TAMKANG JOURNAL OF MATHEMATICS
Volume 43, Number 3, 455-462, Autumn 2012
doi:10.5556/j.tkjm.43.2012.455-462
Available online at http://journals.math.tku.edu.tw/

SECOND HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

AABED MOHAMMED AND MASLINA DARUS

Abstract. By making use of the linear operator $\Theta_m^{\lambda,n}$, $m \in \mathbb{N} = \{1,2,3,\ldots\}$ and $\lambda, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ given by the authors, a class of analytic functions $S_m^{\lambda,n}(\alpha,\sigma)$ $(|\alpha| < \pi/2, 0 \le \sigma < 1)$ is introduced. The object of the present paper is to obtain sharp upper bound for functional $\left| a_2 a_4 - a_3^2 \right|$.

1. Introduction

Let $\mathscr{A}$ denote the class of normalised analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

where $z \in U := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let S denote the class of all functions in $\mathscr{A}$ which are univalent.

Robertson [14] introduced the class of starlike functions of order σ as follows:

Definition 1.1 ([14]). Let $\sigma \in [0,1]$, $f \in S$ and

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \sigma, \quad z \in U.$$

Then, we say that f is a starlike function of order σ on U and we denoted this class by $S^*(\sigma)$.

Spacek [15] introduced the class of spirallike functions of type α as follows:

Theorem 1.1 ([15]). *Let $f \in S$ and $-\pi/2 < \alpha < \pi/2$. Then $f(z)$ is a spirallike function of type α on U if*

$$\Re\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in U.$$

We denoted this class by S_α.

Corresponding author: Maslina Darus.
2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Hankel determinant, Positive real functions, Linear operator.

From Definition 1.1 and Theorem 1.1, it is easy to see ([17])that starlike functions of order σ and spirallike functions of type α have some relationships on geometry. Starlike functions of order σ map U into the right half complex plane whose real part is greater than σ by the mapping $\frac{zf'(z)}{f(z)}$, while spirallike functions of type α map U in to the right half complex plane by the mapping $e^{i\alpha}\frac{zf'(z)}{f(z)}$. Since $\lim_{z\to 0} e^{i\alpha}\frac{zf'(z)}{f(z)} = e^{i\alpha}$, we can deduce that if we restrict the image of the mapping $e^{i\alpha}\frac{zf'(z)}{f(z)}$ in the right complex plane whose real part is greater than a certain constant, then the constant must be smaller than $\cos\alpha$.

Libera [16] introduced and studied the class S_σ^α given as follows:

Definition 1.2 ([16])**.** Let $\sigma \in [0,1[$, $-\pi/2 < \alpha < \pi/2$ and $f \in S$. Then $f \in S_\sigma^\alpha$ if and only if

$$\Re\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > \sigma\cos\alpha, \quad z \in U.$$

Obviously,

$$S_\sigma^0 = S^*(\sigma) \text{ and } S_0^\alpha = S_\alpha.$$

For $f_j \in \mathcal{A}$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \ (j = 1,2),$$

the Hadamard product (or convolution) $f_1 * f_2$ of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k \ (z \in U).$$

We recall that a family of the Hurwitz-Lerch Zeta functions $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ ([12]) is defined by

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s},$$

$$(\mu \in \mathbb{C};\ a,\nu \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \rho,\sigma \in \mathbb{R}^+,\ \rho < \sigma \text{ when } s,z \in \mathbb{C};$$
$$\rho = \sigma \text{ and } s \in \mathbb{C} \text{ when } |z| < 1;\ \rho = \sigma \text{ and}$$
$$\Re(s - \mu + \nu) > 1 \text{ when } |z| = 1),$$

contains as its special cases, not only the Hurwitz-Lerch Zeta function

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) = \Phi_{\mu,\nu}^{(0,0)}(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s},$$

but also the following generalized Hurwitz-Zeta function introduced and studied earlier by Goyal and Laddha ([13]),

$$\Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi_\mu(z,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}, \tag{1.2}$$

which, for convenience, are called the Goyal-Laddha-Hurwitz-Lerch Zeta function. Here $(x)_k$ is Pochhammer symbol (or the shifted factorial, since $(1)_k = k!$) and $(x)_k$ given in terms of the Gamma functions can be written as

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}\begin{cases} 1, & \text{if } k=0 \text{ and } x\in\mathbb{C}\backslash\{0\};\\ x(x+1)\ldots(x+k-1), & \text{if } k\in\mathbb{N} \text{ and } x\in\mathbb{C}.\end{cases}$$

It follows that the authors [1] introduced the linear operator $\Theta_m^{\lambda,n} f(z)$ as the following.

For $a = 1$, in (1.2), we consider the function

$$G(z) = z\Phi_\mu(z,s,1) = z + \sum_{k=2}^{\infty} \frac{(\mu)_{k-1}}{(k-1)!}\frac{z^k}{k^s}.$$

Thus

$$\begin{aligned} G(z) * G(z)^{(-1)} &= \tfrac{z}{(1-z)^{\lambda+1}}, \quad \lambda > -1 \\ &= z + \sum_{k=2}^{\infty} \tfrac{(\lambda+1)_{k-1}}{(k-1)!} z^k. \end{aligned}$$

Now for $s = n, \lambda \in \mathbb{N}_0$ and $\mu = m \in \mathbb{N}$, we define the linear operator

$$\begin{aligned} \Theta_m^{\lambda,n} f(z) &= G(z)^{(-1)} * f(z). \quad \big(f \in \mathscr{A}\big) \\ &= z + \sum_{k=2}^{\infty} \frac{(\lambda+1)_{k-1}}{(m)_{k-1}} k^n a_k z^k. \end{aligned} \tag{1.3}$$

In [10], Noonan and Thomas stated that the qth Hankel determinant of the function f of the form (1.1) is defined for $q \in \mathbb{N}$ by

$$H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q+1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+q+2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+q-1} & a_{k+q} & \cdots & a_{k+2q-2} \end{vmatrix}.$$

We now introduce the following class of functions.

Definition 1.3. The function $f \in \mathscr{A}$ is said to be in the class $S_m^{\lambda,n}(\alpha,\sigma)$, $(|\alpha| < \pi/2, 0 \le \sigma < 1)$ if it satisfies the inequality

$$\Re\left\{e^{i\alpha}\frac{\Theta_m^{\lambda,n} f(z)}{z}\right\} > \sigma\cos\alpha \quad (z \in U). \tag{1.4}$$

As is usually the case, we let P be the family of all functions p analytic in U for which $\Re\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots, \qquad z \in U. \tag{1.5}$$

It follows from (1.4) that

$$f \in S_m^{\lambda,n}(\alpha,\sigma) \Leftrightarrow e^{i\alpha}\frac{\Theta_m^{\lambda,n} f(z)}{z} = [(1-\sigma)p(z)+\sigma]\cos\alpha + i\sin\alpha, \tag{1.6}$$

where α is real, $|\alpha| < \pi/2$ and $p(z) \in P$.

We note that

$$S_1^{0,0}(\alpha,\sigma) = \left\{ f : f \in \mathscr{A} \text{ and } \Re\left\{ e^{i\alpha}\frac{f(z)}{z}\right\} > \sigma\cos\alpha \right\},$$

$$S_1^{0,1}(\alpha,\sigma) = \left\{ f : f \in \mathscr{A} \text{ and } \Re\left\{ e^{i\alpha} f'(z)\right\} > \sigma\cos\alpha \right\},$$

$$S_1^{0,1}(0,0) = S_1^{1,0}(0,0) = S_2^{1,1}(0,0) = \mathscr{R} := \left\{ f : f \in \mathscr{A} \text{ and } \Re\left\{ f'(z)\right\} > 0 \right\}.$$

Remark 1.1 ([6]). The subclass $\mathscr{R}$ was studied systematically by MacGregor ([11]) who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

It is well known ([9]) that for $f \in S$ and given by (1.1) the sharp inequality $\left|a_3 - a_2^2\right| \le 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $k = 1$. For a given family $\mathscr{F}$ of functions in $\mathscr{A}$, the sharp bound for the nonlinear functional $\left|a_2a_4 - a_3^2\right|$ is popularly known as the second Hankel determinant. This corresponds to the Hankel determinant with $q = 2$ and $k = 2$. The second Hankel determinant for some subclasses of analytic and nuivalent functions has been studied by many authors (see [2]-[6], [18], [19]).

In the present paper, we seek upper bound for the functional $\left|a_2a_4 - a_3^2\right| \left(f \in S_m^{\lambda,n}(\alpha,\sigma)\right)$. Our investigation includes a recent result of Janteng et al. [2].

To prove our main result, we need the following lemmas.

Lemma 1.2 ([9]). *Let the function $p \in P$ and be given by the series (1.5). Then, the sharp estimate*

$$|c_k| \le 2 \quad (k \in \mathbb{N})$$

holds.

Lemma 1.3 ([7] and [8]). *Let the function $p \in P$ be given by the series (1.5). Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.7}$$

for some x, $|x| \le 1$ and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \tag{1.8}$$

for some z, $|z| \le 1$.

2. Main results

We prove the following.

Theorem 2.1. *Let the function f given by (1.1) be in the class $S_m^{\lambda,n}(\alpha,\sigma)$. Then*

$$\left|a_2a_4-a_3^2\right| \le \frac{4m^2(1-\sigma)^2(1+m)^2\cos^2\alpha}{3^{2n}(\lambda+1)^2(\lambda+2)^2}. \tag{2.1}$$

The estimate (2.1) is sharp.

Proof. Let $f \in S_m^{\lambda,n}(\alpha,\sigma)$. Then from (1.6)we have

$$e^{i\alpha}\frac{\Theta_m^{\lambda,n}f(z)}{z} = [(1-\sigma)p(z)+\sigma]\cos\alpha + i\sin\alpha,$$

where $p \in P$ and is given by (1.5). Then

$$e^{i\alpha}\left\{1+\sum_{k=2}^{\infty}\frac{(k+\lambda-1)!(m-1)!}{\lambda!(k+m-2)!}k^n a_k z^{k-1}\right\} = [(1-\sigma)(1+\sum_{k=1}^{\infty}c_kz^k)+\sigma]\cos\alpha + i\sin\alpha.$$

Comparing the coefficients, we get

$$\left.\begin{aligned}
&\frac{(\lambda+1)}{m}2^ne^{i\alpha}a_2 = (1-\sigma)c_1\cos\alpha,\\
&\frac{(\lambda+2)(\lambda+1)}{m(m+1)}3^ne^{i\alpha}a_3 = (1-\sigma)c_2\cos\alpha,\\
&\frac{(\lambda+3)(\lambda+2)(\lambda+1)}{m(m+1)(m+2)}4^ne^{i\alpha}a_4 = (1-\sigma)c_3\cos\alpha.
\end{aligned}\right\} \tag{2.2}$$

Therefore, (2.2) yields

$$\left|a_2a_4-a_3^2\right| = \frac{m^2(1-\sigma)^2(1+m)\cos^2\alpha}{(\lambda+1)^2(\lambda+2)}\left|\frac{(m+2)c_1c_3}{2^{3n}(\lambda+3)} - \frac{c_2^2(m+1)}{3^{2n}(\lambda+2)}\right|.$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$, $(\theta \in \mathbb{R})$ are members of the class P simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$, $c \in [0,2]$. Using (1.7) along with (1.8), we get

$$\begin{aligned}
\left|a_2a_4-a_3^2\right| &= \frac{m^2(1-\sigma)^2(1+m)\cos^2\alpha}{4(\lambda+1)^2(\lambda+2)}\Bigg\{\Bigg|\frac{(m+2)}{2^{3n}(\lambda+3)}[c^4+2c^2(4-c^2)x-c^2(4-c^2)x^2\\
&\quad +2c(4-c^2)(1-|x|^2)z] - \frac{(m+1)}{3^{2n}(\lambda+2)}\left[c^4+2c^2(4-c^2)x+x^2(4-c^2)^2\right]\Bigg|\Bigg\}\\
&= \frac{m^2(1-\sigma)^2(1+m)\cos^2\alpha}{4(\lambda+1)^2(\lambda+2)}\Bigg|\left\{\frac{(m+2)}{2^{3n}(\lambda+3)} - \frac{(m+1)}{3^{2n}(\lambda+2)}\right\}c^4\\
&\quad +\left\{\frac{(m+2)}{2^{3n}(\lambda+3)} - \frac{(m+1)}{3^{2n}(\lambda+2)}\right\}2c^2(4-c^2)x
\end{aligned}$$

$$-\left\{\frac{c^2(m+2)}{2^{3n}(\lambda+3)}+\frac{(m+1)(4-c^2)}{3^{2n}(\lambda+2)}\right\}x^2(4-c^2)+\frac{2(m+2)}{2^{3n}(\lambda+3)}c(4-c^2)(1-|x|^2)z\Bigg|.$$

An application of triangle inequality and replacement of $|x|$ by y give

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &\leq \frac{m^2(1-\sigma)^2(1+m)\cos^2\alpha}{4(\lambda+1)^2(\lambda+2)}\left[\left\{\frac{(m+2)}{2^{3n}(\lambda+3)}-\frac{(m+1)}{3^{2n}(\lambda+2)}\right\}c^4\right.\\ &+\left\{\frac{(m+2)}{2^{3n}(\lambda+3)}-\frac{(m+1)}{3^{2n}(\lambda+2)}\right\}2c^2y(4-c^2)\\ &\left.+\left\{\frac{c^2(m+2)}{2^{3n}(\lambda+3)}+\frac{(m+1)(4-c^2)}{3^{2n}(\lambda+2)}\right\}y^2(4-c^2)+\frac{2(m+2)}{2^{3n}(\lambda+3)}c(4-c^2)(1-y^2)\right]\\ &=\frac{m^2(1-\sigma)^2(1+m)\cos^2\alpha}{4(\lambda+1)^2(\lambda+2)}\left[\left\{\frac{(m+2)}{2^{3n}(\lambda+3)}-\frac{(m+1)}{3^{2n}(\lambda+2)}\right\}c^4\right.\\ &+\left\{\frac{(m+2)}{2^{3n}(\lambda+3)}-\frac{(m+1)}{3^{2n}(\lambda+2)}\right\}2c^2y(4-c^2)\\ &\left.+\left\{\frac{c(c-2)(m+2)}{2^{3n}(\lambda+3)}+\frac{(m+1)(4-c^2)}{3^{2n}(\lambda+2)}\right\}y^2(4-c^2)+\frac{2(m+2)}{2^{3n}(\lambda+3)}c(4-c^2)\right]\\ &=G(c,y),\quad 0\leq c\leq 2 \text{ and} 0\leq y\leq 1. \end{aligned}\tag{2.3}$$

We next maximize the function $G(c, y)$ on the closed square $[0,2]\times[0,1]$. Since

$$\begin{aligned}\frac{\partial G}{\partial y}&=\frac{m^2(1-\sigma)^2(1+m)\cos^2\alpha}{4(\lambda+1)^2(\lambda+2)}\left[\left\{\frac{(m+2)}{2^{3n}(\lambda+3)}-\frac{(m+1)}{3^{2n}(\lambda+2)}\right\}2c^2(4-c^2)\right.\\ &\left.+\left\{\frac{c(c-2)(m+2)}{2^{3n}(\lambda+3)}+\frac{(m+1)(4-c^2)}{3^{2n}(\lambda+2)}\right\}2y(4-c^2)\right],\end{aligned}$$

$c-2<0$, $3^{2n}(m+2)(\lambda+2)>2^{3n}(m+1)(\lambda+3)$, we have $\partial G/\partial y>0$ for $0<c<2$, $0<y<1$. Thus $G(c,y)$ cannot have a maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$, $\max_{0\leq y\leq 1}G(c,y)=G(c,1)=F(c)$. Since

$$\begin{aligned}F(c)&=\frac{m^2(1-\sigma)^2(1+m)\cos^2\alpha}{4(\lambda+1)^2(\lambda+2)}\left[\left\{\frac{(m+2)}{2^{3n}(\lambda+3)}-\frac{(m+1)}{3^{2n}(\lambda+2)}\right\}c^4\right.\\ &+\left\{\frac{(m+2)}{2^{3n}(\lambda+3)}-\frac{(m+1)}{3^{2n}(\lambda+2)}\right\}2c^2(4-c^2)\\ &\left.+\left\{\frac{c^2(m+2)}{2^{3n}(\lambda+3)}+\frac{(m+1)(4-c^2)}{3^{2n}(\lambda+2)}\right\}(4-c^2)\right]\end{aligned}$$

Then

$$F'(c)=\frac{2m^2(1-\sigma)^2(1+m)\cos^2\alpha}{(\lambda+1)^2(\lambda+2)}\left\{\frac{c(3-c^2)(m+2)}{2^{3n}(\lambda+3)}-\frac{c(4-c^2)(m+1)}{3^{2n}(\lambda+2)}\right\},$$

so that $F'(c)<0$ for $0<c<2$ and has real critical point at $c=0$. Also $F(c)>F(2)$. Therefore, $\max_{0\leq c\leq 2}F(c)$ occurs at $c=0$. Therefore, the upper bound of (2.3) corresponds to $y=1$, $c=0$. Hence

$$\left|a_2a_4-a_3^2\right|\leq\frac{4m^2(1-\sigma)^2(1+m)^2\cos^2\alpha}{3^{2n}(\lambda+1)^2(\lambda+2)^2}.$$

which is the assertion (2.1).Equality holds for the function

$$f(z)=\left(\sum_{k=1}^{\infty}\frac{(m)_{k-1}}{(\lambda+1)_{k-1}k^n}z^k\right)*e^{-i\alpha}\left[z\left(\frac{1+(1-2\sigma)z^2}{1-z^2}\cos\alpha+i\sin\alpha\right)\right].$$

This completes the proof of the Theorem 2.1. □

Remark 2.1. For $\alpha=0,\sigma=0,\lambda=m=1$, $n=0$ and for $\alpha=0,\sigma=0,\lambda=1,m=2$, $n=1$ we get a resent result due to Janteng et al. [2] as in the following corollary.

Corollary 2.1. *If $f\in\mathcal{R}$ then*

$$\left|a_2a_4-a_3^2\right|\le\frac{4}{9}.$$

The result is sharp.

Acknowledgement

The work here is fully supported by MOHE Grant: UKM-ST-06-FRGS0244-2010, Malaysia.

References

[1] A. Mohammed and M. Darus, *An operator defined by convolution involving the generalized Hurwitz–Lerch zeta function* , Submitted

[2] A. Janteng, S. A. Halim, and M. Darus, *Coefficient inequality for a function whose derivative has positive real part* , J.Ineq. Pure and Appl. Math.,**7(2)**(2006), 1–5.

[3] A. Janteng, S. A. Halim, and M. Darus, *Hankel determinant for starlike and convex functions* , Int. Journal of Math. Analysis,**1(13)**(2007), 619 – 625.

[4] A. K. Mishra and P. Gochhayat, *Second Hankel determinant for a class of ana lytic functions defined by fractional derivative functions* , International Journal of Mathematics and Mathematical Sciences, 2008, Article ID 153280, 10 pages.

[5] S. C. Soh and D. Mohamad, *Coefficient bounds for certain classes of close–to–convex functions*, Int. Journal of Math. Analysis, **2(27)**(2008), 1343 – 1351.

[6] G. Murugusundaramoorthy and N. Magesh, *Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant*, Bulletin of Math. Anal. Appl., **1(3)**(2009),85–89.

[7] R. J. Libera and E. J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc., **85(2)**(1982), 225–230.

[8] R. J. Libera and E. J. Zlotkiewicz, *Coeffcient bounds for the inverse of a function with derivative in P*, Proc. Amer. Math. Soc., **87(2)**(1983), 251–289.

[9] P. L. Duren, Univalent Functions, Springer–Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[10] J. W. Noonan and D.K.Thomas, *On the second Hankel determinant of areally mean $p--$ valent functions*, Trans. Amer. Math. Soc., **223(2)**(1976), 337–346.

[11] T. H. MacGregor,*Functions whose derivative has a positive real part,* Trans. Amer. Math. Soc., **104**(1962), 532–537.

[12] S. D. Lin, H. M.Srivastava, *Some families of the Hurwitz–Lerch Zeta function and associated fractional derivative and other integral representations*, Appl. Math. Comput., **154**(2004), 725–733.

[13] S. Goyal, R. K. Laddha, *On the generalized Riemann zeta function and the generalized Lambert transform*, Ganita Sandesh, **11**(1997), 99–108.

[14] M.S. Robertson,*On the theory of univalent functions*, Ann. Math., **37**(1936), 374–408.

[15] L. Spacek, *Contribution la thorie des fonctions univalentes,* Casopis Pest Math., **62**(1932), 1219, (in Russian).

[16] J. R. Libera, *Univalent* $\alpha --$ *spiral functions,* Canada. J. Math., **19**(1967), 449–456.

[17] Q. Xu and S. Lu, *The Alexander transformation of a subclass of spirallike functions of type* β, J. Ineq. Pure and Appl. Math, **10(1)**(2009), 1–7.

[18] T. Hayami and S. Owa, *Hankel determinant for p–valently starlike and convex functions of order* α, General Math., **17** (2009), 29–44.

[19] T. Hayami and S. Owa, *Generalized Hankel determinant for certain classes*, Int. J. Math. Anal., **4** (2010), 2573–2585.

School of Mathematical Sceinces, Faculty of Science and Technology, Universiti Kebangsaan Malaysia 43600 Bangi, Selangor D. Ehsan, Malaysia.

E-mail: aabedukm@yahoo.com

School of Mathematical Sceinces, Faculty of Science and Technology, Universiti Kebangsaan Malaysia 43600 Bangi, Selangor D. Ehsan, Malaysia.

E-mail: maslina@ukm.my