_n - q_n, Br_n - Bd\rangle
\end{aligned}
$$

$$
\frac{1}{2}\|q_n - d\|^2 \leq \frac{1}{2}\|r_n - d\|^2 - \frac{1}{2}\|r_n - q_n\|^2 - \frac{1}{2}\mu_n^2\|Br_n - Bd\|^2 + \mu_n\langle r_n - q_n, Br_n - Bd\rangle
$$

$$
\|q_n - d\|^2 \leq \|r_n - d\|^2 - \|r_n - q_n\|^2 - \mu_n^2\|Br_n - Bd\|^2 + 2\mu_n\langle r_n - q_n, Br_n - Bd\rangle \tag{3.21}
$$

$$
\begin{aligned}
\|r_n - d\|^2 &= \|T_{\mu_n}(p_n - \mu_n Bp_n) - T_{\mu_n}(d - \mu_n Bd)\|^2 \\
&\leq \langle r_n - d, (p_n - \mu_n Bp_n) - (d - \mu_n Bd)\rangle \\
&= \frac{1}{2}(\|(p_n - \mu_n Bp_n) - (d - \mu_n Bd)\|^2 + \|r_n - d\|^2) \\
&- \frac{1}{2}(\|(p_n - \mu_n Bp_n) - (d - \mu_n Bd) - (r_n - d)\|^2) \\
&= \frac{1}{2}(\|(I - \mu_n B)(p_n - d)\|^2 + \|r_n - d\|^2) - \frac{1}{2}(\|(p_n - r_n) - \mu_n(Bp_n - Bd)\|^2) \\
&\leq \frac{1}{2}(\|p_n - d\|^2 + \|r_n - d\|^2) - \frac{1}{2}(\|(p_n - r_n) - \mu_n(Bp_n - Bd)\|^2) \\
&= \frac{1}{2}(\|p_n - d\|^2 + \|r_n - d\|^2) - \frac{1}{2}[\|p_n - r_n\|^2 + \|\mu_n(Bp_n - Bd)\|^2 \\
&- 2\langle p_n - r_n, \mu_n(Bp_n - Bd)\rangle] \\
&= \frac{1}{2}\|p_n - d\|^2 + \frac{1}{2}\|r_n - d\|^2 - \frac{1}{2}\|p_n - r_n\|^2 - \frac{1}{2}\mu_n^2\|Bp_n - Bd\|^2 \\
&+ \mu_n\langle p_n - r_n, Bp_n - Bd\rangle
\end{aligned}
$$

$$
\frac{1}{2}\|r_n - d\|^2 \leq \frac{1}{2}\|p_n - d\|^2 - \frac{1}{2}\|p_n - r_n\|^2 - \frac{1}{2}\mu_n^2\|Bp_n - Bd\|^2 + \mu_n\langle p_n - r_n, Bp_n - Bd\rangle
$$

$$
\|r_n - d\|^2 \leq \|p_n - d\|^2 - \|p_n - r_n\|^2 - \mu_n^2\|Bp_n - Bd\|^2 + 2\mu_n\langle p_n - r_n, Bp_n - Bd\rangle \tag{3.22}
$$

$$
\begin{aligned}
\therefore \|q_n - d\|^2 &\leq \|p_n - d\|^2 - \|p_n - r_n\|^2 - \mu_n^2\|Bp_n - Bd\|^2 + 2\mu_n\langle p_n - r_n, Bp_n - Bd\rangle \\
&- \|r_n - q_n\|^2 - \mu_n^2\|Br_n - Bd\|^2 + 2\mu_n\langle r_n - q_n, Br_n - Bd\rangle
\end{aligned}
$$

$$
\begin{aligned}
= \|p_n - d\|^2 - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\
+2\mu_n[\langle p_n - r_n, Bp_n - Bd\rangle + \langle r_n - q_n, Br_n - Bd\rangle].
\end{aligned} \tag{3.23}
$$

Putting value from (3.23) into (3.19),

$$
\begin{aligned}
\|p_{n+1} - d\|^2 &\leq \xi_n\|g(p) - d\|^2 + (1 - \xi_n)l_n^2[\|p_n - d\|^2 - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 \\
&- \mu_n^2[\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\
&+ 2\mu_n[\langle p_n - r_n, Bp_n - Bd\rangle + \langle r_n - q_n, Br_n - Bd\rangle]] \\
&= \xi_n\|g(p) - d\|^2 + (1 - \xi_n)l_n^2\|p_n - d\|^2 - (1 - \xi_n)l_n^2[\|p_n - r_n\|^2 + \|r_n - q_n\|^2] \\
&- \mu_n^2(1 - \xi_n)l_n^2[\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\
&+ 2\mu_n(1 - \xi_n)l_n^2[\langle p_n - r_n, Bp_n - Bd\rangle + \langle r_n - q_n, Br_n - Bd\rangle] \\
&\leq \xi_n\|g(p) - d\|^2 + (1 - \xi_n)(l_n^2 - 1)\|p_n - d\|^2 + (1 - \xi_n)\|p_n - d\|^2 \\
&- (1 - \xi_n)l_n^2[\|p_n - r_n\|^2 + \|r_n - q_n\|^2] - \mu_n^2(1 - \xi_n)l_n^2[\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\
&+ 2\mu_n(1 - \xi_n)l_n^2[\|p_n - r_n\|.\|Bp_n - Bd\| + \|r_n - q_n\|.\|Br_n - Bd\|]
\end{aligned}
$$

$$\begin{aligned}&\leq \xi_n\|g(p)-d\|^2+(1-\xi_n)(l_n^2-1)\|p_n-d\|^2+\|p_n-d\|^2\\&-(1-\xi_n)l_n^2[\|p_n-r_n\|^2+\|r_n-q_n\|^2]-\mu_n^2(1-\xi_n)l_n^2[\|Bp_n-Bd\|^2+\|Br_n-Bd\|^2]\\&+2\mu_n(1-\xi_n)l_n^2[\|p_n-r_n\|.\|Bp_n-Bd\|+\|r_n-q_n\|.\|Br_n-Bd\|]\end{aligned}$$

$$\begin{aligned}(1-\xi_n)l_n^2[\|p_n-r_n\|^2+\|r_n-q_n\|^2]\leq{}&\xi_n\|g(p)-d\|^2+(1-\xi_n)(l_n^2-1)\|p_n-d\|^2+\|p_n-d\|^2\\&-\|p_{n+1}-d\|^2-\mu_n^2(1-\xi_n)l_n^2[\|Bp_n-Bd\|^2+\|Br_n-Bd\|^2]\\&+2\mu_n(1-\xi_n)l_n^2[\|p_n-r_n\|.\|Bp_n-Bd\|\\&+\|r_n-q_n\|.\|Br_n-Bd\|].\end{aligned}$$

Since $\xi_n\to 0$, $l_n\to 1$ as $n\to\infty$ and $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ are bounded,
$\lim_{n\to\infty}[\|p_n-d\|^2-\|p_{n+1}-d\|^2]=0$, $lim_{n\to\infty}\|Bp_n-Bd\|=0$ and
$\lim_{n\to\infty}\|Br_n-Bd\|=0$,
$\therefore \lim_{n\to\infty}[\|p_n-r_n\|^2+\|r_n-q_n\|^2]\leq \lim_{n\to\infty}[\|p_n-d\|^2-\|p_{n+1}-d\|^2]=0$
$\Rightarrow \lim_{n\to\infty}\|p_n-r_n\|^2=0$ and $\lim_{n\to\infty}\|r_n-q_n\|^2=0$

$$\lim_{n\to\infty}\|p_n-r_n\|=0 \tag{3.24}$$

$$\lim_{n\to\infty}\|r_n-q_n\|=0 \tag{3.25}$$

$$\begin{aligned}\|p_n-T^nr_n\|&\leq\|p_n-T^{n-1}q_{n-1}\|+\|T^{n-1}q_{n-1}-T^{n-1}r_{n-1}\|\\&+\|T^{n-1}r_{n-1}-T^nr_{n-1}\|+\|T^nr_{n-1}-T^nr_n\|\\&\leq\|\xi_{n-1}g(p_{n-1})+(1-\xi_{n-1})T^{n-1}q_{n-1}-T^{n-1}q_{n-1}\|+l_{n-1}\|q_{n-1}-r_{n-1}\|\\&+\|T^{n-1}r_{n-1}-T^nr_{n-1}\|+l_n\|r_{n-1}-r_n\|\\&=\|\xi_{n-1}g(p_{n-1})-\xi_{n-1}T^{n-1}q_{n-1}\|+l_{n-1}\|T_{\mu_{n-1}}(r_{n-1}-\mu_{n-1}Br_{n-1})\\&-T_{\mu_{n-1}}(p_{n-1}-\mu_{n-1}Bp_{n-1})\|+\|T^{n-1}r_{n-1}-T^nr_{n-1}\|+l_n\|r_{n-1}-r_n\|\\&\leq\xi_{n-1}\|g(p_{n-1})-T^{n-1}q_{n-1}\|+l_{n-1}\|(r_{n-1}-\mu_{n-1}Br_{n-1})\\&-(p_{n-1}-\mu_{n-1}Bp_{n-1})\|+\|T^{n-1}r_{n-1}-T^nr_{n-1}\|+l_n\|r_{n-1}-r_n\|\\&=\xi_{n-1}\|g(p_{n-1})-T^{n-1}q_{n-1}\|+l_{n-1}\|(I-\mu_{n-1}B)r_{n-1}-(I-\mu_{n-1}B)p_{n-1}\|\\&+\|T^{n-1}r_{n-1}-T^nr_{n-1}\|+l_n\|r_{n-1}-r_n\|\\&\leq\xi_{n-1}\|g(p_{n-1})-T^{n-1}q_{n-1}\|+l_{n-1}\|r_{n-1}-p_{n-1}\|\\&+\|T^{n-1}r_{n-1}-T^nr_{n-1}\|+l_n\|r_{n-1}-r_n\|.\end{aligned}$$

Now, as *limit* $n\to\infty$ then $\xi_n\to 0$, $l_n\to 1$ and since T is asymptotically regular on D and using (3.16) and (3.24),

$$\lim_{n\to\infty}\|p_n-T^nr_n\|=0. \tag{3.26}$$

Now,

$$\begin{aligned}\|r_n-T^nr_n\|&=\|r_n-p_n+p_n-T^nr_n\|\\&\leq\|r_n-p_n\|+\|p_n-T^nr_n\|.\end{aligned}$$

Taking *limit* $n\to\infty$ and using (3.24) and (3.26),

$$\lim_{n\to\infty}\|r_n-T^nr_n\|=0. \tag{3.27}$$

Now,

$$\begin{aligned}\|Tr_n - r_n\| &\le \|Tr_n - T^{n+1}r_n\| + \|T^{n+1}r_n - T^{n+1}r_{n+1}\| + \|T^{n+1}r_{n+1} - r_{n+1}\| + \|r_{n+1} - r_n\| \\ &\le l_1\|r_n - T^n r_n\| + l_{n+1}\|r_n - r_{n+1}\| + \|T^{n+1}r_{n+1} - r_{n+1}\| + \|r_{n+1} - r_n\| \\ &= l_1\|r_n - T^n r_n\| + (l_{n+1}+1)\|r_n - r_{n+1}\| + \|T^{n+1}r_{n+1} - r_{n+1}\|.\end{aligned}$$

Taking *limit* $n \to \infty$ and using (3.16) and (3.27),

$$\lim_{n\to\infty} \|Tr_n - r_n\| = 0. \tag{3.28}$$

Now,

$$\begin{aligned}\|Tp_n - p_n\| &= \|Tp_n - Tr_n + Tr_n - r_n + r_n - p_n\| \\ &\le \|Tp_n - Tr_n\| + \|Tr_n - r_n\| + \|r_n - p_n\| \\ &\le l_1\|p_n - r_n\| + \|Tr_n - r_n\| + \|r_n - p_n\| \\ &= (l_1+1)\|p_n - r_n\| + \|Tr_n - r_n\|.\end{aligned}$$

Taking *limit* $n \to \infty$ and using (3.24) and (3.28),

$$\lim_{n\to\infty} \|Tp_n - p_n\| = 0. \tag{3.29}$$

Step 4. Since $P_{F(T)\bigcap EP(G,B)}g : D \to D$ is $\rho-$contraction, using Banach contraction principle, a unique $d_0 \in F(T)\bigcap EP(G,B)$ exists such that $d_0 = P_{F(T)\bigcap EP(G,B)}g(d_0)$. It will be proved that

$$\lim_{n\to\infty} \langle g(d_0) - d_0, p_n - d_0\rangle \le 0. \tag{3.30}$$

Boundedness of $\{p_n\}$ implies that a subsequence $\{p_{n_i}\}$ can be chosen such that

$$\lim_{n\to\infty} sup\langle g(d_0) - d_0, p_n - d_0\rangle = \lim_{i\to\infty} \langle g(d_0) - d_0, p_{n_i} - d_0\rangle. \tag{3.31}$$

W.L.O.G., assume that $p_{n_i} \rightharpoonup w$. Due to convexity and closedness of D, it is weakly closed and hence $w \in D$. Now, it will be proved that $w \in F(T)$.
$\because p_{n_i} \rightharpoonup w$ and $\lim_{n\to\infty} \|p_n - Tp_n\| = 0$, so by lemma (2.6), $w \in F(T)$.
Now, it will be proved that $w \in EP(G,B)$.
$\because p_{n_i} \rightharpoonup w$ and $\lim_{n\to\infty} \|p_n - r_n\| = 0 \Rightarrow r_{n_i} \rightharpoonup w$.
$\because r_n = T_{\mu_n}(p_n - \mu_n Bp_n)$
$\therefore G(r_n, q) + \langle Bp_n, q - r_n\rangle + \frac{1}{\mu_n}\langle q - r_n, r_n - p_n\rangle \ge 0, \forall q \in D.$
$\because G(r_n, q) + G(q, r_n) \le 0$
$\Rightarrow -G(q,r_n) + \langle Bp_n, q - r_n\rangle + \frac{1}{\mu_n}\langle q - r_n, r_n - p_n\rangle \ge G(r_n,q) + \langle Bp_n, q - r_n\rangle + \frac{1}{\mu_n}\langle q - r_n, r_n - p_n\rangle \ge 0$
$\therefore \langle Bp_n, q - r_n\rangle + \frac{1}{\mu_n}\langle q - r_n, r_n - p_n\rangle \ge G(q, r_n), \forall q \in D.$
Replacing n by n_i,

$$\therefore \langle Bp_{n_i}, q - r_{n_i}\rangle + \frac{1}{\mu_{n_i}}\langle q - r_{n_i}, r_{n_i} - p_{n_i}\rangle \ge G(q, r_{n_i}). \tag{3.32}$$

Put $r_\mu = \mu q + (1-\mu)w, \forall \mu \in (0,1]$.
Since D is a convex set and $q, w \in D$, so $r_\mu \in D$ and hence it must satisfy (3.32) i.e.,

$$\langle Bp_{n_i}, r_\mu - r_{n_i}\rangle + \frac{1}{\mu_{n_i}}\langle r_\mu - r_{n_i}, r_{n_i} - p_{n_i}\rangle \ge G(r_\mu, r_{n_i})$$

$$\begin{aligned}
0 &\geq -\langle Bp_{n_i}, r_\mu - r_{n_i}\rangle - \frac{1}{\mu_{n_i}}\langle r_\mu - r_{n_i}, r_{n_i} - p_{n_i}\rangle + G(r_\mu, r_{n_i}) \\
\langle r_\mu - r_{n_i}, Br_\mu\rangle &\geq \langle r_\mu - r_{n_i}, Br_\mu\rangle - \langle Bp_{n_i}, r_\mu - r_{n_i}\rangle - \frac{1}{\mu_{n_i}}\langle r_\mu - r_{n_i}, r_{n_i} - p_{n_i}\rangle + G(r_\mu, r_{n_i}) \\
&= \langle r_\mu - r_{n_i}, Br_\mu\rangle - \langle r_\mu - r_{n_i}, Bp_{n_i}\rangle - \frac{1}{\mu_{n_i}}\langle r_\mu - r_{n_i}, r_{n_i} - p_{n_i}\rangle + G(r_\mu, r_{n_i}) \\
&= \langle r_\mu - r_{n_i}, Br_\mu - Bp_{n_i}\rangle - \langle r_\mu - r_{n_i}, \frac{r_{n_i} - p_{n_i}}{\mu_{n_i}}\rangle + G(r_\mu, r_{n_i}) \\
&= \langle r_\mu - r_{n_i}, Br_\mu - Br_{n_i} + Br_{n_i} - Bp_{n_i}\rangle - \langle r_\mu - r_{n_i}, \frac{r_{n_i} - p_{n_i}}{\mu_{n_i}}\rangle + G(r_\mu, r_{n_i}) \\
&= \langle r_\mu - r_{n_i}, Br_\mu - Br_{n_i}\rangle + \langle r_\mu - r_{n_i}, Br_{n_i} - Bp_{n_i}\rangle \\
&- \langle r_\mu - r_{n_i}, \frac{r_{n_i} - p_{n_i}}{\mu_{n_i}}\rangle + G(r_\mu, r_{n_i})
\end{aligned}$$

$\because \lim_{n\to\infty} \|r_{n_i} - p_{n_i}\| = 0 \;\Rightarrow \|Br_{n_i} - Bp_{n_i}\| = 0$.
$\because$ B is monotonic, so $\langle r_\mu - r_{n_i}, Br_\mu - Br_{n_i}\rangle \geq 0$,
$\therefore \lim_{n\to\infty}\langle r_\mu - r_{n_i}, Br_\mu\rangle \geq \lim_{n\to\infty} G(r_\mu, r_{n_i})$.
It follows from (G4) that

$$G(r_\mu, w) \leq \lim_{n\to\infty} G(r_\mu, r_{n_i}) \leq \lim_{n\to\infty}\langle r_\mu - r_{n_i}, Br_\mu\rangle = \langle r_\mu - w, Br_\mu\rangle. \tag{3.33}$$

Since $r_\mu \in D$ and from (G1), (G4) and (3.33), we have

$$\begin{aligned}
0 &= G(r_\mu, r_\mu) = G(r_\mu, \mu q + (1-\mu)w) \\
&\leq G(r_\mu, \mu q) + G(r_\mu, (1-\mu)w) \\
&= \mu G(r_\mu, q) + (1-\mu)G(r_\mu, w).
\end{aligned}$$

$$\leq \mu G(r_\mu, q) + (1-\mu)\langle r_\mu - w, Br_\mu\rangle. \tag{3.34}$$

Now,

$$\begin{aligned}
r_\mu - w &= \mu q + (1-\mu)w - w \\
&= \mu q + w - \mu w - w \\
&= \mu q - \mu w = \mu(q - w).
\end{aligned}$$

From (3.34),

$$\begin{aligned}
0 &= G(r_\mu, r_\mu) \\
&\leq \mu G(r_\mu, q) + (1-\mu)\langle \mu(q-w), Br_\mu\rangle \\
&= \mu G(r_\mu, q) + (1-\mu)\mu\langle q - w, Br_\mu\rangle \\
0 &\leq G(r_\mu, q) + (1-\mu)\langle q - w, Br_\mu\rangle.
\end{aligned}$$

Taking $\mu \to 0$ and $\forall q \in D$,

$$0 \leq G(w, q) + \langle q - w, Bw\rangle$$

$$\therefore w \in EP(G, B)$$

$$\therefore w \in F(T) \bigcap EP(G, B)$$

From (3.31) and by using metric projection's property,

$$\lim_{n\to\infty} sup\langle g(d_0)-d_0, p_n-d_0\rangle = \lim_{i\to\infty}\langle g(d_0)-d_0, p_{n_i}-d_0\rangle.$$

$$= \langle g(d_0)-d_0, w-d_0\rangle \leq 0. \tag{3.35}$$

Step 5. $\lim_{n\to\infty}\|p_n-d_0\| = 0.$

$$\begin{aligned}
\|p_{n+1}-d_0\|^2 &= \|\xi_n g(p_n)+(1-\xi_n)T^n q_n - d_0\|^2\\
&= \|\xi_n g(p_n)-\xi_n d_0+(1-\xi_n)T^n q_n+\xi_n d_0-d_0\|^2\\
&= \|\xi_n(g(p_n)-d_0)+(1-\xi_n)T^n q_n-(1-\xi_n)d_0\|^2\\
&= \|(1-\xi_n)(T^n q_n-d_0)+\xi_n(g(p_n)-d_0)\|^2\\
&\leq \|(1-\xi_n)(T^n q_n-d_0)\|^2+2\langle\xi_n(g(p_n)-d_0),\\
&(1-\xi_n)(T^n q_n-d_0)+\xi_n(g(p_n)-d_0)\rangle\\
&= (1-\xi_n)^2\|T^n q_n-d_0\|^2+2\xi_n\langle(g(p_n)-d_0), p_{n+1}-d_0\rangle\\
&= (1-\xi_n)^2\|T^n q_n-T^n d_0\|^2+2\xi_n\langle g(p_n)-d_0, p_{n+1}-d_0\rangle\\
&\leq (1-\xi_n)^2 l_n^2\|q_n-d_0\|^2+2\xi_n\langle g(p_n)-g(d_0)+g(d_0)-d_0, p_{n+1}-d_0\rangle\\
&\leq (1-\xi_n)^2 l_n^2[\|p_n-d_0\|^2-\|p_n-r_n\|^2-\|r_n-q_n\|^2-\mu_n^2[\|Bp_n-Bd_0\|^2\\
&+\|Br_n-Bd_0\|^2]+2\mu_n[\langle p_n-r_n, Bp_n-Bd_0\rangle+\langle r_n-q_n, Br_n-Bd_0\rangle]]\\
&+2\xi_n\langle g(p_n)-g(d_0)+g(d_0)-d_0, p_{n+1}-d_0\rangle\\
&= [(1-\xi_n)l_n]^2\|p_n-d_0\|^2+[(1-\xi_n)l_n]^2[2\mu_n[\langle p_n-r_n, Bp_n-Bd_0\rangle\\
&+\langle r_n-q_n, Br_n-Bd_0\rangle]-\|p_n-r_n\|^2-\|r_n-q_n\|^2-\mu_n^2[\|Bp_n-Bd_0\|^2\\
&+\|Br_n-Bd_0\|^2]]+2\xi_n\langle g(p_n)-g(d_0), p_{n+1}-d_0\rangle+\langle g(d_0)-d_0, p_{n+1}-d_0\rangle\\
&\leq [(1-\xi_n)l_n]^2\|p_n-d_0\|^2+[(1-\xi_n)l_n]^2[2\mu_n[\|p_n-r_n\|.\|Bp_n-Bd_0\|\\
&+\|r_n-q_n\|.\|Br_n-Bd_0\|]-\|p_n-r_n\|^2-\|r_n-q_n\|^2-\mu_n^2[\|Bp_n-Bd_0\|^2\\
&+\|Br_n-Bd_0\|^2]]+2\xi_n\|g(p_n)-g(d_0)\|.\|p_{n+1}-d_0\|+2\xi_n\langle g(d_0)-d_0, p_{n+1}-d_0\rangle\\
&\leq [(1-\xi_n)l_n]^2\|p_n-d_0\|^2+[(1-\xi_n)l_n]^2[2\mu_n[\|p_n-r_n\|.\|Bp_n-Bd_0\|\\
&+\|r_n-q_n\|.\|Br_n-Bd_0\|]-\|p_n-r_n\|^2-\|r_n-q_n\|^2-\mu_n^2[\|Bp_n-Bd_0\|^2\\
&+\|Br_n-Bd_0\|^2]]+2\xi_n\rho\|p_n-d_0\|.\|p_{n+1}-d_0\|+2\xi_n\langle g(d_0)-d_0, p_{n+1}-d_0\rangle\\
&\leq [(1-\xi_n)l_n]^2\|p_n-d_0\|^2+[(1-\xi_n)l_n]^2[2\mu_n[\|p_n-r_n\|.\|Bp_n-Bd_0\|\\
&+\|r_n-q_n\|.\|Br_n-Bd_0\|]-\|p_n-r_n\|^2-\|r_n-q_n\|^2-\mu_n^2[\|Bp_n-Bd_0\|^2\\
&+\|Br_n-Bd_0\|^2]]+2\xi_n\rho[\frac{\|p_n-d_0\|^2+\|p_{n+1}-d_0\|^2}{2}]+2\xi_n\langle g(d_0)-d_0, p_{n+1}-d_0\rangle\\
&= [(1-\xi_n)l_n]^2\|p_n-d_0\|^2+\xi_n\rho\|p_n-d_0\|^2+\xi_n\rho\|p_{n+1}-d_0\|^2\\
&+[(1-\xi_n)l_n]^2[2\mu_n[\|p_n-r_n\|.\|Bp_n-Bd_0\|+\|r_n-q_n\|.\|Br_n-Bd_0\|]\\
&-\|p_n-r_n\|^2-\|r_n-q_n\|^2-\mu_n^2[\|Bp_n-Bd_0\|^2+\|Br_n-Bd_0\|^2]]\\
&+2\xi_n\langle g(d_0)-d_0, p_{n+1}-d_0\rangle\\
&= [(1-\xi_n)(l_n-1)+(1-\xi_n)]^2\|p_n-d_0\|^2+\xi_n\rho\|p_n-d_0\|^2+\xi_n\rho\|p_{n+1}-d_0\|^2\\
&+[(1-\xi_n)l_n]^2[2\mu_n[\|p_n-r_n\|.\|Bp_n-Bd_0\|+\|r_n-q_n\|.\|Br_n-Bd_0\|]
\end{aligned}$$

$$
\begin{aligned}
&- \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|B_n - Bd_0\|^2]] \\
&+ 2\xi_n\langle g(d_0) - d_0, p_{n+1} - d_0\rangle \\
&= [(1-\xi_n)^2 + (1-\xi_n)^2(l_n-1)^2 + 2(1-\xi_n)(l_n-1)(1-\xi_n) + \xi_n\rho]\|p_n - d_0\|^2 \\
&+ \xi_n\rho\|p_{n+1} - d_0\|^2 + [(1-\xi_n)l_n]^2[2\mu_n[\|p_n - r_n\|.\|Bp_n - Bd_0\| \\
&+ \|r_n - q_n\|.\|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 \\
&- \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] + 2\xi_n\langle g(d_0) - d_0, p_{n+1} - d_0\rangle \\
&= [1 - (2-\rho)\beta_n + \beta_n^2 + (1-\beta_n)^2(l_n-1)^2 + 2(1-\beta_n)^2(l_n-1)]\|p_n - d_0\|^2 \\
&+ \beta_n\rho\|p_{n+1} - d_0\|^2 + [(1-\beta_n)l_n]^2[2\mu_n[\|p_n - r_n\|.\|Bp_n - Bd_0\| \\
&+ \|r_n - q_n\|.\|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 \\
&- \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] + 2\xi_n\langle g(d_0) - d_0, p_{n+1} - d_0\rangle.
\end{aligned}
$$

$$
\begin{aligned}
\|p_{n+1} - d_0\|^2 - \xi_n\rho\|p_{n+1} - d_0\|^2 \le\ & [1 - (2-\rho)\xi_n + \xi_n^2 + (l_n-1)^2 + 2(l_n-1)]\|p_n - d_0\|^2 \\
& + l_n^2[2\mu_n[\|p_n - r_n\|.\|Bp_n - Bd_0\| + \|r_n - q_n\|.\|Br_n - Bd_0\|] \\
& - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd_0\|^2 \\
& + \|Br_n - Bd_0\|^2]] + 2\xi_n\langle g(d_0) - d_0, p_{n+1} - d_0\rangle.
\end{aligned}
$$

$$
\begin{aligned}
\|p_{n+1} - d_0\|^2 \le\ & [\frac{1-(2-\rho)\xi_n}{1-\rho\xi_n}]\|p_n - d_0\|^2 + [\frac{\xi_n^2 + (l_n-1)^2 + 2(l_n-1)}{1-\rho\xi_n}]\|p_n - d_0\|^2 \\
& + \frac{l_n^2}{1-\rho\xi_n}[2\mu_n[\|p_n - r_n\|.\|Bp_n - Bd_0\| + \|r_n - q_n\|.\|Br_n - Bd_0\|] \\
& - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] \\
& + 2\xi_n\langle g(d_0) - d_0, p_{n+1} - d_0\rangle \\
\le\ & [1 - \frac{(2-\rho)\xi_n}{1-\rho\xi_n}]\|p_n - r_0\|^2 + [\frac{\xi_n^2 + (l_n-1)^2 + 2(l_n-1)}{1-\rho\xi_n}]L_n \\
& + \frac{l_n^2}{1-\rho\xi_n}[2\mu_n[\|p_n - r_n\|.\|Bp_n - Bd_0\| + \|r_n - q_n\|.\|Br_n - Bd_0\|] \\
& - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] \\
& + \frac{2\xi_n}{1-\rho\xi_n}\langle g(r_0) - d_0, p_{n+1} - d_0\rangle.
\end{aligned}
$$

where $L_n = sup_{n\in N}\|p_n - d_0\|^2$

$$
\therefore \|p_{n+1} - d_0\|^2 \le (1-a_n)\|p_n - d_0\|^2 + a_n\tau_n, \tag{3.36}
$$

where $a_n = \frac{(2-\rho)\xi_n}{1-\rho\xi_n}$,

$$
\begin{aligned}
\tau_n =\ & [\frac{\xi_n^2 + (l_n-1)^2 + 2(l_n-1)}{(2-\rho)\xi_n}]L_n + \frac{l_n^2}{1-\rho\xi_n}[2\mu_n[\|p_n - r_n\|.\|Bp_n - Bd_0\| \\
& + \|r_n - q_n\|.\|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 \\
& - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] + \frac{2\xi_n}{1-\rho\xi_n}\langle g(d_0) - d_0, p_{n+1} - d_0\rangle
\end{aligned}
$$

. $\because \lim_{n\to\infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n = \infty$ and $lim_{n\to\infty} sup\, \tau_n \leq 0$,
so, by using (3.36) and lemma (2.7),

$$\lim_{n\to\infty} \|p_n - d_0\| = 0.$$

□

4 Application

Theorem 4.1. *Let $D\ (\neq \phi) \subset H$ be closed and convex and let $G : D \times D \to R$ be a bifunction satisfying (G1), (G2), (G3) and (G4). Let $g : D \to D$ be ρ-contraction and $T : D \to D$ be asymptotically nonexpansive as well as asymptotically regular mapping such that $F(T) \bigcap EP(G) \neq \phi$. Let $\{\xi_n\} \subset [0, 1]$ and $\{\mu_n\} \subset [0, 2\beta]$ satisfying*

(i) $\lim_{n\to\infty} \xi_n = 0$, $\sum_{n=1}^{\infty} \xi_n = \infty$,

(ii) $0 < a_1 \leq \mu_n \leq b_1 \leq 2\beta$,

(iii) $lim_{n\to\infty}(\mu_n - \mu_{n+1}) = 0$,

(iv) $\lim_{n\to\infty} \frac{l_n - 1}{\xi_n} = 0$.

If $p_1 \in D$ and sequence $\{p_n\}$ is defined in (3.1), then $\{p_n\}$ strongly converges to $r = P_{F(T)\bigcap EP(G)} g(r)$.

Proof. Putting $B \equiv 0$ in theorem 3.1, then $G(r_n, q) + \frac{1}{\mu_n}\langle q - r_n, r_n - p_n\rangle \geq 0,\ \forall q \in D$.
Then, the functions $T_{\mu_n} p_n$ and $T_{\mu_n} r_n$ exist and using theorem 3.1, we get the desired result. □

Remark 1. If T which is asymptotically nonexpansive mapping is replaced by a nonexpansive mapping then improved version of theorem by Takahashi and Takahashi [16] is obtained.

5 Numerical Example

The following example is given for the justification of the theorem 3.1.

Example 1. Let $D = [0, \infty) \subset H = R$, $G(p, q) = -2p^2 + pq + q^2$ be a bifunction, $gp = \frac{p}{3}$ be a contraction mapping, $Bp = \frac{p}{2}$ be a $\frac{1}{2}$-inverse stronge monotone and $Tp = \frac{p}{4}$ asymptotically nonexpansive and asymptotically regular.
Then, $F(T) \bigcap EP(G, B) = \{0\} \neq \phi$. Taking $\mu_n = 1$ and $\xi_n = \frac{1}{4n}$, then by algorithm (3.1), the sequence $\{p_n\}$ is given by

$$\begin{cases} r_n = \frac{p_n}{8}, \\ q_n = \frac{r_n}{8}, \\ p_{n+1} = \xi_n g(p_n) + (1 - \xi_n) T^n q_n,\ n = 1, 2, 3, ... \end{cases} \tag{5.1}$$

As $n \to \infty$, $p_n \to 0$.
Table 1 compares the algorithms (5.1) and (1.5) and it reveals that convergence of sequence $\{p_n\}$ is faster by algorithm 5.1 than that by algorithm 1.5.
Figure 1 shows the convergence of sequence $\{p_n\}$ with different initial values.

Table 1: Comparison of algorithms 5.1 and 1.5.

Number of iterations (n)	p_n using algorithm 5.1	p_n using algorithm 1.5
1	90	90
2	11.7773	15.4688
3	0.7435	1.7847
4	0.0310	0.0756
5	0.0010	0.0024
6	0	0.0001

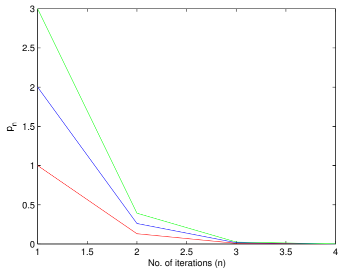

Figure 1: Graphical representation of n and p_n for different initial values of p_n

6 Conclusion

In this paper, asymptotically nonexpansive mapping is used to find out the common solution of the fixed point and generalized equilibrium problem. After that, solution of the equilibrium problem is determined as an application. Further, numerical comparison is done using algorithm (1.5) and (5.1) which shows that the convergence by algorithm (5.1) is faster than algorithm (1.5).

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