



Common solutions of fixed point and generalized equilibrium problems using asymptotically nonexpansive mapping

P. Kumar and A. Panwar

Abstract. In this paper, a common solution of fixed point and generalized equilibrium problems using an asymptotically nonexpansive mapping is determined via iterative approach. Further an application and numerical example of the main result are given.

Keywords. Variational inequality, fixed point, asymptotically nonexpansive mappings, generalized equilibrium problems.

1 Introduction

Let H be a real Hilbert space and $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote inner product and induced norm respectively. Let $D (\neq \emptyset) \subset H$ be closed and convex and $\{q_n\}$ be a sequence in H , then $q_n \rightarrow q$ and $q_n \rightharpoonup q$ denote the strong and weak convergence of sequence $\{q_n\}$ to a point $q \in H$ respectively. For every $q \in H$,

$$\|q - P_D q\| \leq \|q - r\|, \forall r \in D.$$

Then $P_D : H \rightarrow D$ is known as metric projection.

A mapping $T : D \rightarrow D$ is called asymptotically nonexpansive [8] if there exists a sequence $\{l_n\}$ with $\lim_{n \rightarrow \infty} l_n = 1$ and

$$\|T^n q - T^n r\| \leq l_n \|q - r\|, \forall q, r \in D.$$

T is known as l -Lipschitzian ($l \geq 0$) if

$$\|T^n q - T^n r\| \leq l \|q - r\|, \forall q, r \in D, \forall n \in N.$$

If $l_n = 1, \forall n \in N$, then T becomes nonexpansive mapping.

The concept of asymptotically nonexpansive mappings is the generalization of the concept of nonexpansive mappings and it was introduced by Geobel and Kirk [8].

In the fixed point problem (FPP), $q \in D$ is to be determined such that

$$Tq = q. \tag{1.1}$$

Now,

$$F(T) = \{q \in D : Tq = q\}.$$

Consider a bifunction $G : D \times D \rightarrow R$ and a nonlinear mapping $B : D \rightarrow H$.

To find $d \in D$ such that

$$G(d, d_1) + \langle Bd, d_1 - d \rangle \geq 0, \forall d_1 \in D, \quad (1.2)$$

is known as generalized equilibrium problem.

$EP(G, B)$ denotes the solution set of problem (1.2).

If $B \equiv 0$ in (1.2), then the reduced form of problem (1.2) of determining $d \in D$ such that

$$G(d, d_1) \geq 0, \forall d_1 \in D, \quad (1.3)$$

is known as equilibrium problem. $EP(G)$ denotes the solution set of problem (1.3).

If $G \equiv 0$ in (1.2), then the reduced form of problem (1.2) of determining $d \in D$ such that

$$\langle Bd, d_1 - d \rangle \geq 0, \forall d_1 \in D, \quad (1.4)$$

is known as classical variational inequality problem. $VI(D, B)$ denotes the solution set of problem (1.4).

If $G(d, d_1) = \langle Bd, d_1 - d \rangle$, for all $d_1, d \in D$, then $d \in EP(G)$ if and only if $d \in VI(D, B)$.

The problem (1.2) is general and it includes many problems like variational inequality, optimization, Nash equilibrium problems etc. as special cases. In [13, 14], the researcher established fixed point iteration results via nonexpansive mappings. Also, the new iterative algorithms are introduced by [15, 19, 21].

[12] "Let $G : D \times D \rightarrow R$ be a bifunction which satisfies the following conditions:

(G1) $G(p, p) = 0$ for all $p \in D$;

(G2) G is monotone, i.e., $G(p, q) + G(q, p) \leq 0$ for all $p, q \in D$;

(G3) for each $p, q, r \in D$;

$$\limsup_{t \rightarrow 0} G(tr + (1-t)p, q) \leq G(p, q);$$

(G4) for each $p \in D$, $q \mapsto G(p, q)$ is convex and lower semicontinuous."

Definition 1. [12] "A mapping $B : D \rightarrow H$ is called β -inverse strongly monotone if there exists $\beta \geq 0$ such that $\langle Bp - Bq, p - q \rangle \geq \beta \|Bp - Bq\|^2$, $\forall p, q \in D$."

Takahashi and Takahashi [16] proved the following result:

Theorem 1.1. [16] "Let D be a nonempty closed convex subset of a real Hilbert space H and let $G : D \times D \rightarrow R$ be a bifunction satisfying (G1), (G2), (G3) and (G4). Let $B : D \rightarrow H$ be a β -inverse strongly monotone mapping and $T : D \rightarrow D$ be nonexpansive mapping such that $F(T) \cap EP(G, B) \neq \phi$. Let $u \in D$, $p_1 \in D$ and let $\{r_n\} \subset D$ and $\{p_n\} \subset D$ be sequence defined by

$$\begin{cases} G(r_n, q) + \langle Bp_n, q - r_n \rangle + \frac{1}{\mu_n} \langle q - r_n, r_n - p_n \rangle \geq 0, \forall q \in D, \\ p_{n+1} = \beta_n p_n + (1 - \beta_n) T[\alpha_n u + (1 - \alpha_n) r_n], \forall n \in N. \end{cases} \quad (1.5)$$

Let $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{\mu_n\} \subset [0, 2\beta]$ satisfies

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\lim_{n \rightarrow \infty} (\mu_n - \mu_{n+1}) = 0$,

(iii) $0 < c \leq \beta_n \leq d < 1, 0 < a \leq \mu_n \leq b < 2\beta,$

then $\{p_n\}$ converges strongly to $r = P_{F(T)} \cap EP_{(G,B)}(u).$ ”

In this paper, the result of Takahashi and Takahashi is further extended for asymptotically nonexpansive mappings. The common solution of FPP and EP is obtained by iterative approach.

2 Preliminaries

Some definitions and lemmas are as follows:

Definition 2. [12] “A mapping $g : D \rightarrow D$ is known as contraction mapping if there exists $\eta \in [0, 1)$ such that

$$\|g(p) - g(q)\| \leq \|p - q\|, \forall p, q \in D.$$

Definition 3. [4] “Let D be a closed convex subset of a Hilbert space H . A mapping $T : D \rightarrow D$ is called asymptotically regular at p if and only if $\lim_{n \rightarrow \infty} \|T^n p - T^{n+1} p\| = 0.$ ”

Lemma 2.1. ([3]): “Let D be a nonempty closed convex subset of a real Hilbert space H . Let $G : D \times D \rightarrow R$ be a bifunction satisfying (G1), (G2), (G3) and (G4). Let $t > 0$ and $p \in H$. Then, there exists $r \in D$ such that

$$G(r, q) + \frac{1}{t} \langle q - r, r - p \rangle \geq 0, \forall q \in D.$$

Lemma 2.2. ([7]): “Let D be a nonempty closed convex subset of a real Hilbert space H and let $G : D \times D \rightarrow R$ be a bifunction satisfying (G1), (G2), (G3) and (G4). Then, for any $t > 0$ and $p \in H$, there exists $r \in D$ such that

$$G(r, q) + \frac{1}{t} \langle q - r, r - p \rangle \geq 0, \forall q \in D.$$

Also, if

$$T_t p = \{r \in D : G(r, q) + \frac{1}{t} \langle q - r, r - p \rangle \geq 0, \forall q \in D\},$$

then

- (i) T_t is single valued,
- (ii) T_t is firmly nonexpansive, i.e.,

$$\|T_t p - T_t q\|^2 \leq \langle T_t p - T_t q, p - q \rangle, \forall p, q \in H,$$

- (iii) $F(T_t) = EP(G),$
- (iv) $EP(G)$ is closed and convex.”

[12] “Let $G : D \times D \rightarrow R$ be a bifunction which satisfy (G1), (G2), (G3) and (G4). Let $B : D \rightarrow H$ be a β -inverse strong monotone mapping. Then, by lemma 2.2, for each $t > 0$ and $p \in H$, there exists $s \in D$ such that $T_t(p) = \{s\},$ where $T_t p = \{r \in D : G(r, q) + \frac{1}{t} \langle q - r, r - p \rangle, q \in D\} = \{s\}.$ ”

Lemma 2.3. ([17]): “Let $\{r_n\}$ be a sequence of nonnegative real numbers satisfying

$$r_{n+1} \leq (1 - t_n)r_n + t_n\alpha_n + \beta_n, \quad \forall n \geq 0,$$

where $\{t_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions:

- (i) $\{t_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} t_n = \infty$,
- (ii) $\lim_{n \rightarrow \infty} \sup \alpha_n \leq 0$,
- (iii) $\beta_n \geq 0$, $\sum_{n=1}^{\infty} \beta_n \leq \infty$.

Then, $\lim_{n \rightarrow \infty} r_n = 0$.”

Lemma 2.4. ([6]): “Let $G : D \times D \rightarrow R$ be a bifunction satisfying the conditions (G1) and (G2). Let T_{t_1} and T_{t_2} be defined as in lemma 2.2 with $t_1, t_2 > 0$. For any $p, q \in H$, then

$$\|T_{t_1}q - T_{t_2}p\| \leq \|q - p\| + \left| \frac{t_1 - t_2}{t_1} \right| \|T_{t_1}q - q\|.”$$

Lemma 2.5. ([1]): “Let H be a real Hilbert space. Then, for any $u, v \in H$, we have

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v \rangle.”$$

Lemma 2.6. ([5]): “Let T be an asymptotically nonexpansive mapping which is defined on a closed and convex subset D of the real Hilbert space H . Then, $I - T$ is demiclosed at 0 i.e., if $\{u_n\}$ is in D , $u_n \rightarrow u$ and $u_n - Tu_n \rightarrow 0$, then $u \in F(T)$.”

Lemma 2.7. ([18]): “Let $\{r_n\}$ be a sequence of nonnegative real numbers such that

$$r_{n+1} \leq (1 - t_n)r_n + t_n\alpha_n, \quad \forall n \geq 0,$$

where $\{t_n\}$ is a sequence in $(0, 1)$ and $\{\alpha_n\}$ is a sequence in R such that

- (i) $\sum_{n=1}^{\infty} t_n = \infty$,
- (ii) $\lim_{n \rightarrow \infty} \sup \alpha_n \leq 0$.

Then, $\lim_{n \rightarrow \infty} r_n = 0$.”

3 Main Result

The main result is

Theorem 3.1. Let $D (\neq \phi) \subset H$ be closed and convex and let $G : D \times D \rightarrow R$ be a bifunction satisfying (G1), (G2), (G3) and (G4), $g : D \rightarrow D$ be ρ -contraction, $B : D \rightarrow H$ be a β -inverse strong monotone and $T : D \rightarrow D$ be asymptotically nonexpansive as well as asymptotically regular mapping such that $F(T) \cap EP(G, B) \neq \phi$. Let $\{\xi_n\} \subset [0, 1]$ and $\{\mu_n\} \subset [0, 2\beta]$ satisfying

- (i) $\lim_{n \rightarrow \infty} \xi_n = 0$, $\sum_{n=1}^{\infty} \xi_n = \infty$,
- (ii) $0 \leq a_1 \leq \mu_n \leq b_1 \leq 2\beta$,

(iii) $\lim_{n \rightarrow \infty} (\mu_n - \mu_{n+1}) = 0,$

(iv) $\lim_{n \rightarrow \infty} \frac{l_n - 1}{\xi_n} = 0.$

For $p_1 \in D,$ if $\{p_n\}$ be a sequence defined as

$$\begin{cases} r_n = T_{\mu_n}(p_n - \mu_n Bp_n), \\ q_n = T_{\mu_n}(r_n - \mu_n Br_n), \\ p_{n+1} = \xi_n g(p_n) + (1 - \xi_n)T^n q_n, \quad n = 1, 2, 3, \dots, \end{cases} \tag{3.1}$$

then $\{p_n\}$ strongly converges to $r = P_{F(T)} \cap EP(G, B)g(r).$

Proof. Step 1. $\{p_n\}$ is bounded.

Let $d \in F(T) \cap EP(G, B),$ then

$$\begin{aligned} \|p_{n+1} - d\| &= \|\xi_n g(p_n) + (1 - \xi_n)T^n q_n - d\| \\ &= \|\xi_n g(p_n) - \xi_n g(d) + \xi_n g(d) - \xi_n d + \xi_n d + (1 - \xi_n)T^n q_n - d\| \\ &= \|\xi_n [g(p_n) - g(d)] + \xi_n [g(d) - d] + (1 - \xi_n)T^n q_n + (\xi_n - 1)d\| \\ &\leq \xi_n \|g(p_n) - g(d)\| + \xi_n \|g(d) - d\| + (1 - \xi_n) \|T^n q_n - d\| \\ &\leq \xi_n \rho \|p_{n+1} - d\| + \xi_n \|g(d) - d\| + (1 - \xi_n) l_n \|q_n - d\|. \end{aligned} \tag{3.2}$$

Now,

$$\begin{aligned} \|q_n - d\|^2 &= \|T_{\mu_n}(r_n - \mu_n Br_n) - d\|^2 \\ &= \|T_{\mu_n}(r_n - \mu_n Br_n) - T_{\mu_n}(d - \mu_n Bd)\|^2 \\ &\leq \|(r_n - \mu_n Br_n) - (d - \mu_n Bd)\|^2 \\ &= \|(r_n - d) - \mu_n (Br_n - Bd)\|^2 \\ &= \|r_n - d\|^2 - 2\mu_n \langle r_n - d, Br_n - Bd \rangle + \mu_n^2 \|Br_n - Bd\|^2 \\ &\leq \|r_n - d\|^2 - 2\mu_n \beta \|Br_n - Bd\|^2 + \mu_n^2 \|Br_n - Bd\|^2 \\ &\leq \|r_n - d\|^2 + \mu_n (\mu_n - 2\beta) \|Br_n - Bd\|^2 \\ &\leq \|r_n - d\|^2 \\ \therefore \|q_n - d\| &\leq \|r_n - d\|. \end{aligned} \tag{3.3}$$

Now,

$$\|r_n - d\|^2 = \|T_{\mu_n}(p_n - \mu_n Bp_n) - d\|^2.$$

Similarly,

$$\|r_n - d\| \leq \|p_n - d\|. \tag{3.4}$$

So, from (3.2), using (3.3) and (3.4),

$$\begin{aligned} \|p_{n+1} - d\| &\leq \xi_n \rho \|p_n - d\| + \xi_n \|g(d) - d\| + (1 - \xi_n) l_n \|p_n - d\| \\ &= (1 - \xi_n) \|p_n - d\| + \xi_n \rho \|p_n - d\| + \xi_n \|g(d) - d\| \end{aligned}$$

$$\begin{aligned}
& + (1 - \xi_n)l_n\|p_n - d\| - (1 - \xi_n)\|p_n - d\| \\
& = [1 - \xi_n(1 - \rho)]\|p_n - d\| + \xi_n\|g(d) - d\| + (1 - \xi_n)(l_n - 1)\|p_n - d\| \\
& \leq [1 - \xi_n(1 - \rho)]\|p_n - d\| + \xi_n\|g(d) - d\| + \xi_n\delta\|p_n - d\| \\
& = [1 - \xi_n(1 - \rho - \delta)]\|p_n - d\| + \xi_n\|g(d) - d\| \\
& \leq \max\{\|p_n - d\|, \frac{1}{1 - \rho - \delta}\|g(d) - d\|\}.
\end{aligned}$$

So, $\{p_n\}$ is bounded and hence $\{Bp_n\}$, $\{g(p_n)\}$, $\{T^n r_n\}$ and $\{T^n q_n\}$ are bounded.
Step 2.

$$\lim_{n \rightarrow \infty} \|p_{n+1} - p_n\| = 0$$

$$\begin{aligned}
\|p_{n+1} - p_n\| & = \|\xi_n g(p_n) + (1 - \xi_n)T^n q_n - \xi_{n-1}g(p_{n-1}) - (1 - \xi_{n-1})T^{n-1}q_{n-1}\| \\
& = \|\xi_n g(p_n) - \xi_n g(p_{n-1}) + \xi_n g(p_{n-1}) + (1 - \xi_n)T^n q_n \\
& \quad - (1 - \xi_n)T^{n-1}q_{n-1} + (1 - \xi_n)T^{n-1}q_{n-1} - \xi_{n-1}g(p_{n-1}) - (1 - \xi_{n-1})T^{n-1}q_{n-1}\| \\
& = \|\xi_n [g(p_n) - g(p_{n-1})] + (\xi_n - \xi_{n-1})g(p_{n-1}) + (1 - \xi_n)(T^n q_n - T^{n-1}q_{n-1}) \\
& \quad + (1 - \xi_n)T^{n-1}q_{n-1} - (1 - \xi_{n-1})T^{n-1}q_{n-1}\| \\
& = \|\xi_n [g(p_n) - g(p_{n-1})] + (\xi_n - \xi_{n-1})g(p_{n-1}) + (1 - \xi_n)(T^n q_n - T^{n-1}q_{n-1}) \\
& \quad + (\xi_{n-1} - \xi_n)T^{n-1}q_{n-1}\| \\
& = \|\xi_n [g(p_n) - g(p_{n-1})] + (\xi_n - \xi_{n-1})(g(p_{n-1}) - T^{n-1}q_{n-1}) \\
& \quad + (1 - \xi_n)(T^n q_n - T^{n-1}q_{n-1})\| \\
& = \|\xi_n [g(p_n) - g(p_{n-1})] + (\xi_n - \xi_{n-1})(g(p_{n-1}) - T^{n-1}q_{n-1}) \\
& \quad + (1 - \xi_n)(T^n q_n - T^n q_{n-1} + T^n q_{n-1} - T^{n-1}q_{n-1})\| \\
& \leq \xi_n \|g(p_n) - g(p_{n-1})\| + |\xi_n - \xi_{n-1}| \|g(p_{n-1}) - T^{n-1}q_{n-1}\| \\
& \quad + (1 - \xi_n) \|T^n q_n - T^n q_{n-1} + T^n q_{n-1} - T^{n-1}q_{n-1}\| \\
& \leq \xi_n \|g(p_n) - g(p_{n-1})\| + |\xi_n - \xi_{n-1}| [\|g(p_{n-1})\| + \|T^{n-1}q_{n-1}\|] \\
& \quad + (1 - \xi_n) \|T^n q_n - T^n q_{n-1}\| + (1 - \xi_n) \|T^n q_{n-1} - T^{n-1}q_{n-1}\| \\
& \leq \xi_n \rho \|p_n - p_{n-1}\| + |\xi_n - \xi_{n-1}| K + (1 - \xi_n) l_n \|q_n - q_{n-1}\| \\
& \quad + (1 - \xi_n) \|T^n q_{n-1} - T^{n-1}q_{n-1}\| \\
& \leq \xi_n \rho \|p_n - p_{n-1}\| + |\xi_n - \xi_{n-1}| K + (1 - \xi_n)(l_n - 1) \|q_n - q_{n-1}\| \\
& \quad + (1 - \xi_n) \|q_n - q_{n-1}\| + (1 - \xi_n) \|T^n q_{n-1} - T^{n-1}q_{n-1}\|.
\end{aligned} \tag{3.5}$$

Now,

$$q_n = T_{\mu_n}(r_n - \mu_n B r_n) \text{ and } q_{n+1} = T_{\mu_{n+1}}(r_{n+1} - \mu_{n+1} B r_{n+1})$$

$$G(q_n, q) + \langle B r_n, q - q_n \rangle + \frac{1}{\mu_n} \langle q - q_n, q_n - r_n \rangle \geq 0, \quad \forall q \in D \tag{3.6}$$

$$G(q_{n+1}, q) + \langle B r_{n+1}, q - q_{n+1} \rangle + \frac{1}{\mu_{n+1}} \langle q - q_{n+1}, q_{n+1} - r_{n+1} \rangle \geq 0, \quad \forall q \in D \tag{3.7}$$

Replacing $q = q_{n+1}$ in (3.6) and $q = q_n$ in (3.7) and getting

$$G(q_n, q_{n+1}) + \langle B r_n, q_{n+1} - q_n \rangle + \frac{1}{\mu_n} \langle q_{n+1} - q_n, q_n - r_n \rangle \geq 0 \tag{3.8}$$

$$G(q_{n+1}, q_n) + \langle Br_{n+1}, q_n - q_{n+1} \rangle + \frac{1}{\mu_{n+1}} \langle q_n - q_{n+1}, q_{n+1} - r_{n+1} \rangle \geq 0 \quad (3.9)$$

Adding (3.8) and (3.9),

$$G(q_n, q_{n+1}) + G(q_{n+1}, q_n) + \langle Br_n, q_{n+1} - q_n \rangle + \langle Br_{n+1}, q_n - q_{n+1} \rangle + \frac{1}{\mu_n} \langle q_{n+1} - q_n, q_n - r_n \rangle + \frac{1}{\mu_{n+1}} \langle q_n - q_{n+1}, q_{n+1} - r_{n+1} \rangle \geq 0.$$

Using (G2),

$$G(q_n, q_{n+1}) + G(q_{n+1}, q_n) \leq 0$$

$$\begin{aligned} \therefore \langle Br_n, q_{n+1} - q_n \rangle + \langle Br_{n+1}, q_n - q_{n+1} \rangle + \frac{1}{\mu_n} \langle q_{n+1} - q_n, q_n - r_n \rangle \\ + \frac{1}{\mu_{n+1}} \langle q_n - q_{n+1}, q_{n+1} - r_{n+1} \rangle \geq 0 \\ \langle Br_{n+1}, q_n - q_{n+1} \rangle - \langle Br_n, q_n - q_{n+1} \rangle + \langle q_{n+1} - q_n, \frac{1}{\mu_n} (q_n - r_n) \rangle \\ - \langle q_{n+1} - q_n, \frac{1}{\mu_{n+1}} (q_{n+1} - r_{n+1}) \rangle \geq 0 \\ \langle Br_{n+1} - Br_n, q_n - q_{n+1} \rangle + \langle q_{n+1} - q_n, \frac{1}{\mu_n} (q_n - r_n) - \frac{1}{\mu_{n+1}} (q_{n+1} - r_{n+1}) \rangle \geq 0 \\ \langle Br_{n+1} - Br_n, q_n - q_{n+1} \rangle + \langle q_n - q_{n+1}, \frac{1}{\mu_{n+1}} (q_{n+1} - r_{n+1}) - \frac{1}{\mu_n} (r_n - q_n) \rangle \geq 0. \end{aligned}$$

Now,

$$\begin{aligned} \langle q_n - q_{n+1}, \mu_n (Br_{n+1} - Br_n) + \frac{\mu_n}{\mu_{n+1}} (q_{n+1} - r_{n+1}) - (q_n - r_n) \rangle \\ = \langle q_n - q_{n+1}, \mu_n (Br_{n+1} - Br_n) \rangle + \langle q_n - q_{n+1}, \frac{\mu_n}{\mu_{n+1}} (q_{n+1} - r_{n+1}) - (q_n - r_n) \rangle \\ = \mu_n \langle q_n - q_{n+1}, Br_{n+1} - Br_n \rangle + \mu_n \langle q_n - q_{n+1}, \frac{1}{\mu_{n+1}} (q_{n+1} - r_{n+1}) - \frac{1}{\mu_n} (q_n - r_n) \rangle \\ = \mu_n [\langle q_n - q_{n+1}, Br_{n+1} - Br_n \rangle + \langle q_n - q_{n+1}, \frac{1}{\mu_{n+1}} (q_{n+1} - r_{n+1}) - \frac{1}{\mu_n} (q_n - r_n) \rangle] \geq 0 \\ \therefore 0 \leq \langle q_n - q_{n+1}, \mu_n (Br_{n+1} - Br_n) + \frac{\mu_n}{\mu_{n+1}} (q_{n+1} - r_{n+1}) - (q_n - r_n) \rangle \\ = \langle q_{n+1} - q_n, -\mu_n (Br_{n+1} - Br_n) - \frac{\mu_n}{\mu_{n+1}} (q_{n+1} - r_{n+1}) + (q_n - r_n) \rangle \\ = \langle q_{n+1} - q_n, q_n - q_{n+1} + q_{n+1} - \frac{\mu_n}{\mu_{n+1}} q_{n+1} + \frac{\mu_n}{\mu_{n+1}} r_{n+1} - r_{n+1} \\ + r_{n+1} + \mu_n Br_{n+1} + \mu_n Br_n - r_n \rangle \\ = \langle q_{n+1} - q_n, q_n - q_{n+1} + (1 - \frac{\mu_n}{\mu_{n+1}}) q_{n+1} + (\frac{\mu_n}{\mu_{n+1}} - 1) r_{n+1} \\ + (r_{n+1} - \mu_n Br_{n+1}) - (r_n - \mu_n Br_n) \rangle \\ = \langle q_{n+1} - q_n, q_n - q_{n+1} + (1 - \frac{\mu_n}{\mu_{n+1}}) (q_{n+1} - r_{n+1}) + (r_{n+1} - r_n) - \mu_n (Br_{n+1} - Br_n) \rangle \end{aligned}$$

$$\begin{aligned} \therefore \|q_{n+1} - q_n\|^2 &\leq \|q_{n+1} - q_n\| \left\{ \left| 1 - \frac{\mu_n}{\mu_{n+1}} \right| \|q_{n+1} - r_{n+1}\| + \|r_{n+1} - r_n\| \right\} \\ &\Rightarrow \|q_{n+1} - q_n\| \leq \left| 1 - \frac{\mu_n}{\mu_{n+1}} \right| \|q_{n+1} - r_{n+1}\| + \|r_{n+1} - r_n\| \\ &= \frac{|\mu_{n+1} - \mu_n|}{\mu_{n+1}} \|q_{n+1} - r_{n+1}\| + \|r_{n+1} - r_n\| \end{aligned}$$

$$\|q_{n+1} - q_n\| \leq \|r_{n+1} - r_n\| + \frac{1}{a_1} |\mu_{n+1} - \mu_n| M_1, \quad (3.10)$$

where $M_1 = \max\{\|q_{n+1} - r_{n+1}\|\}$

Now,

$$r_n = T_{\mu_n}(q_n - \mu_n B q_n) \text{ and } r_{n+1} = T_{\mu_{n+1}}(q_{n+1} - \mu_{n+1} B q_{n+1})$$

Similarly,

$$\|r_{n+1} - r_n\| \leq \|p_{n+1} - p_n\| + \frac{1}{a_1} |\mu_{n+1} - \mu_n| M_2, \quad (3.11)$$

where $M_2 = \max\{\|r_{n+1} - p_{n+1}\|\}$

So, from (3.10) and (3.11),

$$\|q_{n+1} - q_n\| \leq \|p_{n+1} - p_n\| + \frac{1}{a_1} |\mu_{n+1} - \mu_n| M, \quad (3.12)$$

where $M = M_1 + M_2$.

$$\therefore \|q_n - q_{n-1}\| \leq \|p_n - p_{n-1}\| + \frac{1}{a_1} |\mu_n - \mu_{n-1}| M \quad (3.13)$$

$$\begin{aligned} \|T^n q_n - T^{n-1} q_{n-1}\| &= \|T^n q_n - T^n q_{n-1} + T^n q_{n-1} - T^{n-1} q_{n-1}\| \\ &\leq \|T^n q_n - T^n q_{n-1}\| + \|T^n q_{n-1} - T^{n-1} q_{n-1}\| \\ &\leq l_n \|q_n - q_{n-1}\| + \|T^n q_{n-1} - T^{n-1} q_{n-1}\|. \end{aligned} \quad (3.14)$$

From (3.5),

$$\begin{aligned} \|p_{n+1} - p_n\| &\leq \xi_n \rho \|p_n - p_{n-1}\| + |\xi_n - \xi_{n-1}| K + (1 - \xi_n)(l_n - 1) \|q_n - q_{n-1}\| + (1 - \xi_n) \|q_n - q_{n-1}\| \\ &\quad + (1 - \xi_n) \|T^n q_{n-1} - T^{n-1} q_{n-1}\| \\ &\leq \xi_n \rho \|p_n - p_{n-1}\| + |\xi_n - \xi_{n-1}| K + (l_n - 1) \|q_n - q_{n-1}\| + (1 - \xi_n) \|q_n - q_{n-1}\| \\ &\quad + \|T^n q_{n-1} - T^{n-1} q_{n-1}\| \\ &\leq \xi_n \rho \|p_n - p_{n-1}\| + |\xi_n - \xi_{n-1}| K + (l_n - 1) \left[\|p_n - p_{n-1}\| + \frac{1}{a_1} |\mu_n - \mu_{n-1}| M \right] \\ &\quad + (1 - \xi_n) \left[\|p_n - p_{n-1}\| + \frac{1}{a_1} |\mu_n - \mu_{n-1}| M \right] + \|T^n q_{n-1} - T^{n-1} q_{n-1}\| \\ &= \xi_n \rho \|p_n - p_{n-1}\| + |\xi_n - \xi_{n-1}| K + (l_n - 1) \|p_n - p_{n-1}\| + \frac{(l_n - 1)}{a_1} |\mu_n - \mu_{n-1}| M \\ &\quad + (1 - \xi_n) \|p_n - p_{n-1}\| + \frac{(1 - \xi_n)}{a_1} |\mu_n - \mu_{n-1}| M + \|T^n q_{n-1} - T^{n-1} q_{n-1}\| \\ &\leq \xi_n \rho \|p_n - p_{n-1}\| + |\xi_n - \xi_{n-1}| K + \xi_n \delta \|p_n - p_{n-1}\| + \frac{\xi_n \delta}{a_1} |\mu_n - \mu_{n-1}| M \end{aligned}$$

$$\begin{aligned}
 & + (1 - \xi_n)\|p_n - p_{n-1}\| + \frac{(1 - \xi_n)}{a_1}|\mu_n - \mu_{n-1}|M + \|T^n q_{n-1} - T^{n-1} q_{n-1}\| \\
 & = [1 - \xi_n(1 - \rho - \delta)]\|p_n - p_{n-1}\| + \frac{(\xi_n \delta + 1 - \xi_n)}{a_1}|\mu_n - \mu_{n-1}|M \\
 & + |\xi_n - \xi_{n-1}|K + \|T^n q_{n-1} - T^{n-1} q_{n-1}\| \\
 & = [1 - \xi_n(1 - \rho - \delta)]\|p_n - p_{n-1}\| + \xi_n(1 - \rho - \delta) \frac{|\mu_n - \mu_{n-1}|}{a_1}M \\
 & + \frac{(-2\xi_n + 2\xi_n \delta + \xi_n \rho + 1)}{a_1}|\mu_n - \mu_{n-1}|M + |\xi_n - \xi_{n-1}|K + \|T^n q_{n-1} - T^{n-1} q_{n-1}\| \\
 & \leq [1 - \xi_n(1 - \rho - \delta)]\|p_n - p_{n-1}\| + \xi_n(1 - \rho - \delta) \frac{|\mu_n - \mu_{n-1}|}{a_1}M \\
 & + [1 + \xi_n(2\delta + \rho)] \frac{|\mu_n - \mu_{n-1}|}{a_1}M + |\xi_n - \xi_{n-1}|K + \|T^n q_{n-1} - T^{n-1} q_{n-1}\|,
 \end{aligned}$$

where $K = \sup\{\|g(p_n)\| + \|T^n q_n\|\}$.

Taking $s_n = \xi_n(1 - \rho - \delta)$, $\eta_n = \frac{|\mu_n - \mu_{n-1}|}{a_1}M$ and

$\gamma_n = [1 + \xi_n(2\delta + \rho)] \frac{|\mu_n - \mu_{n-1}|}{a_1}M + |\xi_n - \xi_{n-1}|K + \|T^n q_{n-1} - T^{n-1} q_{n-1}\|$.

Then,

$$\|p_{n+1} - p_n\| \leq (1 - s_n)\|p_n - p_{n-1}\| + s_n \eta_n + \gamma_n,$$

using lemma (2.4),

$$\lim_{n \rightarrow \infty} \|p_{n+1} - p_n\| = 0. \tag{3.15}$$

Using (3.15) and $\lim_{n \rightarrow \infty} (\mu_n - \mu_{n+1}) = 0$, then from (3.11) and (3.12),

$$\lim_{n \rightarrow \infty} \|r_{n+1} - r_n\| = 0 \tag{3.16}$$

and

$$\lim_{n \rightarrow \infty} \|q_{n+1} - q_n\| = 0. \tag{3.17}$$

Step 3. $\lim_{n \rightarrow \infty} \|Tp_n - p_n\| = 0$.

Now,

$$\begin{aligned}
 \|q_n - d\|^2 & = \|T_{\mu_n}(I - \mu_n B)r_n - T_{\mu_n}(I - \mu_n B)d\|^2 \\
 & \leq \|(I - \mu_n B)r_n - (I - \mu_n B)d\|^2 \\
 & = \|(r_n - d) - \mu_n(Br_n - Bd)\|^2 \\
 & = \|r_n - d\|^2 - 2\mu_n \langle r_n - d, Br_n - Bd \rangle + \mu_n^2 \|Br_n - Bd\|^2 \\
 & \leq \|r_n - d\|^2 - 2\mu_n \beta \|Br_n - Bd\|^2 + \mu_n^2 \|Br_n - Bd\|^2 \\
 & = \|q_n - d\|^2 + \mu_n(\mu_n - 2\beta) \|Br_n - Bd\|^2
 \end{aligned} \tag{3.18}$$

$$\begin{aligned}
 \|p_{n+1} - d\|^2 & = \|\xi_n g(p_n) + (1 - \xi_n)T^n q_n - d\|^2 \\
 & = \|\xi_n g(p_n) - \xi_n d + \xi_n d + (1 - \xi_n)T^n q_n - d\|^2 \\
 & = \|\xi_n [g(p_n) - d] + (1 - \xi_n)T^n q_n - (\xi_n - 1)d\|^2 \\
 & = \|\xi_n [g(p_n) - d] + (1 - \xi_n)[T^n q_n - d]\|^2 \\
 & \leq \xi_n \|g(p_n) - d\|^2 + (1 - \xi_n) \|T^n q_n - d\|^2
 \end{aligned}$$

$$\leq \xi_n \|p_{n+1} - d\|^2 + (1 - \xi_n) l_n^2 \|q_n - d\|^2 \quad (3.19)$$

Putting value of $\|q_n - d\|^2$,

$$\begin{aligned} &\leq \xi_n \|g(p_n) - d\|^2 + (1 - \xi_n) l_n^2 [\|r_n - d\|^2 + \mu_n(\mu_n - 2\beta) \|Br_n - Bd\|^2] \\ &\leq \xi_n \|g(p_n) - d\|^2 + (1 - \xi_n) l_n^2 \|r_n - d\|^2 + \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2. \end{aligned} \quad (3.20)$$

Putting value of $\|r_n - d\|^2$,

$$\begin{aligned} &\leq \xi_n \|g(p_n) - d\|^2 + (1 - \xi_n) l_n^2 [\|p_n - d\|^2 + \mu_n(\mu_n - 2\beta) \|Bp_n - Bd\|^2] \\ &\quad + \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2 \\ &= \xi_n \|g(p_n) - d\|^2 + (1 - \xi_n) l_n^2 \|p_n - d\|^2 + (1 - \xi_n) l_n^2 \mu_n(\mu_n - 2\beta) \|Bp_n - Bd\|^2 \\ &\quad + \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2 \\ &= \xi_n \|g(p_n) - d\|^2 + (1 - \xi_n) (l_n^2 - 1) \|p_n - d\|^2 + (1 - \xi_n) l_n^2 \mu_n(\mu_n - 2\beta) \|Bp_n - Bd\|^2 \\ &\quad + \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2 + (1 - \xi_n) \|p_n - d\|^2 \\ &\leq \beta_n \|g(p_n) - d\|^2 + (1 - \xi_n) (l_n^2 - 1) \|p_n - d\|^2 + \|p_n - d\|^2 \\ &\quad + (1 - \xi_n) l_n^2 \mu_n(\mu_n - 2\beta) \|Bp_n - Bd\|^2 + \mu_n(\mu_n - 2\beta) l_n^2 \|Br_n - Bd\|^2 \end{aligned}$$

$$\begin{aligned} \therefore (1 - \xi_n) l_n^2 \mu_n(2\beta - \mu_n) \|Bp_n - Bd\|^2 + \mu_n(2\beta - \mu_n) l_n^2 \|Br_n - Bd\|^2 &\leq \xi_n \|g(p_n) - d\|^2 \\ &\quad + (1 - \xi_n) (l_n^2 - 1) \|p_n - d\|^2 + \|p_n - d\|^2 - \|p_{n+1} - d\|^2. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \xi_n = 0$, $\lim_{n \rightarrow \infty} l_n = 1$, $\{g(p_n)\}$ and $\{p_n\}$ are bounded, so

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mu_n(2\beta - \mu_n) \|Bp_n - Bd\|^2 + \lim_{n \rightarrow \infty} \mu_n(2\beta - \mu_n) \|Br_n - Bd\|^2 \\ &\leq \lim_{n \rightarrow \infty} [\|p_n - d\|^2 - \|p_{n+1} - d\|^2]. \end{aligned}$$

Since $\{p_n\}$ and $\{q_n\}$ are bounded, so

$$\lim_{n \rightarrow \infty} [\|p_n - d\|^2 - \|p_{n+1} - d\|^2] = 0.$$

$\therefore \lim_{n \rightarrow \infty} \|Bp_n - Bd\| = 0$ and $\lim_{n \rightarrow \infty} \|Br_n - Bd\| = 0$.

$$\begin{aligned} \|q_n - d\|^2 &= \|T_{\mu_n}(r_n - \mu_n Br_n) - T_{\mu_n}(d - \mu_n Bd)\|^2 \\ &\leq \langle q_n - d, (r_n - \mu_n Br_n) - (d - \mu_n Bd) \rangle \\ &= \frac{1}{2} (\|(r_n - \mu_n Br_n) - (d - \mu_n Bd)\|^2 + \|q_n - d\|^2) \\ &\quad - \frac{1}{2} (\|(r_n - \mu_n Br_n) - (d - \mu_n Bd) - (q_n - d)\|^2) \\ &= \frac{1}{2} (\|(I - \mu_n B)(r_n - d)\|^2 + \|q_n - d\|^2) - \frac{1}{2} (\|(r_n - q_n) - \mu_n (Br_n - Bd)\|^2) \\ &\leq \frac{1}{2} (\|r_n - d\|^2 + \|q_n - d\|^2) - \frac{1}{2} (\|(r_n - q_n) - \mu_n (Br_n - Bd)\|^2) \\ &= \frac{1}{2} (\|r_n - d\|^2 + \|q_n - d\|^2) - \frac{1}{2} [\|r_n - q_n\|^2 + \|\mu_n (Br_n - Bd)\|^2 \\ &\quad - 2\langle r_n - q_n, \mu_n (Br_n - Bd) \rangle] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2}\|r_n - d\|^2 + \frac{1}{2}\|q_n - d\|^2 - \frac{1}{2}\|r_n - q_n\|^2 - \frac{1}{2}\mu_n^2\|Br_n - Bd\|^2 \\
 &\quad + \mu_n\langle r_n - q_n, Br_n - Bd \rangle \\
 \frac{1}{2}\|q_n - d\|^2 &\leq \frac{1}{2}\|r_n - d\|^2 - \frac{1}{2}\|r_n - q_n\|^2 - \frac{1}{2}\mu_n^2\|Br_n - Bd\|^2 + \mu_n\langle r_n - q_n, Br_n - Bd \rangle \\
 \|q_n - d\|^2 &\leq \|r_n - d\|^2 - \|r_n - q_n\|^2 - \mu_n^2\|Br_n - Bd\|^2 + 2\mu_n\langle r_n - q_n, Br_n - Bd \rangle \quad (3.21)
 \end{aligned}$$

$$\begin{aligned}
 \|r_n - d\|^2 &= \|T_{\mu_n}(p_n - \mu_n Bp_n) - T_{\mu_n}(d - \mu_n Bd)\|^2 \\
 &\leq \langle r_n - d, (p_n - \mu_n Bp_n) - (d - \mu_n Bd) \rangle \\
 &= \frac{1}{2}(\|(p_n - \mu_n Bp_n) - (d - \mu_n Bd)\|^2 + \|r_n - d\|^2) \\
 &\quad - \frac{1}{2}(\|(p_n - \mu_n Bp_n) - (d - \mu_n Bd) - (r_n - d)\|^2) \\
 &= \frac{1}{2}(\|(I - \mu_n B)(p_n - d)\|^2 + \|r_n - d\|^2) - \frac{1}{2}(\|(p_n - r_n) - \mu_n(Bp_n - Bd)\|^2) \\
 &\leq \frac{1}{2}(\|p_n - d\|^2 + \|r_n - d\|^2) - \frac{1}{2}(\|(p_n - r_n) - \mu_n(Bp_n - Bd)\|^2) \\
 &= \frac{1}{2}(\|p_n - d\|^2 + \|r_n - d\|^2) - \frac{1}{2}[\|p_n - r_n\|^2 + \|\mu_n(Bp_n - Bd)\|^2 \\
 &\quad - 2\langle p_n - r_n, \mu_n(Bp_n - Bd) \rangle] \\
 &= \frac{1}{2}\|p_n - d\|^2 + \frac{1}{2}\|r_n - d\|^2 - \frac{1}{2}\|p_n - r_n\|^2 - \frac{1}{2}\mu_n^2\|Bp_n - Bd\|^2 \\
 &\quad + \mu_n\langle p_n - r_n, Bp_n - Bd \rangle \\
 \frac{1}{2}\|r_n - d\|^2 &\leq \frac{1}{2}\|p_n - d\|^2 - \frac{1}{2}\|p_n - r_n\|^2 - \frac{1}{2}\mu_n^2\|Bp_n - Bd\|^2 + \mu_n\langle p_n - r_n, Bp_n - Bd \rangle \\
 \|r_n - d\|^2 &\leq \|p_n - d\|^2 - \|p_n - r_n\|^2 - \mu_n^2\|Bp_n - Bd\|^2 + 2\mu_n\langle p_n - r_n, Bp_n - Bd \rangle \quad (3.22)
 \end{aligned}$$

$$\begin{aligned}
 \therefore \|q_n - d\|^2 &\leq \|p_n - d\|^2 - \|p_n - r_n\|^2 - \mu_n^2\|Bp_n - Bd\|^2 + 2\mu_n\langle p_n - r_n, Bp_n - Bd \rangle \\
 &\quad - \|r_n - q_n\|^2 - \mu_n^2\|Br_n - Bd\|^2 + 2\mu_n\langle r_n - q_n, Br_n - Bd \rangle \\
 &= \|p_n - d\|^2 - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\
 &\quad + 2\mu_n[\langle p_n - r_n, Bp_n - Bd \rangle + \langle r_n - q_n, Br_n - Bd \rangle]. \quad (3.23)
 \end{aligned}$$

Putting value from (3.23) into (3.19),

$$\begin{aligned}
 \|p_{n+1} - d\|^2 &\leq \xi_n \|g(p) - d\|^2 + (1 - \xi_n)l_n^2[\|p_n - d\|^2 - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 \\
 &\quad - \mu_n^2[\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\
 &\quad + 2\mu_n[\langle p_n - r_n, Bp_n - Bd \rangle + \langle r_n - q_n, Br_n - Bd \rangle]] \\
 &= \xi_n \|g(p) - d\|^2 + (1 - \xi_n)l_n^2\|p_n - d\|^2 - (1 - \xi_n)l_n^2[\|p_n - r_n\|^2 + \|r_n - q_n\|^2] \\
 &\quad - \mu_n^2(1 - \xi_n)l_n^2[\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\
 &\quad + 2\mu_n(1 - \xi_n)l_n^2[\langle p_n - r_n, Bp_n - Bd \rangle + \langle r_n - q_n, Br_n - Bd \rangle] \\
 &\leq \xi_n \|g(p) - d\|^2 + (1 - \xi_n)(l_n^2 - 1)\|p_n - d\|^2 + (1 - \xi_n)\|p_n - d\|^2 \\
 &\quad - (1 - \xi_n)l_n^2[\|p_n - r_n\|^2 + \|r_n - q_n\|^2] - \mu_n^2(1 - \xi_n)l_n^2[\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\
 &\quad + 2\mu_n(1 - \xi_n)l_n^2[\|p_n - r_n\| \cdot \|Bp_n - Bd\| + \|r_n - q_n\| \cdot \|Br_n - Bd\|]
 \end{aligned}$$

$$\begin{aligned} &\leq \xi_n \|g(p) - d\|^2 + (1 - \xi_n)(l_n^2 - 1) \|p_n - d\|^2 + \|p_n - d\|^2 \\ &- (1 - \xi_n)l_n^2 [\|p_n - r_n\|^2 + \|r_n - q_n\|^2] - \mu_n^2 (1 - \xi_n)l_n^2 [\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\ &+ 2\mu_n(1 - \xi_n)l_n^2 [\|p_n - r_n\| \cdot \|Bp_n - Bd\| + \|r_n - q_n\| \cdot \|Br_n - Bd\|] \end{aligned}$$

$$\begin{aligned} (1 - \xi_n)l_n^2 [\|p_n - r_n\|^2 + \|r_n - q_n\|^2] &\leq \xi_n \|g(p) - d\|^2 + (1 - \xi_n)(l_n^2 - 1) \|p_n - d\|^2 + \|p_n - d\|^2 \\ &- \|p_{n+1} - d\|^2 - \mu_n^2 (1 - \xi_n)l_n^2 [\|Bp_n - Bd\|^2 + \|Br_n - Bd\|^2] \\ &+ 2\mu_n(1 - \xi_n)l_n^2 [\|p_n - r_n\| \cdot \|Bp_n - Bd\| \\ &+ \|r_n - q_n\| \cdot \|Br_n - Bd\|]. \end{aligned}$$

Since $\xi_n \rightarrow 0$, $l_n \rightarrow 1$ as $n \rightarrow \infty$ and $\{p_n\}, \{q_n\}, \{r_n\}$ are bounded,

$\lim_{n \rightarrow \infty} [\|p_n - d\|^2 - \|p_{n+1} - d\|^2] = 0$, $\lim_{n \rightarrow \infty} \|Bp_n - Bd\| = 0$ and

$\lim_{n \rightarrow \infty} \|Br_n - Bd\| = 0$,

$\therefore \lim_{n \rightarrow \infty} [\|p_n - r_n\|^2 + \|r_n - q_n\|^2] \leq \lim_{n \rightarrow \infty} [\|p_n - d\|^2 - \|p_{n+1} - d\|^2] = 0$

$\Rightarrow \lim_{n \rightarrow \infty} \|p_n - r_n\|^2 = 0$ and $\lim_{n \rightarrow \infty} \|r_n - q_n\|^2 = 0$

$$\lim_{n \rightarrow \infty} \|p_n - r_n\| = 0 \tag{3.24}$$

$$\lim_{n \rightarrow \infty} \|r_n - q_n\| = 0 \tag{3.25}$$

$$\begin{aligned} \|p_n - T^n r_n\| &\leq \|p_n - T^{n-1} q_{n-1}\| + \|T^{n-1} q_{n-1} - T^{n-1} r_{n-1}\| \\ &+ \|T^{n-1} r_{n-1} - T^n r_{n-1}\| + \|T^n r_{n-1} - T^n r_n\| \\ &\leq \|\xi_{n-1} g(p_{n-1}) + (1 - \xi_{n-1}) T^{n-1} q_{n-1} - T^{n-1} q_{n-1}\| + l_{n-1} \|q_{n-1} - r_{n-1}\| \\ &+ \|T^{n-1} r_{n-1} - T^n r_{n-1}\| + l_n \|r_{n-1} - r_n\| \\ &= \|\xi_{n-1} g(p_{n-1}) - \xi_{n-1} T^{n-1} q_{n-1}\| + l_{n-1} \|T_{\mu_{n-1}}(r_{n-1} - \mu_{n-1} B r_{n-1}) \\ &- T_{\mu_{n-1}}(p_{n-1} - \mu_{n-1} B p_{n-1})\| + \|T^{n-1} r_{n-1} - T^n r_{n-1}\| + l_n \|r_{n-1} - r_n\| \\ &\leq \xi_{n-1} \|g(p_{n-1}) - T^{n-1} q_{n-1}\| + l_{n-1} \|(r_{n-1} - \mu_{n-1} B r_{n-1}) \\ &- (p_{n-1} - \mu_{n-1} B p_{n-1})\| + \|T^{n-1} r_{n-1} - T^n r_{n-1}\| + l_n \|r_{n-1} - r_n\| \\ &= \xi_{n-1} \|g(p_{n-1}) - T^{n-1} q_{n-1}\| + l_{n-1} \|(I - \mu_{n-1} B)r_{n-1} - (I - \mu_{n-1} B)p_{n-1}\| \\ &+ \|T^{n-1} r_{n-1} - T^n r_{n-1}\| + l_n \|r_{n-1} - r_n\| \\ &\leq \xi_{n-1} \|g(p_{n-1}) - T^{n-1} q_{n-1}\| + l_{n-1} \|r_{n-1} - p_{n-1}\| \\ &+ \|T^{n-1} r_{n-1} - T^n r_{n-1}\| + l_n \|r_{n-1} - r_n\|. \end{aligned}$$

Now, as $limit\ n \rightarrow \infty$ then $\xi_n \rightarrow 0$, $l_n \rightarrow 1$ and since T is asymptotically regular on D and using (3.16) and (3.24),

$$\lim_{n \rightarrow \infty} \|p_n - T^n r_n\| = 0. \tag{3.26}$$

Now,

$$\begin{aligned} \|r_n - T^n r_n\| &= \|r_n - p_n + p_n - T^n r_n\| \\ &\leq \|r_n - p_n\| + \|p_n - T^n r_n\|. \end{aligned}$$

Taking $limit\ n \rightarrow \infty$ and using (3.24) and (3.26),

$$\lim_{n \rightarrow \infty} \|r_n - T^n r_n\| = 0. \tag{3.27}$$

Now,

$$\begin{aligned} \|Tr_n - r_n\| &\leq \|Tr_n - T^{n+1}r_n\| + \|T^{n+1}r_n - T^{n+1}r_{n+1}\| + \|T^{n+1}r_{n+1} - r_{n+1}\| + \|r_{n+1} - r_n\| \\ &\leq l_1\|r_n - T^n r_n\| + l_{n+1}\|r_n - r_{n+1}\| + \|T^{n+1}r_{n+1} - r_{n+1}\| + \|r_{n+1} - r_n\| \\ &= l_1\|r_n - T^n r_n\| + (l_{n+1} + 1)\|r_n - r_{n+1}\| + \|T^{n+1}r_{n+1} - r_{n+1}\|. \end{aligned}$$

Taking *limit* $n \rightarrow \infty$ and using (3.16) and (3.27),

$$\lim_{n \rightarrow \infty} \|Tr_n - r_n\| = 0. \tag{3.28}$$

Now,

$$\begin{aligned} \|Tp_n - p_n\| &= \|Tp_n - Tr_n + Tr_n - r_n + r_n - p_n\| \\ &\leq \|Tp_n - Tr_n\| + \|Tr_n - r_n\| + \|r_n - p_n\| \\ &\leq l_1\|p_n - r_n\| + \|Tr_n - r_n\| + \|r_n - p_n\| \\ &= (l_1 + 1)\|p_n - r_n\| + \|Tr_n - r_n\|. \end{aligned}$$

Taking *limit* $n \rightarrow \infty$ and using (3.24) and (3.28),

$$\lim_{n \rightarrow \infty} \|Tp_n - p_n\| = 0. \tag{3.29}$$

Step 4. Since $P_{F(T)} \cap EP(G, B)g : D \rightarrow D$ is ρ -contraction, using Banach contraction principle, a unique $d_0 \in F(T) \cap EP(G, B)$ exists such that $d_0 = P_{F(T)} \cap EP(G, B)g(d_0)$. It will be proved that

$$\lim_{n \rightarrow \infty} \langle g(d_0) - d_0, p_n - d_0 \rangle \leq 0. \tag{3.30}$$

Boundedness of $\{p_n\}$ implies that a subsequence $\{p_{n_i}\}$ can be chosen such that

$$\lim_{n \rightarrow \infty} \sup \langle g(d_0) - d_0, p_n - d_0 \rangle = \lim_{i \rightarrow \infty} \langle g(d_0) - d_0, p_{n_i} - d_0 \rangle. \tag{3.31}$$

W.L.O.G., assume that $p_{n_i} \rightharpoonup w$. Due to convexity and closedness of D , it is weakly closed and hence $w \in D$. Now, it will be proved that $w \in F(T)$.

$\therefore p_{n_i} \rightharpoonup w$ and $\lim_{n \rightarrow \infty} \|p_n - Tp_n\| = 0$, so by lemma (2.6), $w \in F(T)$.

Now, it will be proved that $w \in EP(G, B)$.

$\therefore p_{n_i} \rightharpoonup w$ and $\lim_{n \rightarrow \infty} \|p_n - r_n\| = 0 \Rightarrow r_{n_i} \rightharpoonup w$.

$\therefore r_n = T_{\mu_n}(p_n - \mu_n Bp_n)$

$\therefore G(r_n, q) + \langle Bp_n, q - r_n \rangle + \frac{1}{\mu_n} \langle q - r_n, r_n - p_n \rangle \geq 0, \forall q \in D$.

$\therefore G(r_n, q) + G(q, r_n) \leq 0$

$\Rightarrow -G(q, r_n) + \langle Bp_n, q - r_n \rangle + \frac{1}{\mu_n} \langle q - r_n, r_n - p_n \rangle \geq G(r_n, q) + \langle Bp_n, q - r_n \rangle + \frac{1}{\mu_n} \langle q - r_n, r_n - p_n \rangle \geq 0$

$\therefore \langle Bp_n, q - r_n \rangle + \frac{1}{\mu_n} \langle q - r_n, r_n - p_n \rangle \geq G(q, r_n), \forall q \in D$.

Replacing n by n_i ,

$$\therefore \langle Bp_{n_i}, q - r_{n_i} \rangle + \frac{1}{\mu_{n_i}} \langle q - r_{n_i}, r_{n_i} - p_{n_i} \rangle \geq G(q, r_{n_i}). \tag{3.32}$$

Put $r_\mu = \mu q + (1 - \mu)w, \forall \mu \in (0, 1]$.

Since D is a convex set and $q, w \in D$, so $r_\mu \in D$ and hence it must satisfy (3.32) i.e.,

$$\langle Bp_{n_i}, r_\mu - r_{n_i} \rangle + \frac{1}{\mu_{n_i}} \langle r_\mu - r_{n_i}, r_{n_i} - p_{n_i} \rangle \geq G(r_\mu, r_{n_i})$$

$$\begin{aligned}
0 &\geq -\langle Bp_{n_i}, r_\mu - r_{n_i} \rangle - \frac{1}{\mu_{n_i}} \langle r_\mu - r_{n_i}, r_{n_i} - p_{n_i} \rangle + G(r_\mu, r_{n_i}) \\
\langle r_\mu - r_{n_i}, Br_\mu \rangle &\geq \langle r_\mu - r_{n_i}, Br_\mu \rangle - \langle Bp_{n_i}, r_\mu - r_{n_i} \rangle - \frac{1}{\mu_{n_i}} \langle r_\mu - r_{n_i}, r_{n_i} - p_{n_i} \rangle + G(r_\mu, r_{n_i}) \\
&= \langle r_\mu - r_{n_i}, Br_\mu \rangle - \langle r_\mu - r_{n_i}, Bp_{n_i} \rangle - \frac{1}{\mu_{n_i}} \langle r_\mu - r_{n_i}, r_{n_i} - p_{n_i} \rangle + G(r_\mu, r_{n_i}) \\
&= \langle r_\mu - r_{n_i}, Br_\mu - Bp_{n_i} \rangle - \langle r_\mu - r_{n_i}, \frac{r_{n_i} - p_{n_i}}{\mu_{n_i}} \rangle + G(r_\mu, r_{n_i}) \\
&= \langle r_\mu - r_{n_i}, Br_\mu - Br_{n_i} + Br_{n_i} - Bp_{n_i} \rangle - \langle r_\mu - r_{n_i}, \frac{r_{n_i} - p_{n_i}}{\mu_{n_i}} \rangle + G(r_\mu, r_{n_i}) \\
&= \langle r_\mu - r_{n_i}, Br_\mu - Br_{n_i} \rangle + \langle r_\mu - r_{n_i}, Br_{n_i} - Bp_{n_i} \rangle \\
&\quad - \langle r_\mu - r_{n_i}, \frac{r_{n_i} - p_{n_i}}{\mu_{n_i}} \rangle + G(r_\mu, r_{n_i})
\end{aligned}$$

$\therefore \lim_{n \rightarrow \infty} \|r_{n_i} - p_{n_i}\| = 0 \Rightarrow \|Br_{n_i} - Bp_{n_i}\| = 0.$

$\therefore B$ is monotonic, so $\langle r_\mu - r_{n_i}, Br_\mu - Br_{n_i} \rangle \geq 0,$

$\therefore \lim_{n \rightarrow \infty} \langle r_\mu - r_{n_i}, Br_\mu \rangle \geq \lim_{n \rightarrow \infty} G(r_\mu, r_{n_i}).$

It follows from (G4) that

$$G(r_\mu, w) \leq \lim_{n \rightarrow \infty} G(r_\mu, r_{n_i}) \leq \lim_{n \rightarrow \infty} \langle r_\mu - r_{n_i}, Br_\mu \rangle = \langle r_\mu - w, Br_\mu \rangle. \quad (3.33)$$

Since $r_\mu \in D$ and from (G1), (G4) and (3.33), we have

$$\begin{aligned}
0 &= G(r_\mu, r_\mu) = G(r_\mu, \mu q + (1 - \mu)w) \\
&\leq G(r_\mu, \mu q) + G(r_\mu, (1 - \mu)w) \\
&= \mu G(r_\mu, q) + (1 - \mu)G(r_\mu, w). \\
&\leq \mu G(r_\mu, q) + (1 - \mu)\langle r_\mu - w, Br_\mu \rangle.
\end{aligned} \quad (3.34)$$

Now,

$$\begin{aligned}
r_\mu - w &= \mu q + (1 - \mu)w - w \\
&= \mu q + w - \mu w - w \\
&= \mu q - \mu w = \mu(q - w).
\end{aligned}$$

From (3.34),

$$\begin{aligned}
0 &= G(r_\mu, r_\mu) \\
&\leq \mu G(r_\mu, q) + (1 - \mu)\langle \mu(q - w), Br_\mu \rangle \\
&= \mu G(r_\mu, q) + (1 - \mu)\mu\langle q - w, Br_\mu \rangle \\
0 &\leq G(r_\mu, q) + (1 - \mu)\langle q - w, Br_\mu \rangle.
\end{aligned}$$

Taking $\mu \rightarrow 0$ and $\forall q \in D,$

$$\begin{aligned}
0 &\leq G(w, q) + \langle q - w, Bw \rangle \\
&\therefore w \in EP(G, B) \\
&\therefore w \in F(T) \cap EP(G, B)
\end{aligned}$$

From (3.31) and by using metric projection's property,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup \langle g(d_0) - d_0, p_n - d_0 \rangle &= \lim_{i \rightarrow \infty} \langle g(d_0) - d_0, p_{n_i} - d_0 \rangle. \\ &= \langle g(d_0) - d_0, w - d_0 \rangle \leq 0. \end{aligned} \tag{3.35}$$

Step 5. $\lim_{n \rightarrow \infty} \|p_n - d_0\| = 0$.

$$\begin{aligned} \|p_{n+1} - d_0\|^2 &= \|\xi_n g(p_n) + (1 - \xi_n)T^n q_n - d_0\|^2 \\ &= \|\xi_n g(p_n) - \xi_n d_0 + (1 - \xi_n)T^n q_n + \xi_n d_0 - d_0\|^2 \\ &= \|\xi_n(g(p_n) - d_0) + (1 - \xi_n)T^n q_n - (1 - \xi_n)d_0\|^2 \\ &= \|(1 - \xi_n)(T^n q_n - d_0) + \xi_n(g(p_n) - d_0)\|^2 \\ &\leq \|(1 - \xi_n)(T^n q_n - d_0)\|^2 + 2\langle \xi_n(g(p_n) - d_0), \\ &\quad (1 - \xi_n)(T^n q_n - d_0) + \xi_n(g(p_n) - d_0) \rangle \\ &= (1 - \xi_n)^2 \|T^n q_n - d_0\|^2 + 2\xi_n \langle (g(p_n) - d_0), p_{n+1} - d_0 \rangle \\ &= (1 - \xi_n)^2 \|T^n q_n - T^n d_0\|^2 + 2\xi_n \langle g(p_n) - d_0, p_{n+1} - d_0 \rangle \\ &\leq (1 - \xi_n)^2 l_n^2 \|q_n - d_0\|^2 + 2\xi_n \langle g(p_n) - g(d_0) + g(d_0) - d_0, p_{n+1} - d_0 \rangle \\ &\leq (1 - \xi_n)^2 l_n^2 [\|p_n - d_0\|^2 - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2 [\|Bp_n - Bd_0\|^2 \\ &\quad + \|Br_n - Bd_0\|^2] + 2\mu_n [\langle p_n - r_n, Bp_n - Bd_0 \rangle + \langle r_n - q_n, Br_n - Bd_0 \rangle]] \\ &\quad + 2\xi_n \langle g(p_n) - g(d_0) + g(d_0) - d_0, p_{n+1} - d_0 \rangle \\ &= [(1 - \xi_n)l_n]^2 \|p_n - d_0\|^2 + [(1 - \xi_n)l_n]^2 [2\mu_n [\langle p_n - r_n, Bp_n - Bd_0 \rangle \\ &\quad + \langle r_n - q_n, Br_n - Bd_0 \rangle] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2 [\|Bp_n - Bd_0\|^2 \\ &\quad + \|Br_n - Bd_0\|^2]] + 2\xi_n \langle g(p_n) - g(d_0), p_{n+1} - d_0 \rangle + \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle \\ &\leq [(1 - \xi_n)l_n]^2 \|p_n - d_0\|^2 + [(1 - \xi_n)l_n]^2 [2\mu_n [\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| \\ &\quad + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2 [\|Bp_n - Bd_0\|^2 \\ &\quad + \|Br_n - Bd_0\|^2]] + 2\xi_n \|g(p_n) - g(d_0)\| \cdot \|p_{n+1} - d_0\| + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle \\ &\leq [(1 - \xi_n)l_n]^2 \|p_n - d_0\|^2 + [(1 - \xi_n)l_n]^2 [2\mu_n [\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| \\ &\quad + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2 [\|Bp_n - Bd_0\|^2 \\ &\quad + \|Br_n - Bd_0\|^2]] + 2\xi_n \rho \|p_n - d_0\| \cdot \|p_{n+1} - d_0\| + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle \\ &\leq [(1 - \xi_n)l_n]^2 \|p_n - d_0\|^2 + [(1 - \xi_n)l_n]^2 [2\mu_n [\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| \\ &\quad + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2 [\|Bp_n - Bd_0\|^2 \\ &\quad + \|Br_n - Bd_0\|^2]] + 2\xi_n \rho \left[\frac{\|p_n - d_0\|^2 + \|p_{n+1} - d_0\|^2}{2} \right] + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle \\ &= [(1 - \xi_n)l_n]^2 \|p_n - d_0\|^2 + \xi_n \rho \|p_n - d_0\|^2 + \xi_n \rho \|p_{n+1} - d_0\|^2 \\ &\quad + [(1 - \xi_n)l_n]^2 [2\mu_n [\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] \\ &\quad - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2 [\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] \\ &\quad + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle \\ &= [(1 - \xi_n)(l_n - 1) + (1 - \xi_n)]^2 \|p_n - d_0\|^2 + \xi_n \rho \|p_n - d_0\|^2 + \xi_n \rho \|p_{n+1} - d_0\|^2 \\ &\quad + [(1 - \xi_n)l_n]^2 [2\mu_n [\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] \end{aligned}$$

$$\begin{aligned}
& - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|B_n - Bd_0\|^2] \\
& + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle \\
& = [(1 - \xi_n)^2 + (1 - \xi_n)^2(l_n - 1)^2 + 2(1 - \xi_n)(l_n - 1)(1 - \xi_n) + \xi_n \rho] \|p_n - d_0\|^2 \\
& + \xi_n \rho \|p_{n+1} - d_0\|^2 + [(1 - \xi_n)l_n]^2 [2\mu_n[\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| \\
& + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 \\
& - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle \\
& = [1 - (2 - \rho)\beta_n + \beta_n^2 + (1 - \beta_n)^2(l_n - 1)^2 + 2(1 - \beta_n)^2(l_n - 1)] \|p_n - d_0\|^2 \\
& + \beta_n \rho \|p_{n+1} - d_0\|^2 + [(1 - \beta_n)l_n]^2 [2\mu_n[\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| \\
& + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 \\
& - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle.
\end{aligned}$$

$$\begin{aligned}
\|p_{n+1} - d_0\|^2 - \xi_n \rho \|p_{n+1} - d_0\|^2 & \leq [1 - (2 - \rho)\xi_n + \xi_n^2 + (l_n - 1)^2 + 2(l_n - 1)] \|p_n - d_0\|^2 \\
& + l_n^2 [2\mu_n[\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] \\
& - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2] \\
& + \|Br_n - Bd_0\|^2] + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle.
\end{aligned}$$

$$\begin{aligned}
\|p_{n+1} - d_0\|^2 & \leq \left[\frac{1 - (2 - \rho)\xi_n}{1 - \rho\xi_n} \right] \|p_n - d_0\|^2 + \left[\frac{\xi_n^2 + (l_n - 1)^2 + 2(l_n - 1)}{1 - \rho\xi_n} \right] \|p_n - d_0\|^2 \\
& + \frac{l_n^2}{1 - \rho\xi_n} [2\mu_n[\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] \\
& - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] \\
& + 2\xi_n \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle \\
& \leq \left[1 - \frac{(2 - \rho)\xi_n}{1 - \rho\xi_n} \right] \|p_n - r_0\|^2 + \left[\frac{\xi_n^2 + (l_n - 1)^2 + 2(l_n - 1)}{1 - \rho\xi_n} \right] L_n \\
& + \frac{l_n^2}{1 - \rho\xi_n} [2\mu_n[\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] \\
& - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] \\
& + \frac{2\xi_n}{1 - \rho\xi_n} \langle g(r_0) - d_0, p_{n+1} - d_0 \rangle.
\end{aligned}$$

where $L_n = \sup_{p_n \in N} \|p_n - d_0\|^2$

$$\therefore \|p_{n+1} - d_0\|^2 \leq (1 - a_n) \|p_n - d_0\|^2 + a_n \tau_n, \quad (3.36)$$

where $a_n = \frac{(2 - \rho)\xi_n}{1 - \rho\xi_n}$,

$$\begin{aligned}
\tau_n & = \left[\frac{\xi_n^2 + (l_n - 1)^2 + 2(l_n - 1)}{(2 - \rho)\xi_n} \right] L_n + \frac{l_n^2}{1 - \rho\xi_n} [2\mu_n[\|p_n - r_n\| \cdot \|Bp_n - Bd_0\| \\
& + \|r_n - q_n\| \cdot \|Br_n - Bd_0\|] - \|p_n - r_n\|^2 - \|r_n - q_n\|^2 \\
& - \mu_n^2[\|Bp_n - Bd_0\|^2 + \|Br_n - Bd_0\|^2]] + \frac{2\xi_n}{1 - \rho\xi_n} \langle g(d_0) - d_0, p_{n+1} - d_0 \rangle
\end{aligned}$$

∴ $\lim_{n \rightarrow \infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n = \infty$ and $\lim_{n \rightarrow \infty} \sup \tau_n \leq 0$,
 so, by using (3.36) and lemma (2.7),

$$\lim_{n \rightarrow \infty} \|p_n - d_0\| = 0.$$

□

4 Application

Theorem 4.1. Let $D (\neq \phi) \subset H$ be closed and convex and let $G : D \times D \rightarrow R$ be a bifunction satisfying (G1), (G2), (G3) and (G4). Let $g : D \rightarrow D$ be ρ -contraction and $T : D \rightarrow D$ be asymptotically nonexpansive as well as asymptotically regular mapping such that $F(T) \cap EP(G) \neq \phi$. Let $\{\xi_n\} \subset [0, 1]$ and $\{\mu_n\} \subset [0, 2\beta]$ satisfying

- (i) $\lim_{n \rightarrow \infty} \xi_n = 0$, $\sum_{n=1}^{\infty} \xi_n = \infty$,
- (ii) $0 < a_1 \leq \mu_n \leq b_1 \leq 2\beta$,
- (iii) $\lim_{n \rightarrow \infty} (\mu_n - \mu_{n+1}) = 0$,
- (iv) $\lim_{n \rightarrow \infty} \frac{b_n - 1}{\xi_n} = 0$.

If $p_1 \in D$ and sequence $\{p_n\}$ is defined in (3.1), then $\{p_n\}$ strongly converges to $r = P_{F(T) \cap EP(G)}g(r)$.

Proof. Putting $B \equiv 0$ in theorem 3.1, then $G(r_n, q) + \frac{1}{\mu_n} \langle q - r_n, r_n - p_n \rangle \geq 0, \forall q \in D$. Then, the functions $T_{\mu_n p_n}$ and $T_{\mu_n r_n}$ exist and using theorem 3.1, we get the desired result. □

Remark 1. If T which is asymptotically nonexpansive mapping is replaced by a nonexpansive mapping then improved version of theorem by Takahashi and Takahashi [16] is obtained.

5 Numerical Example

The following example is given for the justification of the theorem 3.1.

Example 1. Let $D = [0, \infty) \subset H = R$, $G(p, q) = -2p^2 + pq + q^2$ be a bifunction, $gp = \frac{p}{3}$ be a contraction mapping, $Bp = \frac{p}{2}$ be a $\frac{1}{2}$ -inverse strong monotone and $Tp = \frac{p}{4}$ asymptotically nonexpansive and asymptotically regular.

Then, $F(T) \cap EP(G, B) = \{0\} \neq \phi$. Taking $\mu_n = 1$ and $\xi_n = \frac{1}{4n}$, then by algorithm (3.1), the sequence $\{p_n\}$ is given by

$$\begin{cases} r_n = \frac{p_n}{8}, \\ q_n = \frac{r_n}{8}, \\ p_{n+1} = \xi_n g(p_n) + (1 - \xi_n) T^n q_n, \quad n = 1, 2, 3, \dots \end{cases} \tag{5.1}$$

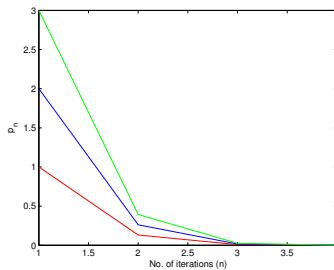
As $n \rightarrow \infty, p_n \rightarrow 0$.

Table 1 compares the algorithms (5.1) and (1.5) and it reveals that convergence of sequence $\{p_n\}$ is faster by algorithm 5.1 than that by algorithm 1.5.

Figure 1 shows the convergence of sequence $\{p_n\}$ with different initial values.

Table 1: Comparison of algorithms 5.1 and 1.5.

Number of iterations (n)	p_n using algorithm 5.1	p_n using algorithm 1.5
1	90	90
2	11.7773	15.4688
3	0.7435	1.7847
4	0.0310	0.0756
5	0.0010	0.0024
6	0	0.0001

Figure 1: Graphical representation of n and p_n for different initial values of p_n

6 Conclusion

In this paper, asymptotically nonexpansive mapping is used to find out the common solution of the fixed point and generalized equilibrium problem. After that, solution of the equilibrium problem is determined as an application. Further, numerical comparison is done using algorithm (1.5) and (5.1) which shows that the convergence by algorithm (5.1) is faster than algorithm (1.5).

References

- [1] R. P. Agarwal, D. O'Regan and D. R. Sahu, Fixed point theory for Lipschitzian-type mappings with applications, 1st edition, Springer, New York, (2009).
- [2] M. O. Aibinu, S. C. Thakur and S. Moyo, The implicit midpoint procedures for asymptotically nonexpansive mappings, Journal of Mathematics, (2020).
- [3] E. Blum, From optimization and variational inequalities to equilibrium problems, Math. student, **63** (1994), 123-145.
- [4] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, Journal of Mathematical Analysis and Applications, **20(2)** (1967), 197-228.
- [5] Y. J. Cho, H. Zhou and G. Guo, Weak and strong convergence theorems for three-step iter-

- ations with errors for asymptotically nonexpansive mappings, *Computers and Mathematics with Applications*, **47(4-5)** (2004), 707-717.
- [6] F. Cianciaruso, G. Marino, L. Muglia and Y. Yao, A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem, *Fixed Point Theory and Applications*, (2009), 1-19.
- [7] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal*, **6(1)** (2005), 117-136.
- [8] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proceedings of the American Mathematical Society*, **35(1)** (1972), 171-174.
- [9] K. R. Kazmi and S. H. Rizvi, Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem, *Journal of the Egyptian Mathematical society*, **21(1)** (2013), 44-51.
- [10] T. H. Meche, M. G. Sangago and H. Zegeye, Approximating a common solution of a finite family of generalized equilibrium and fixed point problems, *SINET: Ethiopian Journal of Science*, **38(1)** (2015), 17-28.
- [11] A. Moudafi and M. Thera, Proximal and dynamical approaches to equilibrium problems, In *Ill-posed variational problems and regularization techniques*, Springer, Berlin, Heidelberg, (1999), 187-201.
- [12] R. Osward, S. Kumar and M. G. Sangago, Approximation of common solutions for a fixed point problem of asymptotically nonexpansive mapping and a generalized equilibrium problem in Hilbert space, *Journal of the Egyptian Mathematical Society*, **27(1)** (2019), 1-16.
- [13] A. Panwar and R. P. Bhokal, Fixed point results for K-iteration using non-linear type mappings, *Open Access Library Journal*, **6(3)** (2019), 1-13.
- [14] A. Panwar, R. Morwal and S. Kumar, Fixed points of ρ -nonexpansive mappings using MP iterative process, *Advances in the Theory of Nonlinear Analysis and its Application*, **6(2)** (2022), 229-245.
- [15] T. M. M. Sow, A new iterative approach for solving generalized equilibrium and fixed point problems with nonlinear mappings, *J. Adv. Math. Stud.*, **15** (2022), 399-414.
- [16] S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, *Nonlinear Analysis: Theory, Methods and Applications*, **69(3)** (2008), 1025-1033.
- [17] H. K. Xu, Iterative algorithms for nonlinear operators, *Journal of the London Mathematical Society*, **66(1)** (2002), 240-256.
- [18] H. K. Xu, Another control condition in an iterative method for nonexpansive mappings, *Bulletin of the Australian Mathematical Society*, **65(1)** (2002), 109-113.
- [19] Y. Yao, J. C. Yao and M. Postolache, An iterate for solving quasi-variational inclusions and nonmonotone equilibrium problems, *J. Nonlinear Convex Anal.*, **24** (2023), 463-474.

- [20] H. Zegeye, T. H. Meche and M. G. Sangago, Algorithms of common solutions for a fixed point of hemicontractive-type mapping and a generalized equilibrium problem, *Inter. J. Adv. Math. Sci*, **5** (2017), 20-26.
- [21] X. Zhao and Y. Yao, Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems, *Optim.*, **69** (2020), 1987-2002.

Pradeep Kumar Maharshi Dayanand University Rohtak, Haryana, India.

E-mail: pradeep.rs.maths@mdurohtak.ac.in

Anju Panwar Maharshi Dayanand University Rohtak, Haryana, India.

E-mail: anjupanwar15@gmail.com, dranjupanwar.math@mdurohtak.ac.in