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Certain coefficient problems of $\mathcal{S}_e^*$ and $\mathcal{C}_e$

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Abstract. In this current study, we consider the classes $\mathcal{S}_e^*$ and $\mathcal{C}_e$ to obtain sharp bounds for the third Hankel determinant for functions within these classes. Additionally, we provide estimates for the sixth and seventh coefficients while establishing the fourth-order Hankel determinant as well.

Keywords. Coefficient estimate, exponential function, Starlike function, sharp third Hankel

1 Introduction

Consider the set of normalized analytic functions, denoted as $\mathcal{A}$, which are defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. These functions are represented by the expansion:

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots . \tag{1.1}$$

Within this class, we define a subclass $\mathcal{S}$, which comprises univalent functions. Also, assume a class of analytic functions defined on the unit disk $\mathbb{D}$, which possess a positive real part. This class is represented as $\mathcal{P}$ whose elements are of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. We use the notation $h_1 \prec h_2$ to indicate that function h_1 is subordinate to h_2, which implies the existence of a Schwarz function w with the properties $w(0) = 0$ and $|w(z)| \leq |z|$, such that $h_1(z) = h_2(w(z))$.

The Bieberbach's conjecture, as discussed in [6, Page no. 17] has made a substantial contribution to the advancement of geometric function theory and the emergence of coefficient-related challenges. In the wake of this, numerous additional subclasses of $\mathcal{S}$, encompassing starlike functions denoted as $\mathcal{S}^*$ and convex functions denoted as $\mathcal{C}$, have been introduced. Notably, in 1992, Ma and Minda [18] introduced the following two classes:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \tag{1.2}$$

and

$$\mathcal{C}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}, \tag{1.3}$$

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which unifies various subclasses of $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Here φ is an analytic univalent function satisfying the conditions $\varphi(z) > 0$, $\varphi(\mathbb{D})$ is symmetric about the real axis and starlike with respect to $\varphi(0) = 1$ with $\varphi'(0) > 0$. Some of these classes are outlined in Table 1 for ready reference.

The notion of Hankel determinants was introduced in [21]. Remarkably, this concept continues to captivate the attention of numerous researchers to this very day. Encompassing a broad spectrum of applications and implications, the qth Hankel determinants $H_q(n)$ of analytic functions belonging to the class $\mathcal{A}$, as represented in (1.1), have been defined under the premise that a_1 takes the value 1. For $n, q \in \mathbb{N}$, this definition unfolds as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.4}$$

The specific expression for the third-order Hankel determinant, denoted as $H_3(1)$, is obtained by substituting $q = 3$ and $n = 1$ into (1.4). This determinant can be precisely defined as:

$$H_3(1) = 2a_2a_3a_4 - a_3^3 - a_4^2 - a_2^2a_5 + a_3a_5. \tag{1.5}$$

Over the time, several authors established sharp bound of second-order Hankel determinants, see [1, 4, 10, 11]. However, the task of computing bounds for third-order Hankel determinants, proves to be considerably more intricate, can be observed from [15, 28, 27]. In the context of the class $\mathcal{S}^*$, Kwon et al. [15] established the inequality $|H_3(1)| \leq 8/9$, which has recently been best improved to the bound of $4/9$ in [2, 9] independently. Furthermore, Lecko et al. [16] successfully derived the bound $|H_3(1)| \leq 1/9$, a result that stands as sharp for functions in $\mathcal{S}^*(1/2)$. For a more comprehensive exploration of Hankel determinants, interested readers can turn to works such as [3, 9, 16, 26].

Below, we enlist specific subclasses of $\mathcal{S}^*$ and $\mathcal{C}$, resulting from diverse selections of $\varphi(z)$. In a similar manner, Mendiratta et al. [19] introduced and analyzed the classes $\mathcal{S}_e^*$ and $\mathcal{C}_e$ by selecting $\varphi(z) = e^z$ in (1.2) and (1.3), respectively. These classes are defined as follows:

$$\mathcal{S}_e^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z \right\} \quad \text{and} \quad \mathcal{C}_e = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec e^z \right\}.$$

Table 1: List of subclasses of $\mathcal{S}^*$ and $\mathcal{C}$

$\mathcal{S}^*(\varphi)$	$\mathcal{C}(\varphi)$	$\varphi(z)$	Author(s)	Reference
$\mathcal{S}^*[C,D]$	$\mathcal{C}[C,D]$	$(1+Cz)/(1+Dz)$	Janowski	[7]
$\mathcal{S}^*_{SG}$	$\mathcal{C}_{SG}$	$2/(1+e^{-z})$	Goel and Kumar	[5]
$\mathcal{S}^*_{\wp}$	$\mathcal{C}_{\wp}$	$1+ze^z$	Kumar and Kamaljeet	[13]
$\mathcal{S}^*_{q}$	$\mathcal{C}_{q}$	$z+\sqrt{1+z^2}$	Raina and Sokół	[22]
$\mathcal{S}^*_{L}$	$\mathcal{C}_{L}$	$\sqrt{1+z}$	Sokół and Stankiewicz	[24]

Numerous studies have addressed radius problems [19] and investigated implications of first and higher-order differential subordination [20, 25] for the subclasses associated with the exponential function. Zaprawa [27] established bounds for the third Hankel determinants, yielding values of 0.385 and 0.021 for the classes $\mathcal{S}_e^*$ and $\mathcal{C}_e$, respectively, although the results were not sharp.

In our present investigation, we contribute by establishing sharp bounds for $|H_3(1)|$ for functions in the classes $\mathcal{S}_e^*$ and $\mathcal{C}_e$. Additionally, in the upcoming sections, we will provide estimations for the bounds of the sixth and seventh coefficients for the functions belonging to the classes, $\mathcal{S}_e^*$ and $\mathcal{C}_e$ and also evaluate the fourth Hankel determinant.

2 Hankel Determinants for $\mathcal{S}_e^*$

2.1 Preliminaries

In this part of the section, we derive the expressions of a_i $(i = 2, 3, \dots, 7)$ in terms of Carathéodory coefficients. For this, let $f \in \mathcal{S}_e^*$, then there exists a Schwarz function $w(z)$ such that

$$\frac{zf'(z)}{f(z)} = e^{w(z)}. \tag{2.1}$$

Suppose that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$ and consider $w(z) = (p(z) - 1)/(p(z) + 1)$. Further, by substituting the expansions of $w(z)$, $p(z)$ and $f(z)$ in (2.1) and then comparing the coefficients, we obtain the expressions of a_i $(i = 2, 3, ..., 7)$ in terms of p_j $(j = 1, 2, ..., 5)$, given as follows:

$$a_2 = \frac{1}{2}p_1, \quad a_3 = \frac{1}{16}\Big(4p_2 + p_1^2\Big), \quad a_4 = \frac{1}{288}\Big(-p_1^3 + 12p_1p_2 + 48p_3\Big), \tag{2.2}$$

$$a_5 = \frac{1}{1152}\Big(p_1^4 - 12p_1^2p_2 + 24p_1p_3 + 144p_4\Big), \tag{2.3}$$

$$a_6 = \frac{1}{57600}\Big(-17p_1^5 + 220p_1^3p_2 - 480p_1p_2^2 - 480p_1^2p_3 - 480p_2p_3 + 720p_1p_4 + 5760p_5\Big), \tag{2.4}$$

and

$$\begin{aligned} a_7 = \frac{1}{8294400}\Big(&881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 14400p_2^3 + 29040p_1^3p_3 \\ &- 106560p_1p_2p_3 - 57600p_3^2 - 56160p_1^2p_4 - 86400p_2p_4 \\ &+ 69120p_1p_5\Big). \end{aligned} \tag{2.5}$$

The formula for p_j $(j = 2, 3, 4)$, which is included in Lemma 2.1 below, plays a vital role in establishing the sharp bound for Hankel determinants and forms the foundation for our main results.

Lemma 2.1. *[17, 14] Let $p \in \mathcal{P}$ has the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then*

$$2p_2 = p_1^2 + \gamma(4 - p_1^2), \tag{2.6}$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta, \tag{2.7}$$

and

$$\begin{aligned} 8p_4 = {}& p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma) - 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta \\ &+ \bar{\gamma}\eta^2 - (1 - |\eta|^2)\rho), \end{aligned} \tag{2.8}$$

for some γ, η and ρ such that $|\gamma| \le 1$, $|\eta| \le 1$ and $|\rho| \le 1$.

2.2 Sharp Third Hankel Determinant for $\mathcal{S}_e^*$

In this subsection, we present the sharp bound for $|H_3(1)|$ for functions belonging to the class $\mathcal{S}_e^*$.

Theorem 2.1. *Let $f \in \mathcal{S}_e^*$, then*

$$|H_3(1)| \leq \frac{1}{9}. \tag{2.9}$$

This result is sharp.

Proof. Since the class $\mathcal{P}$ is invariant under rotation, the value of p_1 belongs to the interval [0,2]. Let $p := p_1$ and then substitute the values of a_i $(i = 2, 3, 4, 5)$ in (1.5) from (2.2) and (2.3). We get

$$H_3(1) = \frac{1}{331776}\Big(-211p^6 + 420p^4p_2 - 1872p^2p_2^2 - 5184p_2^3 + 2544p^3p_3 + 10944pp_2p_3 - 9216p_3^2 - 7776p^2p_4 + 10368p_2p_4\Big).$$

After simplifying the calculations through (2.6)-(2.8), we obtain

$$H_3(1) = \frac{1}{331776}\Big(\beta_1(p,\gamma) + \beta_2(p,\gamma)\eta + \beta_3(p,\gamma)\eta^2 + \phi(p,\gamma,\eta)\rho\Big),$$

for $\gamma, \eta, \rho \in \mathbb{D}$, where

$$\begin{aligned}
\beta_1(p,\gamma) :&= -13p^6 - 36\gamma^2p^2(4-p^2)^2 - 360\gamma^3p^2(4-p^2)^2 + 72\gamma^4p^2(4-p^2)^2 \\
&\quad + 78\gamma p^4(4-p^2) + 120p^4\gamma^2(4-p^2) - 324p^4\gamma^3(4-p^2) \\
&\quad - 1296\gamma^2p^2(4-p^2), \\
\beta_2(p,\gamma) :&= 24(1-|\gamma|^2)(4-p^2)(17p^3 + 54\gamma p^3 + 30p\gamma(4-p^2) - 12p\gamma^2(4-p^2)), \\
\beta_3(p,\gamma) :&= 144(1-|\gamma|^2)(4-p^2)(-16(4-p^2) - 2|\gamma|^2(4-p^2) + 9p^2\bar{\gamma}), \\
\phi(p,\gamma,\eta) :&= 1296(1-|\gamma|^2)(4-p^2)(1-|\eta|^2)(2(4-p^2)\gamma - p^2).
\end{aligned}$$

By choosing $x = |\gamma|$, $y = |\eta|$ and utilizing the fact that $|\rho| \leq 1$, the above expression reduces to the following:

$$|H_3(1)| \leq \frac{1}{331776}\Big(|\beta_1(p,\gamma)| + |\beta_2(p,\gamma)|y + |\beta_3(p,\gamma)|y^2 + |\phi(p,\gamma,\eta)|\Big) \leq M(p,x,y),$$

where

$$M(p,x,y) = \frac{1}{331776}\Big(M_1(p,x) + M_2(p,x)y + M_3(p,x)y^2 + M_4(p,x)(1-y^2)\Big), \tag{2.10}$$

with

$$\begin{aligned}
M_1(p,x) :&= 13p^6 + 36x^2p^2(4-p^2)^2 + 360x^3p^2(4-p^2)^2 + 72x^4p^2(4-p^2)^2 \\
&\quad + 78xp^4(4-p^2) + 120p^4x^2(4-p^2) + 324p^4x^3(4-p^2) + 1296x^2p^2(4-p^2), \\
M_2(p,x) :&= 24(1-x^2)(4-p^2)(17p^3 + 54xp^3 + 30px(4-p^2) + 12px^2(4-p^2)),
\end{aligned}$$

$$
\begin{aligned}
M_3(p,x) &:= 144(1-x^2)(4-p^2)(16(4-p^2)+2x^2(4-p^2)+9p^2x),\\
M_4(p,x) &:= 1296(1-x^2)(4-p^2)(2x(4-p^2)+p^2).
\end{aligned}
$$

To maximize $M(p,x,y)$ where $p\in[0,2]$ and $x,y\in[0,1]$, we consider the closed cuboid $U:[0,2]\times[0,1]\times[0,1]$ with six faces and twelve edges. The maximum will be attained at a point that is either on the edges, on the faces, or in the interior of the cuboid. Let's first consider all the interior points of the cuboid U.

1. Let (p,x,y) be an interior point in U, at which M attains its maximum. Clearly $(p,x,y)\in(0,2)\times(0,1)\times(0,1)$ and at (p,x,y), we have $\partial M/\partial p=\partial M/\partial x=\partial M/\partial y=0$. Upon expanding, we get

$$
\begin{aligned}
\frac{\partial M}{\partial p} = \frac{(1-x^2)}{13824}\Big(&96x(5+2x)y+5p^4(-17-24x+12x^2)y-12p^2(-17+6x+24x^2)y\\
&+24p^3(-9+25y^2+2x^2y^2-9x(-2+3y^2))\\
&-48p(-9+41y^2+4x^2y^2-9x(-4+5y^2))\Big)=0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial M}{\partial x} = \frac{(4-p^2)}{6912}\Big(&-12p(-5-4x+15x^2+8x^3)y+p^3(12-29x-36x^2+24x^3)y\\
&-24(14xy^2+4x^3y^2+9(-1+y^2)-27x^2(-1+y^2))+3p^2(8x^3y^2\\
&+x^2(54-81y^2)+9(-2+3y^2)+2x(-9+23y^2))\Big)=0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial M}{\partial y} = \frac{(4-p^2)(1-x^2)}{13824}\Big(&24px(5+2x)+p^3(17+24x-12x^2)+96(8-9x+x^2)y\\
&-12p^2(25-27x+2x^2)y\Big)=0.
\end{aligned}
$$

Upon solving the above equations through the software Mathematica, we get the following critical points (p,x,y), given by

$$(-2.09652, 64.2765, 2.81098);\quad (0, 0.57735, 0);\quad (2.09652, 64.2765, -2.81098);$$

and

$$(-1.11368, 9.14102, 2.7143);\quad (0, -0.57735, 0);\quad (1.11368, 9.14102, -2.7143).$$

However, none of them belong to the interior of cuboid U. Therefore, we conclude that M does not attain its maximum value inside U.

Next, we consider the faces $p=0$, $p=2$, $x=0$, $x=1$, $y=0$ and $y=1$ of U for the point of maxima of M.

2. On the face $p=0$: We have $x,y\in(0,1)$ and

$$M(0,x,y)=\frac{(1-x^2)(8y^2+x^2y^2+9x(1-y^2))}{72}=:s_1(x,y). \tag{2.11}$$

Since

$$\frac{\partial s_1}{\partial y} = \frac{(1-x^2)(x-1)(x-8)y}{36}.$$

Now, $\partial s_1/\partial y = 0$ implies $x = \pm 1$ or $x = 8$ or $y = 0$. Clearly, all these points are not in $(0,1)$. Thus, M does not attain its maximum on the face $p = 0$.
On the face $p = 2$: We have

$$M(2,x,y) := \frac{13}{5184}, \quad x, y \in (0,1). \tag{2.12}$$

On the face $x = 0$: We have $p \in (0,2)$, $y \in (0,1)$ and

$$M(p,0,y) = \frac{13p^6 + (4-p^2)(408p^3y + 2304y^2(4-p^2) + 1296p^2(1-y^2)}{331776} =: s_2(p,y). \tag{2.13}$$

To determine the points of maxima, we solve $\partial s_2/\partial p = 0$ and $\partial s_2/\partial y = 0$. After solving $\partial s_2/\partial y = 0$, we get

$$y = \frac{17p^3}{12(25p^2-64)} =: y_p. \tag{2.14}$$

In order to have $y_p \in (0,1)$ for the given range of y, $p >\approx 1.68218 =: p_0$ is required. Based on calculations, $\partial s_2/\partial p = 0$ gives

$$1728p - 864p^3 + 13p^5 + 816p^2y - 340p^4y - 7872py^2 + 2400p^3y^2 = 0. \tag{2.15}$$

After substituting (2.14) into (2.15), we have

$$21233664p - 27205632p^3 + 11472192p^5 - 1613016p^7 + 2700p^9 = 0. \tag{2.16}$$

A numerical calculation suggests that $p \approx 1.35596 \in (0,2)$ is the solution of (2.16) i.e., $p < p_0$, which contradicts the range of p obtained by solving (2.14). Therefore, we conclude that s_2 does not have any critical point in $(0,2) \times (0,1)$.

On the face $x = 1$: We have $p \in (0,2)$ and

$$M(p,1,y) = \frac{12672p^2 - 2952p^4 - 41p^6}{331776} =: s_3(p). \tag{2.17}$$

Upon solving $\partial s_3/\partial p = 0$, we get the solution $p \approx 1.43461 =: p_0$. Since, $p_0 \in (0,2)$, it is a critical point at which s_3 attains its maximum value, given by ≈ 0.0398426.
On the face $y = 0$: We have $p \in (0,2)$, $x \in (0,1)$ and

$$\begin{aligned} M(p,x,0) = \frac{1}{331776}\Big(& 41472x(1-x^2) + 576p^2(9 - 36x + x^2 + 46x^3 + 2x^4) \\ & - 24p^4(54 - 121x - 8x^2 + 174x^3 + 24x^4) \\ & + p^6(13 - 78x - 84x^2 + 36x^3 + 72x^4)\Big) =: s_4(p,x). \end{aligned}$$

After undergoing further computations such as

$$\frac{\partial s_4}{\partial x} = \frac{1}{331776}\Big(41472(1-x^2) - 82944x^2 + 576p^2(-36 + 2x + 138x^2 + 8x^3)$$

$$+ 24p^4(121 + 16x - 522x^2 - 96x^3) - p^6(78 + 168x - 108x^2 - 288x^3)\Big)$$

and

$$\frac{\partial s_4}{\partial p} = \frac{1}{331776}\Big(6p^5(13 - 78x - 84x^2 + 36x^3 + 72x^4) - 96p^3(54 - 121x - 8x^2 + 174x^3 + 24x^4) + 1152p(9 - 36x + x^2 + 46x^3 + 2x^4)\Big),$$

we observe from Mathematica that all the real solutions (p, x) of this system of equations are as follows:

$$(4.29674, 2.34253); \quad (-6.69897, -0.446594); \quad (-6.80446, -0.196205);$$

and

$$(6.69897, -0.446594); \quad (-4.29674, 2.34253); \quad (-2, -1.00369);$$

and

$$(2, -1.00369); \quad (0, 0.57735); \quad (1.57971, -0.993637);$$

and

$$(6.80446, -0.196205); \quad (0, -0.57735); \quad (-1.57971, -0.993637); \quad (-1.37155, 0.946054).$$

Thus, none of the above critical point lies in $(0, 2) \times (0, 1)$. Therefore, M does not attain its maximum on this face.

On the face $y = 1$: We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$M(p, x, 1) = \frac{1}{331776}\Big(2304px(5 + 2x - 5x^2 - 2x^3) - 4608(-8 + 7x^2 + x^4) + 576p^2(-32 + 9x + 38x^2 + x^3 + 6x^4) - 24p^5(17 + 24x - 29x^2 - 24x^3 + 12x^4) + 96p^3(17 - 6x - 41x^2 + 6x^3 + 24x^4) - 24p^4(-96 + 41x + 130x^2 + 12x^3 + 36x^4) + p^6(13 - 78x - 84x^2 + 36x^3 + 72x^4)\Big) =: s_5(p, x).$$

Upon numerical computations through Mathematica, we have a system of equations

$$\frac{\partial s_5}{\partial x} = \frac{1}{331776}\Big(2304p((2 - 10x - 6x^2)x + (5 + 2x - 5x^2 - 2x^3)) - 4608(14x + 4x^3) + 576p^2(9 + 76x + 3x^2 + 24x^3) - 24p^5(24 - 58x - 72x^2 + 48x^3) + 96p^3(-6 - 82x + 18x^2 + 96x^3) - 24p^4(41 + 260x + 36x^2 + 144x^3) + p^6(-78 - 168x + 108x^2 + 288x^3)\Big)$$

and

$$\frac{\partial s_5}{\partial p} = \frac{1}{331776}\Big(2304x(5 + 2x - 5x^2 - 2x^3) + 1152p(-32 + 9x + 38x^2 + x^3 + 6x^4)$$

$$- 120p^4(17 + 24x - 29x^2 - 24x^3 + 12x^4) + 288p^2(17 - 6x - 41x^2 + 6x^3 + 24x^4) - 96p^3(-96 + 41x + 130x^2 + 12x^3 + 36x^4) + 6p^5(13 - 78x - 84x^2 + 36x^3 + 72x^4)\Big)$$

having all the real solutions (p, x) given by:

$$(2.63476, 8.33714); \quad (-1.3852, -8.27637); \quad (2, 1.53448);$$

and

$$(2, -1.00729); \quad (-2.11632, -0.880438); \quad (-2, -0.606518);$$

and

$$(2, -0.166303); \quad (-3.02448, 3.66443); \quad (-1.84344, 0.190567);$$

and

$$(-2, 0.613808); \quad (1.76914, -0.679457); \quad (2, -0.997809);$$

and

$$(-1.08065, 0.86173); \quad (1.36894, -0.996513); \quad (0, 0).$$

Thus, none of the above critical points lies inside the face $y = 1$.

3. We next examine the maxima attained by $M(p, x, y)$ on the edges of the cuboid U. From (2.13), we have $M(p, 0, 0) = (5184p^2 - 1296p^4 + 13p^6)/331776 =: r_1(p)$. It is easy to observe that $r_1'(p) = 0$ whenever $p = 0$ and $p = 1.4367 \in [0, 2]$ as its points of minima and maxima, respectively. Hence,

$$M(p, 0, 0) \leq 0.0159535, \quad p \in [0, 2].$$

Now considering (2.13) at $y = 1$, we get $M(p, 0, 1) = (36864 - 18432p^2 + 1632p^3 + 2304p^4 - 408p^5 + 13p^6)/331776 =: r_2(p)$. It is easy to observe that $r_2'(p) < 0$ in $[0, 2]$ and hence $p = 0$ serves as the point of maxima. So,

$$M(p, 0, 1) \leq \frac{1}{9}, \quad p \in [0, 2].$$

Through computations, (2.13) shows that $M(0, 0, y)$ attains its maxima at $y = 1$. This implies that

$$M(0, 0, y) \leq \frac{1}{9}, \quad y \in [0, 1].$$

Since, (2.17) does not involve x, we have $M(p, 1, 1) = M(p, 1, 0) = (12672p^2 - 2952p^4 - 41p^6)/331776 =: r_3(p)$. Now, $r_3'(p) = 4224p - 1968p^3 - 41p^5 = 0$ when $p = 0$ and $p = 1.43461$ in the interval $[0, 2]$ as points of minima and maxima, respectively. Hence

$$M(p, 1, 1) = M(p, 1, 0) \leq 0.0398426, \quad p \in [0, 2].$$

After considering $p = 0$ in (2.17), we get, $M(0, 1, y) = 0$. From (2.12), we see that M is a constant function. Therefore, on the edges, the maximum value of $M(p, x, y)$ is

$$M(2, 1, y) = M(2, 0, y) = M(2, x, 0) = M(2, x, 1) = \frac{13}{5184}, \quad x, y \in [0, 1].$$

Using (2.11), we obtain $M(0, x, 1) = (8 - 7x^2 - x^4)/72 =: r_4(x)$. A computation shows that $r_4(x)$ is a decreasing function in $[0, 1]$ and attains its maxima at $x = 0$. Hence

$$M(0, x, 1) \leq \frac{1}{9}, \quad x \in [0, 1].$$

Again utilizing (2.11), we get $M(0,x,0)=x(1-x^2)/8=:r_5(x)$. Further, $r_5'(x)=0$, then $x=1/\sqrt{3}=:\delta_0$. Clearly, δ_0 is a point of maxima for $r_5(x)$. Thus

$$M(0,x,0)\leq 0.0481125, \quad x\in[0,1].$$

From the above all possible cases, we observe that optimal value of $|H_3(1)|$ is $1/9$. Therefore, the inequality (2.9) holds. Now, consider the following function $f_1\in\mathcal{S}_e^*$, defined by

$$f_1(z)=z\exp\left(\int_0^z \frac{e^{t^3}-1}{t}dt\right)=z+\frac{z^4}{3}+\frac{5z^7}{36}+\cdots,$$

with $f_1(0)=0$ and $f_1'(0)=1$, acts as an extremal function for the bound of $|H_3(1)|$ for $a_2=a_3=a_5=0$ and $a_4=1/3$. □

2.3 Fourth Hankel Determinant for $\mathcal{S}_e^*$

In this subsection, we derive the bounds of sixth and seventh coefficients and consequently $|H_4(1)|$ for functions belonging to the class $\mathcal{S}_e^*$. We need the following lemma for deriving our results.

Lemma 2.2. *[12, 23] Let $p=1+\sum_{n=1}^{\infty}p_nz^n\in\mathcal{P}$. Then*

$$|p_n|\leq 2, \quad n\geq 1,$$

$$|p_{n+k}-\nu p_np_k|\leq\begin{cases}2, & 0\leq\nu\leq 1;\\ 2|2\nu-1|, & \text{otherwise},\end{cases}$$

and

$$|p_1^3-\nu p_3|\leq\begin{cases}2|\nu-4|, & \nu\leq 4/3;\\ 2\nu\sqrt{\dfrac{\nu}{\nu-1}}, & 4/3<\nu.\end{cases}$$

We derive the expression of the fourth Hankel determinant when $q=4$ and $n=1$ are put into (1.4) as follows :

$$H_4(1)=a_7H_3(1)-a_6T_1+a_5T_2-a_4T_3, \tag{2.18}$$

where

$$T_1:=a_6(a_3-a_2^2)+a_3(a_2a_5-a_3a_4)-a_4(a_5-a_2a_4), \tag{2.19}$$

$$T_2:=a_3(a_3a_5-a_4^2)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3), \tag{2.20}$$

and

$$T_3:=a_4(a_3a_5-a_4^2)-a_5(a_2a_5-a_3a_4)+a_6(a_4-a_2a_3). \tag{2.21}$$

Now, using Lemma 2.2, we first determine the bounds of T_1, T_2, and T_3.
By substituting the values of a_i's $(i=2,3,...,6)$ in (2.19) using (2.2)-(2.4), we obtain

$$\begin{aligned}5529600T_1=&\,581p_1^7+5040p_1^4p_3+25920p_1^2p_2p_3-7068p_1^5p_2+11040p_1^3p_4\\&-115200p_3p_4+7920p_1^3p_2^2-69120p_2^2p_3+74880p_1p_2p_4-25920p_1p_2^3\\&+57600p_1p_3^2+138240p_2p_5-103680p_1^2p_5\end{aligned}$$

or

$$\begin{aligned}5529600|T_1| \leq\ & |p_1^4(581p_1^3+5040p_3)| + |p_1^2p_2(25920p_3-7068p_1^3)| + |57600p_1p_3^2| \\ & + |p_2^2(7920p_1^3-69120p_3)| + |p_1p_2(74880p_4-25920p_2^2)| \\ & + |p_4(11040p_1^3-115200p_3)| + |p_5(138240p_2-103680p_1^2)|.\end{aligned}$$

Using Lemma 2.2 and the triangle inequality, we arrive at

$$\begin{aligned}|T_1| &\leq \frac{1848448+4976640\sqrt{\frac{15}{1571}}+1843200\sqrt{\frac{15}{217}}+442368\sqrt{\frac{30}{17}}}{5529600} \\ &\approx 0.616137.\end{aligned}$$

Now, we calculate the bound of T_2 in the similar way by substituting the values of a_i's ($i = 2, 3, ..., 6$) in (2.20) from (2.2)-(2.4), as follows:

$$\begin{aligned}22118400T_2 =\ & 235p_1^8 + 8712p_1^5p_3 + 37440p_1^3p_2p_3 - 1156p_1^6p_2 - 63360p_1p_2^2p_3 \\ & - 14640p_1^4p_2^2 + 161280p_1p_3p_4 - 8400p_1^4p_4 + 368640p_3p_5 \\ & - 76800p_1^3p_5 - 8640p_1^2p_2^3 + 172800p_2^2p_4 - 345600p_4^2 - 40320p_1^2p_3^2 \\ & - 184320p_2p_3^2 + 178560p_1^2p_2p_4 - 184320p_1p_2p_5\end{aligned}$$

or

$$\begin{aligned}22118400|T_2| \leq\ & |p_1^5(235p_1^3+8712p_3)| + |p_1^3p_2(37440p_3-1156p_1^3)| + |8640p_1^2p_2^3| \\ & + |p_1p_2^2(63360p_3+14640p_1^3)| + |p_1p_4(161280p_3-8400p_1^3)| \\ & + |p_5(368640p_3-76800p_1^3)| + |p_4(172800p_2^2-345600p_4)| \\ & + |p_3^2(184320p_2+40320p_1^2)| + |p_1p_2(178560p_1p_4-184320p_5)|.\end{aligned}$$

Lemma 2.2 and the triangle inequality lead us to

$$\begin{aligned}|T_2| &\leq \frac{7821568+14376960\sqrt{\frac{65}{9071}}+2949120\sqrt{\frac{6}{19}}+737280\sqrt{\frac{42}{13}}}{22118400} \\ &\approx 0.543487.\end{aligned}$$

Next, we determine the bound of T_3, by replacing the values of a_i's ($i = 2, 3, ..., 6$) from (2.2)-(2.4) in (2.21), as follows:

$$\begin{aligned}597196800T_3 =\ & 6120p_1^8 + 143424p_1^5p_3 - 425p_1^9 - 9000p_1^6p_3 + 9000p_1^7p_2 + 172800p_1^4p_2p_3 \\ & + 302400p_1^3p_3^2 - 2764800p_3^3 + 1036800p_1^3p_2p_4 + 6220800p_2p_3p_4 \\ & - 17280p_1^4p_2^2 + 9953280p_3p_5 - 2073600p_1^3p_5 + 967680p_1^3p_2p_3 \\ & - 64512p_1^6p_2 - 1036800p_1p_2p_3^2 - 32400p_1^5p_2^2 - 777600p_1^2p_2^2p_3 \\ & + 1244160p_1p_3p_4 - 259200p_1^4p_4 - 97200p_1^5p_4 + 1555200p_1p_2^2p_4 \\ & - 4665600p_1p_4^2 - 414720p_1p_2^2p_3 - 172800p_1^3p_2^3 - 829440p_2p_3^2 \\ & - 829440p_1^2p_3^2 + 414720p_1^2p_2^3 - 622080p_1^2p_2p_4 - 4976640p_1p_2p_5\end{aligned}$$

or

$$597196800|T_3| \leq |p_1^5(6120p_1^3+143424p_3)| + |p_1^6(425p_1^3+9000p_3)| + |17280p_1^4p_2^2|$$

$$\begin{aligned}&+|p_1^4p_2(9000p_1^3+172800p_3)|+|p_3^2(302400p_1^3-2764800p_3)|\\&+|p_2p_4(1036800p_1^3+6220800p_3)|+|p_5(9953280p_3-2073600p_1^3)|\\&+|p_1^3p_2(967680p_3-64512p_1^3)|+|1036800p_1p_2p_3^2|+|97200p_1^5p_4|\\&+|p_1^2p_2^2(32400p_1^3+777600p_3)|+|p_1p_4(1244160p_3-259200p_1^3)|\\&+|p_1p_4(1555200p_2^2-4665600p_4)|+|p_1^2p_2(414720p_2^2-622080p_4)|\\&+|p_3^2(829440p_2+829440p_1^2)|+|172800p_1^3p_2^3|\\&+|p_1p_2(414720p_2p_3+4976640p_5)|.\end{aligned}$$

By applying Lemma 2.2 and the triangle inequality,

$$\begin{aligned}|T_3|&\leq\frac{286061056+58982400\sqrt{\frac{3}{19}}+99532800\sqrt{\frac{6}{19}}+2211840\sqrt{210}}{597196800}\\&\approx 0.665582.\end{aligned}$$

Remark 1. On the basis of the above calculations, the bounds of T_1, T_2 and T_3 are 0.616137, 0.543487 and 0.665582, respectively.

To progress further, our next objective is to determine the bounds of the initial coefficients a_i where $i = 2, 3, 4, 5$. These bounds, as derived in [27], are summarized in the following remark.

Remark 2. For $f \in \mathcal{S}_e^*$, $|a_2| \leq 1$, $|a_3| \leq 3/4$, $|a_4| \leq 17/36$ and $|a_5| \leq 25/72$. Here the first three bounds are sharp.

Finding coefficient bounds for $n > 5$ becomes notably more challenging. In order to overcome this difficulty, we employ Lemma 2.2 to deduce the bounds for the sixth and seventh coefficients within the class of functions $\mathcal{S}_e^*$, as demonstrated in the subsequent theorem.

Theorem 2.2. *Let $f \in \mathcal{S}_e^*$. Then $|a_6| \leq 587/1800 \approx 0.326111$ and $|a_7| \leq 1397/4320 \approx 0.32338$.*

Proof. By suitably rearranging the terms given in (2.4), we have

$$57600a_6 = 220p_1^3p_2 - 480p_1^2p_3 - 480p_1p_2^2 + 720p_1p_4 - 17p_1^5 - 480p_2p_3 + 5760p_5.$$

Using triangle inequality, it can be viewed as

$$\begin{aligned}57600|a_6| \leq\ &|p_1^2(220p_1p_2-480p_3)|+|p_1(720p_4-480p_2^2)|+|-17p_1^5|\\&+|5760p_5-480p_2p_3|.\end{aligned} \tag{2.22}$$

Using Lemma 2.2, we arrive at the following inequality:

$$|a_6| \leq \frac{587}{1800} \approx 0.326111.$$

Similarly, considering (2.5), we have

$$\begin{aligned}8294400a_7 =\ &881p_1^6-13260p_1^4p_2+48240p_1^2p_2^2-14400p_2^3+29040p_1^3p_3-56160p_1^2p_4\\&+69120p_1p_5-106560p_1p_2p_3-57600p_3^2-86400p_2p_4.\end{aligned}$$

Through the triangle inequality, it can also be seen as

$$8294400|a_7| \leq |p_1^4(881p_1^2-13260p_2)|+|p_2^2(48240p_1^2-14400p_2)|$$

$$+|p_1(69120p_5-106560p_2p_3)|+|p_1^2(29040p_1p_3-56160p_4)|$$
$$+|57600p_3^2|+|86400p_2p_4|.$$

Lemma 2.2 implies that $|a_7|\leq 1397/4320\approx 0.32338$. □

Theorem 2.3. *Let $f\in\mathcal{S}_e^*$, then*

$$|H_4(1)|\leq 0.29059.$$

The proof of the above theorem follows by substituting the values obtained from Theorem 2.1, Remark 1, Remark 2 and Theorem 2.2 in (2.18), therefore, it is skipped here.

3 Hankel Determinants for $\mathcal{C}_e$

3.1 Preliminaries

In this segment, we express the expressions of initial coefficients a_i $(i=2,3,\ldots,7)$ involving Carathéodory coefficients. When $f\in\mathcal{C}_e$, we replace the L.H.S of (2.1) by $1+zf''(z)/f'(z)$ and arrive at the following equation

$$1+\frac{zf''(z)}{f'(z)}=e^{w(z)}.$$

Proceeding on the similar lines as done for the class $\mathcal{S}_e^*$, we obtain a_i $(i=2,3,...,7)$ in terms of p_j $(j=1,2,...,5)$, we obtain

$$a_2=\frac{1}{4}p_1,\quad a_3=\frac{1}{48}\Big(p_1^2+4p_2\Big),\quad a_4=\frac{1}{1152}\Big(-p_1^3+12p_1p_2+48p_3\Big), \tag{3.1}$$

$$a_5=\frac{1}{5760}\Big(p_1^4-12p_1^2p_2+24p_1p_3+144p_4\Big), \tag{3.2}$$

$$a_6=\frac{1}{345600}\Big(-17p_1^5+220p_1^3p_2-480p_1p_2^2-480p_1^2p_3-480p_2p_3+720p_1p_4$$
$$+5760p_5\Big), \tag{3.3}$$

and

$$a_7=\frac{1}{58060800}\Big(881p_1^6-13260p_1^4p_2+48240p_1^2p_2^2-14400p_2^3+29040p_1^3p_3-106560p_1p_2p_3$$
$$-57600p_3^2-56160p_1^2p_4-86400p_2p_4+69120p_1p_5\Big). \tag{3.4}$$

3.2 Sharp Third Hankel Determinant for $\mathcal{C}_e$

In this subsection, we establish the sharp bound of $|H_3(1)|$ for functions that belong to the class $\mathcal{C}_e$.

Theorem 3.1. *Let $f\in\mathcal{C}_e$, then*

$$|H_3(1)|\leq\frac{1}{144}. \tag{3.5}$$

This bound is sharp.

Proof. We follow the same steps which were used to prove Theorem 2.1. The values of $a_i' s$ $(i = 2, 3, 4, 5)$ from (3.1) and (3.2) are substituted into (1.5). Thus

$$H_3(1) = \frac{1}{6635520}\Big(-173p^6 + 552p^4p_2 - 1872p^2p_2^2 - 3840p_2^3 + 2208p^3p_3 + 8064pp_2p_3 - 11520p_3^2 - 6912p^2p_4 + 13824p_2p_4\Big).$$

Using (2.6)-(2.8) for simplification, we arrive at

$$H_3(1) = \frac{1}{6635520}\Big(\alpha_1(p,\gamma) + \alpha_2(p,\gamma)\eta + \alpha_3(p,\gamma)\eta^2 + \psi(p,\gamma,\eta)\rho\Big),$$

where $\gamma, \eta, \rho \in \mathbb{D}$,

$$\begin{aligned}
\alpha_1(p,\gamma) :&= -5p^6 - 180\gamma^2p^2(4-p^2)^2 + 1536\gamma^3(4-p^2)^2 - 240\gamma^3p^2(4-p^2)^2 \\
&\quad + 144\gamma^4p^2(4-p^2)^2 + 12\gamma p^4(4-p^2) - 120p^4\gamma^2(4-p^2),\\
\alpha_2(p,\gamma) :&= (1-|\gamma|^2)(4-p^2)(240p^3 - 288p\gamma(4-p^2) - 576p\gamma^2(4-p^2)),\\
\alpha_3(p,\gamma) :&= (1-|\gamma|^2)(4-p^2)(-2880(4-p^2) - 576|\gamma|^2(4-p^2)),\\
\psi(p,\gamma,\eta) :&= 3456\gamma(1-|\gamma|^2)(4-p^2)^2(1-|\eta|^2).
\end{aligned}$$

Since $|\rho| \le 1$, also for the simplicity of the calculations, assume $x = |\gamma|$ and $y = |\eta|$,

$$|H_3(1)| \le \frac{1}{6635520}\Big(|\alpha_1(p,\gamma)| + |\alpha_2(p,\gamma)|y + |\alpha_3(p,\gamma)|y^2 + |\psi(p,\gamma,\eta)|\Big) \le N(p,x,y),$$

where

$$N(p,x,y) = \frac{1}{6635520}\Big(N_1(p,x) + N_2(p,x)y + N_3(p,x)y^2 + N_4(p,x)(1-y^2)\Big), \tag{3.6}$$

with

$$\begin{aligned}
N_1(p,x) :&= 5p^6 + 180x^2p^2(4-p^2)^2 + 1536x^3(4-p^2)^2 + 240x^3p^2(4-p^2)^2 \\
&\quad + 144x^4p^2(4-p^2)^2 + 12xp^4(4-p^2) + 120p^4x^2(4-p^2),\\
N_2(p,x) :&= (1-x^2)(4-p^2)(240p^3 + 288px(4-p^2) + 576px^2(4-p^2)),\\
N_3(p,x) :&= (1-x^2)(4-p^2)(2880(4-p^2) + 576x^2(4-p^2)),\\
N_4(p,x) :&= 3456x(1-x^2)(4-p^2)^2.
\end{aligned}$$

We must maximise $N(p,x,y)$ in the closed cuboid $V : [0,2] \times [0,1] \times [0,1]$. By identifying the maximum values on the twelve edges, the interior of V, and the interiors of the six faces, we can prove this.

1. We start by taking into account, every interior point of V. Assume that $(p,x,y) \in (0,2) \times (0,1) \times (0,1)$. We partially differentiate (3.6) with respect to p, x and y to locate the points of maxima in the interior of V. We obtain

$$\frac{\partial N}{\partial p} = \frac{(1-x^2)}{138240}\Big(96x(1+2x)y + 5p^4(-5+6x+12x^2)y - 12p^2(-5+12x+24x^2)y$$

$$- 192p(5y^2 + x^2y^2 + 6x(1 - y^2)) + 48p^3(5y^2 + x^2y^2 + 6x(1 - y^2))\Big),$$

$$\begin{aligned}\frac{\partial N}{\partial x} = \frac{(4 - p^2)}{138240}\Big(&12p(1 + 4x - 3x^2 - 8x^3)y + p^3(-3 - 17x + 9x^2 + 24x^3)y \\ &- 48(4xy^2 + 2x^3y^2 + 3(-1 + y^2) + 9x^2(1 - y^2)) \\ &+ 12p^2(4xy^2 + 2x^3y^2 - 3(1 - y^2) + 9x^2(1 - y^2))\Big)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial N}{\partial y} = \frac{(1 - x^2)(4 - p^2)}{138240}\Big(&24px(1 + 2x) - p^3(-5 + 6x + 12x^2) + 96(5 - 6x + x^2)y \\ &- 24p^2(5 - 6x + x^2)y\Big).\end{aligned}$$

By solving $\partial N/\partial p = \partial N/\partial x = \partial N/\partial y = 0$, we get the common solutions (p, x, y) using the mathematical software, Mathematica as

$$(0, -0.57735, 0); \quad (-1.35285, 6.98337, 2.99877); \quad (1.35285, 6.98337, -2.99877);$$

and

$$(2.01517, 5.04402, -1.09029); \quad (-2.01517, 5.04402, 1.09029); \quad (-2.08219, 0.122286, -1.29086);$$

and

$$(0, 0.57735, 0); \quad (-1.92198, -0.0913794, 0.863409); \quad (1.92198, -0.0913794, -0.863409);$$

and

$$(-2, -1, -0.171635); \quad (2.08219, 0.122286, -1.29086).$$

However, none of them belong to the interior of cuboid V. Therefore, we conclude that N does not attain its maximum value inside V. Next, we proceed to find the critical points on each of the interior of faces of V.

2. Now, we study the interior of each of the six faces of the cuboid V.
On the face $p = 0$: We get $x, y \in (0, 1)$ and

$$N(0, x, y) = \frac{y^2(15 - 12x^2 - 3x^4) + 18x(1 - y^2) - 2x^3(5 - 9y^2)}{2160} =: c_1(x, y). \tag{3.7}$$

Since

$$\frac{\partial c_1}{\partial y} = \frac{y(1 - x)^2(x + 1)(5 - x)}{360}, \quad x, y \in (0, 1),$$

Since, $\partial c_1/\partial y$ becomes zero only when $x = \pm 1$ or at $x = 5$ or at $y = 0$. Clearly, all these points are not in $(0, 1)$. Thus, c_1 does not have any critical point in $(0, 1) \times (0, 1)$.
On the face $p = 2$: We obtain

$$N(2, x, y) := \frac{1}{20736}, \quad x, y \in (0, 1). \tag{3.8}$$

On the face $x=0$: We have $p\in(0,2)$, $y\in(0,1)$ and

$$N(p,0,y)=\frac{(p^3+96y-24p^2y)^2}{1327104}=:c_2(p,y). \tag{3.9}$$

We solve $\partial c_2/\partial p=0$ and $\partial c_2/\partial y=0$ to locate the points of maxima. On solving $\partial c_2/\partial y=0$, we obtain

$$y=-\frac{p^3}{24(4-p^2)}=:y_p.$$

Upon calculations, we observe that such y_p does not belong to $(0,1)$. Consequently, no such critical point of c_2 exists in $(0,2)\times(0,1)$.
On the face $x=1$: We get $p\in(0,2)$ and

$$N(p,1,y)=\frac{24576-3264p^2-2448p^4+437p^6}{6635520}=:c_3(p). \tag{3.10}$$

After computing $\partial c_3/\partial p=0$, we notice that c_3 has no critical point in $(0,2)$.
On the face $y=0$: We have $p\in(0,2)$, $x\in(0,1)$ and

$$\begin{aligned}N(p,x,0)=\frac{1}{6635520}\Big(&6144x(9-5x^2)+192p^2x(-144+15x+100x^2+12x^3)\\&-48p^4x(-73+20x+80x^2+24x^3)\\&+p^6(5-12x+60x^2+240x^3+144x^4)\Big)=:c_4(p,x).\end{aligned}$$

Calculations through Mathematica lead to

$$\begin{aligned}\frac{\partial c_4}{\partial x}=\frac{1}{6635520}\Big(&-61440x^2+6144(9-5x^2)+192p^2x(15+200x+36x^2)\\&-48p^4x(20+160x+72x^2)+192p^2(-144+15x+100x^2\\&+12x^3)-48p^4(-73+20x+80x^2+24x^3)+p^6(-12\\&+120x+720x^2+576x^3)\Big)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial c_4}{\partial p}=\frac{1}{6635520}\Big(&384px(-144+15x+100x^2+12x^3)-192p^3x(-73+20x\\&+80x^2+24x^3)+6p^5(5-12x+60x^2+240x^3+144x^4)\Big)\end{aligned}$$

having real solutions (p,x) as follows:

$$(2.04471,3.58205);\quad(-2.03352,-2.98401);\quad(1.47565,1.6649);$$

and

$$(0,0.774597);\quad(0,-0.774597);\quad(2.03352,-2.98401);$$

and

$$(2,0.307071);\quad(-2.04471,3.58205);\quad(-2,0.307071);$$

and

$$(2,-0.407071); \quad (-2,-0.407071); \quad (-1.47565, 1.6649).$$

Thus, no solution exist for the system of equations, $\partial c_4/\partial x = 0$ and $\partial c_4/\partial p = 0$ in $(0,2)\times(0,1)$. On the face $y = 1$: We have $p \in (0,2)$, $x \in (0,1)$ and

$$\begin{aligned} N(p,x,1) = \frac{1}{6635520}\Big(&5p^6 + (4-p^2)(12p^4x + 120p^4x^2 + 180p^2(4-p^2)x^2 \\ &+ 1536(4-p^2)x^3 + 240p^2(4-p^2)x^3 + 144p^2(4-p^2)x^4 \\ &+ 576(4-p^2)(5+x^2)(1-x^2) + 48(1-x^2)(p^3(5-6x \\ &- 12x^2) + 24px(1+2x)))\Big) =: c_5(p,x). \end{aligned}$$

After numerical computations done through Mathematica, we note that the real solutions (p, x) of the system of equations

$$\begin{aligned} \frac{\partial c_5}{\partial x} = \frac{(4-p^2)}{552960}\Big(&768x(-2+2x-x^2) + 24p^2x(21-6x+16x^2) + 96p(1+4x-3x^2 \\ &- 8x^3) - 8p^3(3+17x-9x^2-24x^3) + p^4(1-10x-60x^2-48x^3)\Big) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial c_5}{\partial p} = \frac{1}{1105920}\Big(&768x(1+2x-x^2-2x^3) - 40p^4(5-6x-17x^2+6x^3+12x^4) \\ &+ 96p^2(5-12x-29x^2+12x^3+24x^4) - 64p(120-111x^2 \\ &+ 44x^3 - 36x^4) + 32p^3(60+x-68x^2-8x^3-36x^4) \\ &+ p^5(5-12x+60x^2+240x^3+144x^4)\Big) \end{aligned}$$

are

$$(2,-17.9049); \quad (23.831, 0); \quad (15.9898, -0.0325369);$$

and

$$(2,-8.75); \quad (8.16565,-2.24092); \quad (2.0765, 1.0018);$$

and

$$(-1.84536,-0.877352); \quad (-2,-0.775417); \quad (2.09099, 0.254719);$$

and

$$(0,0); \quad (2.09338, 0); \quad (-1.92434, 0);$$

and

$$(-0.644804, 1.10265); \quad (-2, 0.725417).$$

Thus, we conclude that these two equations $\partial c_5/\partial x = 0$ and $\partial c_5/\partial p = 0$ also do not assume any solution in $(0,2)\times(0,1)$.

3. Next, we check the maximum values of $N(p,x,y)$ obtained on the edges of the cuboid V. From (3.9), we have $N(p,0,0) = p^6/1327104 =: t_1(p)$. It is easy to observe that $t_1'(p) = 0$ for $p = 0$ in the interval $[0,2]$. The maximum value of $t_1(p)$ is 0. Now, (3.9) reduces to

$N(p,0,1) = (96-24p^2+p^3)^2/1327104 =: t_2(p)$ at $y=1$. Since, $t_2'(p)<0$ in $[0,2]$, hence $p=0$ is the point of maxima. Thus

$$N(p,0,1) \leq \frac{1}{144}, \quad p \in [0,2].$$

Through computations, (3.9) shows that $N(0,0,y)$ attains its maxima at $y=1$. Hence

$$N(0,0,y) \leq \frac{1}{144}, \quad y \in [0,1].$$

Since, (3.10) is free from x, we have $N(p,1,1) = N(p,1,0) = (24576-3264p^2-2448p^4+437p^6)/6635520 =: t_3(p)$. Now, we observe that $t_3'(p)<0$ in $[0,2]$, consequently, $t_3(p)$ attains its maximum at $p=0$. Hence

$$N(p,1,1) = N(p,1,0) \leq 0.0037037, \quad p \in [0,2].$$

On substituting $p=0$ in (3.10), we get, $N(0,1,y)=1/270$. The equation (3.8) does not contain any variable such as p, x and y. Therefore, the maxima of $N(p,x,y)$ on the edges is given by

$$N(2,1,y) = N(2,0,y) = N(2,x,0) = N(2,x,1) = \frac{1}{20736}, \quad x,y \in [0,1].$$

Using (3.7), we obtain $N(0,x,1) = (15-12x^2+8x^3-3x^4)/2160 =: t_4(x)$. Upon calculations, we see that t_4 is a decreasing function in $[0,1]$ and its maximum value is achieved at $x=0$. Hence

$$N(0,x,1) \leq \frac{1}{144}, \quad x \in [0,1].$$

On again using (3.7), we get $N(0,x,0) = x(9-5x^2)/1080 =: t_5(x)$. On further calculations, we get $t_5'(x)=0$ when $x=\sqrt{3/5}$, the point of maxima. Thus

$$N(0,x,0) \leq 0.00430331, \quad x \in [0,1].$$

From the above all possible cases, we observe that optimal value of $|H_3(1)|$ is $1/144$. Therefore, the inequality (3.5) holds. Now, consider the following function $f_2 \in \mathcal{C}_e$, defined by

$$f_2(z) = \int_0^z \left(\exp\left(\int_0^y \frac{e^{t^3}-1}{t} dt \right) \right) dy = z + \frac{z^4}{12} + \frac{5z^7}{252} + \cdots,$$

with $f_2(0) = f_2'(0)-1 = 0$, plays the role of an extremal function for the bounds of $|H_3(1)|$ having values $a_3=a_5=0$ and $a_4=1/12$. □

3.3 Fourth Hankel Determinant for $\mathcal{C}_e$

In this part of the section, we derive the bound of $|H_4(1)|$ including finding the bounds of sixth and seventh coefficients for functions in the class $\mathcal{C}_e$. By selecting $q=4$ and $n=1$ in (1.4), the expression of $H_4(1)$ can be obtained for functions in the class $\mathcal{C}_e$, which is given as follows:

$$H_4(1) = a_7H_3(1) - a_6U_1 + a_5U_2 - a_4U_3. \tag{3.11}$$

Here

$$U_1 := a_6(a_3-a_2^2) + a_3(a_2a_5-a_3a_4) - a_4(a_5-a_2a_4), \tag{3.12}$$

$$U_2 := a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3), \tag{3.13}$$

and

$$U_3 := a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3). \tag{3.14}$$

We start by determining the bounds for U_1, U_2, and U_3.
By substituting the values of a_i's $(i = 2, 3, ..., 6)$ in (3.12) from (3.1)-(3.3), we obtain

$$\begin{aligned}132710400U_1 = {} & 487p_1^7 - 6304p_1^5p_2 + 11440p_1^3p_2^2 - 24960p_1p_2^3 + 5280p_1^4p_3 \\ & + 34560p_1p_3^2 + 19200p_1^2p_2p_3 - 53760p_2^2p_3 + 57600p_1p_2p_4 \\ & - 138240p_3p_4 + 184320p_2p_5 - 92160p_1^2p_5 + 8640p_1^3p_4,\end{aligned}$$

can also be viewed as the following, due to the triangle inequality,

$$\begin{aligned}132710400|U_1| \leq {} & |p_1^5(487p_1^2 - 6304p_2)| + |p_1p_2^2(11440p_1^2 - 24960p_2)| \\ & + |p_1p_3(5280p_1^3 + 34560p_3)| + |p_2p_3(19200p_1^2 - 53760p_2)| \\ & + |p_4(57600p_1p_2 - 138240p_3)| + |p_5(184320p_2 - 92160p_1^2)| \\ & + |8640p_1^3p_4|.\end{aligned}$$

Using Lemma 2.2, we arrive at

$$|U_1| \leq \frac{4121}{345600} \approx 0.0119242.$$

We replace the values of a_i's $(i = 2, 3, ..., 6)$ from (3.1)-(3.4) in (3.13) and proceed on the same lines to obtain the bound of U_2

$$\begin{aligned}1592524800U_2 = {} & 463p_1^8 - 2732p_1^6p_2 - 23472p_1^4p_2^2 - 14400p_1^2p_2^3 + 14592p_1^5p_3 \\ & - 108288p_1^2p_3^2 + 92928p_1^3p_2p_3 - 138240p_1p_2^2p_3 + 1105920p_3p_5 \\ & - 25344p_1^4p_4 + 276480p_2^2p_4 - 995328p_4^2 + 373248p_1^2p_2p_4 \\ & - 276480p_1p_2p_5 + 221184p_1p_3p_4 - 161280p_1^3p_5 - 322560p_2p_3^2,\end{aligned}$$

by implementing the triangle inequality,

$$\begin{aligned}1592524800|U_2| \leq {} & |p_1^6(463p_1^2 - 2732p_2)| + |p_1^2p_2^2(-23472p_1^2 - 14400p_2)| \\ & + |p_1^2p_3(14592p_1^3 - 108288p_3)| + |161280p_1^3p_5| \\ & + |p_1^2p_4(373248p_2 - 25344p_1^2)| + |p_4(276480p_2^2 - 995328p_4)| \\ & + |322560p_2p_3^2| + |p_1p_2p_3(92928p_1^2 - 138240p_2)| \\ & + |221184p_1p_3p_4| + |p_5(1105920p_3 - 276480p_1p_2)|.\end{aligned}$$

By applying Lemma 2.2, we have

$$|U_2| \leq \frac{24947200 + 866304\sqrt{\frac{282}{61}}}{1592524800} \approx 0.0168348.$$

Again, substitute the values of a_i's $(i = 2, 3, ..., 6)$ from (3.1)-(3.4) in (3.14) and proceed to calculate the bound of U_3 in the same manner.

$$38220595200U_3 = 11424p_1^8 - 128256p_1^6p_2 + 10812p_1^7p_2 - 503p_1^9 + 69120p_1^4p_2^2$$

$$
\begin{aligned}
&+552960p_1^2p_2^3-42192p_1^5p_2^2-181440p_1^3p_2^3+206208p_1^4p_2p_3\\
&-11664p_1^6p_3+1889280p_1^3p_2p_3-1658880p_1p_2^2p_3-2211840p_1^2p_3^2\\
&-2211840p_2p_3^2+283392p_1^3p_3^2-967680p_1p_2p_3^2+3317760p_1p_3p_4\\
&-483840p_1^4p_4+1271808p_1^3p_2p_4-117504p_1^5p_4+1658880p_1p_2^2p_4\\
&-5971968p_1p_4^2+6635520p_2p_3p_4-331776p_1^2p_3p_4+26542080p_3p_5\\
&-6635520p_1p_2p_5+244224p_1^5p_3-794880p_1^2p_2^2p_3-2764800p_3^3\\
&-829440p_1^2p_2p_4-3870720p_1^3p_5,
\end{aligned}
$$

can be visualized as the following with the help of the triangle inequality,

$$
\begin{aligned}
38220595200|U_3| \leq\ & |p_1^6(11424p_1^2-128256p_2)|+|p_1^7(10812p_2-503p_1^2)|\\
&+|p_1^2p_2^2(69120p_1^2+552960p_2)|+|p_1^3p_2^2(42192p_1^2+181440p_2)|\\
&+|p_1^4p_3(206208p_2-11664p_1^2)|+|p_1p_2p_3(1889280p_1^2-1658880p_2)|\\
&+|p_3^2(2211840p_1^2+2211840p_2)|+|p_1p_3^2(283392p_1^2-967680p_2)|\\
&+|p_1p_4(3317760p_3-483840p_1^3)|+|p_1^3p_4(1271808p_2-117504p_1^2)|\\
&+|p_1p_4(1658880p_2^2-5971968p_4)|+|p_3p_4(6635520p_2-331776p_1^2)|\\
&+|p_5(26542080p_3-6635520p_1p_2)|+|244224p_1^5p_3-794880p_1^2p_2^2p_3\\
&-2764800p_3^3-829440p_1^2p_2p_4-3870720p_1^3p_5|.
\end{aligned}
$$

By applying Lemma 2.2, we get

$$
|U_3| \leq \frac{560108544+106168320\sqrt{\frac{3}{41}}}{38220595200} \approx 0.015406.
$$

Remark 3. The bounds of U_1, U_2 and U_3, based on the above calculations, are 0.0119242, 0.0168348, and 0.015406, respectively.

The bounds of a_i's ($i=2,3,4,5$) for functions in the class $\mathcal{C}_e$ are obtained in [27], presented below in the following remark:

Remark 4. For $f \in \mathcal{C}_e$, $|a_2| \leq 1/2$, $|a_3| \leq 1/4$, $|a_4| \leq 17/144$ and $|a_5| \leq 5/72$. The first three bounds are sharp.

Next, we calculate the bounds of the sixth and seventh coefficient of functions belonging to the class $\mathcal{C}_e$ to establish our main result along the lines of Theorem 2.2.

Theorem 3.2. *Let $f \in \mathcal{C}_e$. Then $|a_6| \leq 587/10800 \approx 0.0543519$ and $|a_7| \leq 0.0343723$.*

Proof. A suitable rearrangement of the terms given in (3.3) provides us

$$
345600a_6 = 5760p_5-480p_2p_3+720p_1p_4-480p_1p_2^2-17p_1^5+220p_1^3p_2-480p_1^2p_3.
$$

Further, through the triangle inequality, it can be viewed as

$$
\begin{aligned}
345600|a_6| \leq\ & |5760p_5-480p_2p_3|+|p_1(720p_4-480p_2^2)|+|17p_1^5|\\
&+|p_1^2(220p_1p_2-480p_3)|.
\end{aligned}
$$

Using Lemma 2.2, we arrive at

$$|a_6| \leq \frac{587}{10800} \approx 0.0543519.$$

Similarly, considering (3.4), we have

$$\begin{aligned}58060800a_7 = {} & 881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 106560p_1p_2p_3 + 29040p_1^3p_3 \\ & - 57600p_3^2 + 69120p_1p_5 - 56160p_1^2p_4 - 86400p_2p_4 - 14400p_2^3.\end{aligned}$$

It can also be seen as with the aid of the triangle inequality,

$$\begin{aligned}58060800|a_7| \leq {} & |p_1^4(881p_1^2 - 13260p_2)| + |p_1p_2(48240p_1p_2 - 106560p_3)| \\ & + |p_3(29040p_1^3 - 57600p_3)| + |p_1(69120p_5 - 56160p_1p_4)| \\ & + |p_2(86400p_4 + 14400p_2^2)|. \qquad (3.15)\end{aligned}$$

Lemma 2.2 takes us at

$$|a_7| \leq \frac{2014080 + 921600\sqrt{\frac{15}{119}}}{58060800} \approx 0.0403246.$$

□

We obtain the following result by omitting the proof as it directly follows from Theorem 3.1, Remark 3, Remark 4, Theorem 3.2 and (3.11).

Theorem 3.3. *Let $f \in \mathcal{C}_e$, then*

$$|H_4(1)| \leq 0.00101775.$$

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