



Certain coefficient problems of \mathcal{S}_e^* and \mathcal{C}_e

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Abstract. In this current study, we consider the classes \mathcal{S}_e^* and \mathcal{C}_e to obtain sharp bounds for the third Hankel determinant for functions within these classes. Additionally, we provide estimates for the sixth and seventh coefficients while establishing the fourth-order Hankel determinant as well.

Keywords. Coefficient estimate, exponential function, Starlike function, sharp third Hankel

1 Introduction

Consider the set of normalized analytic functions, denoted as \mathcal{A} , which are defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. These functions are represented by the expansion:

$$f(z) = z + a_2z^2 + a_3z^3 + \dots \tag{1.1}$$

Within this class, we define a subclass \mathcal{S} , which comprises univalent functions. Also, assume a class of analytic functions defined on the unit disk \mathbb{D} , which possess a positive real part. This class is represented as \mathcal{P} whose elements are of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. We use the notation $h_1 \prec h_2$ to indicate that function h_1 is subordinate to h_2 , which implies the existence of a Schwarz function w with the properties $w(0) = 0$ and $|w(z)| \leq |z|$, such that $h_1(z) = h_2(w(z))$.

The Bieberbach’s conjecture, as discussed in [6, Page no. 17] has made a substantial contribution to the advancement of geometric function theory and the emergence of coefficient-related challenges. In the wake of this, numerous additional subclasses of \mathcal{S} , encompassing starlike functions denoted as \mathcal{S}^* and convex functions denoted as \mathcal{C} , have been introduced. Notably, in 1992, Ma and Minda [18] introduced the following two classes:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \tag{1.2}$$

and

$$\mathcal{C}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}, \tag{1.3}$$

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which unifies various subclasses of \mathcal{S}^* and \mathcal{C} , respectively. Here φ is an analytic univalent function satisfying the conditions $\varphi(z) > 0$, $\varphi(\mathbb{D})$ is symmetric about the real axis and starlike with respect to $\varphi(0) = 1$ with $\varphi'(0) > 0$. Some of these classes are outlined in Table 1 for ready reference.

The notion of Hankel determinants was introduced in [21]. Remarkably, this concept continues to captivate the attention of numerous researchers to this very day. Encompassing a broad spectrum of applications and implications, the q th Hankel determinants $H_q(n)$ of analytic functions belonging to the class \mathcal{A} , as represented in (1.1), have been defined under the premise that a_1 takes the value 1. For $n, q \in \mathbb{N}$, this definition unfolds as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.4}$$

The specific expression for the third-order Hankel determinant, denoted as $H_3(1)$, is obtained by substituting $q = 3$ and $n = 1$ into (1.4). This determinant can be precisely defined as:

$$H_3(1) = 2a_2a_3a_4 - a_3^3 - a_4^2 - a_2^2a_5 + a_3a_5. \tag{1.5}$$

Over the time, several authors established sharp bound of second-order Hankel determinants, see [1, 4, 10, 11]. However, the task of computing bounds for third-order Hankel determinants, proves to be considerably more intricate, can be observed from [15, 28, 27]. In the context of the class \mathcal{S}^* , Kwon et al. [15] established the inequality $|H_3(1)| \leq 8/9$, which has recently been best improved to the bound of $4/9$ in [2, 9] independently. Furthermore, Lecko et al. [16] successfully derived the bound $|H_3(1)| \leq 1/9$, a result that stands as sharp for functions in $\mathcal{S}^*(1/2)$. For a more comprehensive exploration of Hankel determinants, interested readers can turn to works such as [3, 9, 16, 26].

Below, we enlist specific subclasses of \mathcal{S}^* and \mathcal{C} , resulting from diverse selections of $\varphi(z)$. In a similar manner, Mendiratta et al. [19] introduced and analyzed the classes \mathcal{S}_e^* and \mathcal{C}_e by selecting $\varphi(z) = e^z$ in (1.2) and (1.3), respectively. These classes are defined as follows:

$$\mathcal{S}_e^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z \right\} \quad \text{and} \quad \mathcal{C}_e = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec e^z \right\}.$$

Table 1: List of subclasses of \mathcal{S}^* and \mathcal{C}

$\mathcal{S}^*(\varphi)$	$\mathcal{C}(\varphi)$	$\varphi(z)$	Author(s)	Reference
$\mathcal{S}^*[C, D]$	$\mathcal{C}[C, D]$	$(1 + Cz)/(1 + Dz)$	Janowski	[7]
\mathcal{S}_{SG}^*	\mathcal{C}_{SG}	$2/(1 + e^{-z})$	Goel and Kumar	[5]
\mathcal{S}_φ^*	\mathcal{C}_φ	$1 + ze^z$	Kumar and Kamaljeet	[13]
\mathcal{S}_q^*	\mathcal{C}_q	$z + \sqrt{1 + z^2}$	Raina and Sokół	[22]
\mathcal{S}_L^*	\mathcal{C}_L	$\sqrt{1 + z}$	Sokół and Stankiewicz	[24]

Numerous studies have addressed radius problems [19] and investigated implications of first and higher-order differential subordination [20, 25] for the subclasses associated with the exponential function. Zaprawa [27] established bounds for the third Hankel determinants, yielding values of 0.385 and 0.021 for the classes \mathcal{S}_e^* and \mathcal{C}_e , respectively, although the results were not sharp.

In our present investigation, we contribute by establishing sharp bounds for $|H_3(1)|$ for functions in the classes \mathcal{S}_e^* and \mathcal{C}_e . Additionally, in the upcoming sections, we will provide estimations for the bounds of the sixth and seventh coefficients for the functions belonging to the classes, \mathcal{S}_e^* and \mathcal{C}_e and also evaluate the fourth Hankel determinant.

2 Hankel Determinants for \mathcal{S}_e^*

2.1 Preliminaries

In this part of the section, we derive the expressions of a_i ($i = 2, 3, \dots, 7$) in terms of Carathéodory coefficients. For this, let $f \in \mathcal{S}_e^*$, then there exists a Schwarz function $w(z)$ such that

$$\frac{zf'(z)}{f(z)} = e^{w(z)}. \tag{2.1}$$

Suppose that $p(z) = 1 + p_1z + p_2z^2 + \dots \in \mathcal{P}$ and consider $w(z) = (p(z) - 1)/(p(z) + 1)$. Further, by substituting the expansions of $w(z)$, $p(z)$ and $f(z)$ in (2.1) and then comparing the coefficients, we obtain the expressions of a_i ($i = 2, 3, \dots, 7$) in terms of p_j ($j = 1, 2, \dots, 5$), given as follows:

$$a_2 = \frac{1}{2}p_1, \quad a_3 = \frac{1}{16}\left(4p_2 + p_1^2\right), \quad a_4 = \frac{1}{288}\left(-p_1^3 + 12p_1p_2 + 48p_3\right), \tag{2.2}$$

$$a_5 = \frac{1}{1152}\left(p_1^4 - 12p_1^2p_2 + 24p_1p_3 + 144p_4\right), \tag{2.3}$$

$$a_6 = \frac{1}{57600}\left(-17p_1^5 + 220p_1^3p_2 - 480p_1p_2^2 - 480p_1^2p_3 - 480p_2p_3 + 720p_1p_4 + 5760p_5\right), \tag{2.4}$$

and

$$a_7 = \frac{1}{8294400}\left(881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 14400p_2^3 + 29040p_1^3p_3 - 106560p_1p_2p_3 - 57600p_3^2 - 56160p_1^2p_4 - 86400p_2p_4 + 69120p_1p_5\right). \tag{2.5}$$

The formula for p_j ($j = 2, 3, 4$), which is included in Lemma 2.1 below, plays a vital role in establishing the sharp bound for Hankel determinants and forms the foundation for our main results.

Lemma 2.1. [17, 14] *Let $p \in \mathcal{P}$ has the form $1 + \sum_{n=1}^\infty p_n z^n$. Then*

$$2p_2 = p_1^2 + \gamma(4 - p_1^2), \tag{2.6}$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta, \tag{2.7}$$

and

$$8p_4 = p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma) - 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta + \bar{\gamma}\eta^2 - (1 - |\eta|^2)\rho), \tag{2.8}$$

for some γ, η and ρ such that $|\gamma| \leq 1, |\eta| \leq 1$ and $|\rho| \leq 1$.

2.2 Sharp Third Hankel Determinant for \mathcal{S}_e^*

In this subsection, we present the sharp bound for $|H_3(1)|$ for functions belonging to the class \mathcal{S}_e^* .

Theorem 2.1. *Let $f \in \mathcal{S}_e^*$, then*

$$|H_3(1)| \leq \frac{1}{9}. \quad (2.9)$$

This result is sharp.

Proof. Since the class \mathcal{P} is invariant under rotation, the value of p_1 belongs to the interval $[0, 2]$. Let $p := p_1$ and then substitute the values of a_i ($i = 2, 3, 4, 5$) in (1.5) from (2.2) and (2.3). We get

$$H_3(1) = \frac{1}{331776} \left(-211p^6 + 420p^4p_2 - 1872p^2p_2^2 - 5184p_2^3 + 2544p^3p_3 \right. \\ \left. + 10944pp_2p_3 - 9216p_3^2 - 7776p^2p_4 + 10368p_2p_4 \right).$$

After simplifying the calculations through (2.6)-(2.8), we obtain

$$H_3(1) = \frac{1}{331776} \left(\beta_1(p, \gamma) + \beta_2(p, \gamma)\eta + \beta_3(p, \gamma)\eta^2 + \phi(p, \gamma, \eta)\rho \right),$$

for $\gamma, \eta, \rho \in \mathbb{D}$, where

$$\beta_1(p, \gamma) := -13p^6 - 36\gamma^2p^2(4-p^2)^2 - 360\gamma^3p^2(4-p^2)^2 + 72\gamma^4p^2(4-p^2)^2 \\ + 78\gamma p^4(4-p^2) + 120p^4\gamma^2(4-p^2) - 324p^4\gamma^3(4-p^2) \\ - 1296\gamma^2p^2(4-p^2),$$

$$\beta_2(p, \gamma) := 24(1 - |\gamma|^2)(4 - p^2)(17p^3 + 54\gamma p^3 + 30p\gamma(4 - p^2) - 12p\gamma^2(4 - p^2)),$$

$$\beta_3(p, \gamma) := 144(1 - |\gamma|^2)(4 - p^2)(-16(4 - p^2) - 2|\gamma|^2(4 - p^2) + 9p^2\bar{\gamma}),$$

$$\phi(p, \gamma, \eta) := 1296(1 - |\gamma|^2)(4 - p^2)(1 - |\eta|^2)(2(4 - p^2)\gamma - p^2).$$

By choosing $x = |\gamma|$, $y = |\eta|$ and utilizing the fact that $|\rho| \leq 1$, the above expression reduces to the following:

$$|H_3(1)| \leq \frac{1}{331776} \left(|\beta_1(p, \gamma)| + |\beta_2(p, \gamma)|y + |\beta_3(p, \gamma)|y^2 + |\phi(p, \gamma, \eta)| \right) \leq M(p, x, y),$$

where

$$M(p, x, y) = \frac{1}{331776} \left(M_1(p, x) + M_2(p, x)y + M_3(p, x)y^2 + M_4(p, x)(1 - y^2) \right), \quad (2.10)$$

with

$$M_1(p, x) := 13p^6 + 36x^2p^2(4 - p^2)^2 + 360x^3p^2(4 - p^2)^2 + 72x^4p^2(4 - p^2)^2 \\ + 78xp^4(4 - p^2) + 120p^4x^2(4 - p^2) + 324p^4x^3(4 - p^2) + 1296x^2p^2(4 - p^2),$$

$$M_2(p, x) := 24(1 - x^2)(4 - p^2)(17p^3 + 54xp^3 + 30px(4 - p^2) + 12px^2(4 - p^2)),$$

$$M_3(p, x) := 144(1 - x^2)(4 - p^2)(16(4 - p^2) + 2x^2(4 - p^2) + 9p^2x),$$

$$M_4(p, x) := 1296(1 - x^2)(4 - p^2)(2x(4 - p^2) + p^2).$$

To maximize $M(p, x, y)$ where $p \in [0, 2]$ and $x, y \in [0, 1]$, we consider the closed cuboid $U : [0, 2] \times [0, 1] \times [0, 1]$ with six faces and twelve edges. The maximum will be attained at a point that is either on the edges, on the faces, or in the interior of the cuboid. Let's first consider all the interior points of the cuboid U .

1. Let (p, x, y) be an interior point in U , at which M attains its maximum. Clearly $(p, x, y) \in (0, 2) \times (0, 1) \times (0, 1)$ and at (p, x, y) , we have $\partial M/\partial p = \partial M/\partial x = \partial M/\partial y = 0$. Upon expanding, we get

$$\begin{aligned} \frac{\partial M}{\partial p} = \frac{(1 - x^2)}{13824} & \left(96x(5 + 2x)y + 5p^4(-17 - 24x + 12x^2)y - 12p^2(-17 + 6x + 24x^2)y \right. \\ & + 24p^3(-9 + 25y^2 + 2x^2y^2 - 9x(-2 + 3y^2)) \\ & \left. - 48p(-9 + 41y^2 + 4x^2y^2 - 9x(-4 + 5y^2)) \right) = 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial M}{\partial x} = \frac{(4 - p^2)}{6912} & \left(-12p(-5 - 4x + 15x^2 + 8x^3)y + p^3(12 - 29x - 36x^2 + 24x^3)y \right. \\ & - 24(14xy^2 + 4x^3y^2 + 9(-1 + y^2) - 27x^2(-1 + y^2)) + 3p^2(8x^3y^2 \\ & \left. + x^2(54 - 81y^2) + 9(-2 + 3y^2) + 2x(-9 + 23y^2)) \right) = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial M}{\partial y} = \frac{(4 - p^2)(1 - x^2)}{13824} & \left(24px(5 + 2x) + p^3(17 + 24x - 12x^2) + 96(8 - 9x + x^2)y \right. \\ & \left. - 12p^2(25 - 27x + 2x^2)y \right) = 0. \end{aligned}$$

Upon solving the above equations through the software Mathematica, we get the following critical points (p, x, y) , given by

$$(-2.09652, 64.2765, 2.81098); \quad (0, 0.57735, 0); \quad (2.09652, 64.2765, -2.81098);$$

and

$$(-1.11368, 9.14102, 2.7143); \quad (0, -0.57735, 0); \quad (1.11368, 9.14102, -2.7143).$$

However, none of them belong to the interior of cuboid U . Therefore, we conclude that M does not attain its maximum value inside U .

Next, we consider the faces $p = 0$, $p = 2$, $x = 0$, $x = 1$, $y = 0$ and $y = 1$ of U for the point of maxima of M .

2. On the face $p = 0$: We have $x, y \in (0, 1)$ and

$$M(0, x, y) = \frac{(1 - x^2)(8y^2 + x^2y^2 + 9x(1 - y^2))}{72} =: s_1(x, y). \tag{2.11}$$

Since

$$\frac{\partial s_1}{\partial y} = \frac{(1-x^2)(x-1)(x-8)y}{36}.$$

Now, $\partial s_1/\partial y = 0$ implies $x = \pm 1$ or $x = 8$ or $y = 0$. Clearly, all these points are not in $(0, 1)$. Thus, M does not attain its maximum on the face $p = 0$.

On the face $p = 2$: We have

$$M(2, x, y) := \frac{13}{5184}, \quad x, y \in (0, 1). \quad (2.12)$$

On the face $x = 0$: We have $p \in (0, 2)$, $y \in (0, 1)$ and

$$M(p, 0, y) = \frac{13p^6 + (4-p^2)(408p^3y + 2304y^2(4-p^2) + 1296p^2(1-y^2))}{331776} =: s_2(p, y). \quad (2.13)$$

To determine the points of maxima, we solve $\partial s_2/\partial p = 0$ and $\partial s_2/\partial y = 0$. After solving $\partial s_2/\partial y = 0$, we get

$$y = \frac{17p^3}{12(25p^2 - 64)} =: y_p. \quad (2.14)$$

In order to have $y_p \in (0, 1)$ for the given range of y , $p > \approx 1.68218 =: p_0$ is required. Based on calculations, $\partial s_2/\partial p = 0$ gives

$$1728p - 864p^3 + 13p^5 + 816p^2y - 340p^4y - 7872py^2 + 2400p^3y^2 = 0. \quad (2.15)$$

After substituting (2.14) into (2.15), we have

$$21233664p - 27205632p^3 + 11472192p^5 - 1613016p^7 + 2700p^9 = 0. \quad (2.16)$$

A numerical calculation suggests that $p \approx 1.35596 \in (0, 2)$ is the solution of (2.16) i.e., $p < p_0$, which contradicts the range of p obtained by solving (2.14). Therefore, we conclude that s_2 does not have any critical point in $(0, 2) \times (0, 1)$.

On the face $x = 1$: We have $p \in (0, 2)$ and

$$M(p, 1, y) = \frac{12672p^2 - 2952p^4 - 41p^6}{331776} =: s_3(p). \quad (2.17)$$

Upon solving $\partial s_3/\partial p = 0$, we get the solution $p \approx 1.43461 =: p_0$. Since, $p_0 \in (0, 2)$, it is a critical point at which s_3 attains its maximum value, given by ≈ 0.0398426 .

On the face $y = 0$: We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$M(p, x, 0) = \frac{1}{331776} \left(41472x(1-x^2) + 576p^2(9-36x+x^2+46x^3+2x^4) - 24p^4(54-121x-8x^2+174x^3+24x^4) + p^6(13-78x-84x^2+36x^3+72x^4) \right) =: s_4(p, x).$$

After undergoing further computations such as

$$\frac{\partial s_4}{\partial x} = \frac{1}{331776} \left(41472(1-x^2) - 82944x^2 + 576p^2(-36+2x+138x^2+8x^3) \right)$$

$$+ 24p^4(121 + 16x - 522x^2 - 96x^3) - p^6(78 + 168x - 108x^2 - 288x^3) \Big)$$

and

$$\frac{\partial s_4}{\partial p} = \frac{1}{331776} \left(6p^5(13 - 78x - 84x^2 + 36x^3 + 72x^4) - 96p^3(54 - 121x - 8x^2 + 174x^3 + 24x^4) + 1152p(9 - 36x + x^2 + 46x^3 + 2x^4) \right),$$

we observe from Mathematica that all the real solutions (p, x) of this system of equations are as follows:

$$(4.29674, 2.34253); \quad (-6.69897, -0.446594); \quad (-6.80446, -0.196205);$$

and

$$(6.69897, -0.446594); \quad (-4.29674, 2.34253); \quad (-2, -1.00369);$$

and

$$(2, -1.00369); \quad (0, 0.57735); \quad (1.57971, -0.993637);$$

and

$$(6.80446, -0.196205); \quad (0, -0.57735); \quad (-1.57971, -0.993637); \quad (-1.37155, 0.946054).$$

Thus, none of the above critical point lies in $(0, 2) \times (0, 1)$. Therefore, M does not attain its maximum on this face.

On the face $y = 1$: We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$\begin{aligned} M(p, x, 1) = \frac{1}{331776} \Big(& 2304px(5 + 2x - 5x^2 - 2x^3) - 4608(-8 + 7x^2 + x^4) \\ & + 576p^2(-32 + 9x + 38x^2 + x^3 + 6x^4) - 24p^5(17 + 24x \\ & - 29x^2 - 24x^3 + 12x^4) + 96p^3(17 - 6x - 41x^2 + 6x^3 \\ & + 24x^4) - 24p^4(-96 + 41x + 130x^2 + 12x^3 + 36x^4) \\ & + p^6(13 - 78x - 84x^2 + 36x^3 + 72x^4) \Big) =: s_5(p, x). \end{aligned}$$

Upon numerical computations through Mathematica, we have a system of equations

$$\begin{aligned} \frac{\partial s_5}{\partial x} = \frac{1}{331776} \Big(& 2304p((2 - 10x - 6x^2)x + (5 + 2x - 5x^2 - 2x^3)) - 4608(14x + 4x^3) \\ & + 576p^2(9 + 76x + 3x^2 + 24x^3) - 24p^5(24 - 58x - 72x^2 + 48x^3) \\ & + 96p^3(-6 - 82x + 18x^2 + 96x^3) - 24p^4(41 + 260x + 36x^2 + 144x^3) \\ & + p^6(-78 - 168x + 108x^2 + 288x^3) \Big) \end{aligned}$$

and

$$\frac{\partial s_5}{\partial p} = \frac{1}{331776} \left(2304x(5 + 2x - 5x^2 - 2x^3) + 1152p(-32 + 9x + 38x^2 + x^3 + 6x^4) \right)$$

$$\begin{aligned}
& -120p^4(17 + 24x - 29x^2 - 24x^3 + 12x^4) + 288p^2(17 - 6x - 41x^2 \\
& + 6x^3 + 24x^4) - 96p^3(-96 + 41x + 130x^2 + 12x^3 + 36x^4) \\
& + 6p^5(13 - 78x - 84x^2 + 36x^3 + 72x^4)
\end{aligned}$$

having all the real solutions (p, x) given by:

$$(2.63476, 8.33714); \quad (-1.3852, -8.27637); \quad (2, 1.53448);$$

and

$$(2, -1.00729); \quad (-2.11632, -0.880438); \quad (-2, -0.606518);$$

and

$$(2, -0.166303); \quad (-3.02448, 3.66443); \quad (-1.84344, 0.190567);$$

and

$$(-2, 0.613808); \quad (1.76914, -0.679457); \quad (2, -0.997809);$$

and

$$(-1.08065, 0.86173); \quad (1.36894, -0.996513); \quad (0, 0).$$

Thus, none of the above critical points lies inside the face $y = 1$.

3. We next examine the maxima attained by $M(p, x, y)$ on the edges of the cuboid U . From (2.13), we have $M(p, 0, 0) = (5184p^2 - 1296p^4 + 13p^6)/331776 =: r_1(p)$. It is easy to observe that $r_1'(p) = 0$ whenever $p = 0$ and $p = 1.4367 \in [0, 2]$ as its points of minima and maxima, respectively. Hence,

$$M(p, 0, 0) \leq 0.0159535, \quad p \in [0, 2].$$

Now considering (2.13) at $y = 1$, we get $M(p, 0, 1) = (36864 - 18432p^2 + 1632p^3 + 2304p^4 - 408p^5 + 13p^6)/331776 =: r_2(p)$. It is easy to observe that $r_2'(p) < 0$ in $[0, 2]$ and hence $p = 0$ serves as the point of maxima. So,

$$M(p, 0, 1) \leq \frac{1}{9}, \quad p \in [0, 2].$$

Through computations, (2.13) shows that $M(0, 0, y)$ attains its maxima at $y = 1$. This implies that

$$M(0, 0, y) \leq \frac{1}{9}, \quad y \in [0, 1].$$

Since, (2.17) does not involve x , we have $M(p, 1, 1) = M(p, 1, 0) = (12672p^2 - 2952p^4 - 41p^6)/331776 =: r_3(p)$. Now, $r_3'(p) = 4224p - 1968p^3 - 41p^5 = 0$ when $p = 0$ and $p = 1.43461$ in the interval $[0, 2]$ as points of minima and maxima, respectively. Hence

$$M(p, 1, 1) = M(p, 1, 0) \leq 0.0398426, \quad p \in [0, 2].$$

After considering $p = 0$ in (2.17), we get, $M(0, 1, y) = 0$. From (2.12), we see that M is a constant function. Therefore, on the edges, the maximum value of $M(p, x, y)$ is

$$M(2, 1, y) = M(2, 0, y) = M(2, x, 0) = M(2, x, 1) = \frac{13}{5184}, \quad x, y \in [0, 1].$$

Using (2.11), we obtain $M(0, x, 1) = (8 - 7x^2 - x^4)/72 =: r_4(x)$. A computation shows that $r_4(x)$ is a decreasing function in $[0, 1]$ and attains its maxima at $x = 0$. Hence

$$M(0, x, 1) \leq \frac{1}{9}, \quad x \in [0, 1].$$

Again utilizing (2.11), we get $M(0, x, 0) = x(1 - x^2)/8 =: r_5(x)$. Further, $r_5'(x) = 0$, then $x = 1/\sqrt{3} =: \delta_0$. Clearly, δ_0 is a point of maxima for $r_5(x)$. Thus

$$M(0, x, 0) \leq 0.0481125, \quad x \in [0, 1].$$

From the above all possible cases, we observe that optimal value of $|H_3(1)|$ is $1/9$. Therefore, the inequality (2.9) holds. Now, consider the following function $f_1 \in \mathcal{S}_e^*$, defined by

$$f_1(z) = z \exp\left(\int_0^z \frac{e^{t^3} - 1}{t} dt\right) = z + \frac{z^4}{3} + \frac{5z^7}{36} + \dots,$$

with $f_1(0) = 0$ and $f_1'(0) = 1$, acts as an extremal function for the bound of $|H_3(1)|$ for $a_2 = a_3 = a_5 = 0$ and $a_4 = 1/3$. □

2.3 Fourth Hankel Determinant for \mathcal{S}_e^*

In this subsection, we derive the bounds of sixth and seventh coefficients and consequently $|H_4(1)|$ for functions belonging to the class \mathcal{S}_e^* . We need the following lemma for deriving our results.

Lemma 2.2. [12, 23] *Let $p = 1 + \sum_{n=1}^\infty p_n z^n \in \mathcal{P}$. Then*

$$|p_n| \leq 2, \quad n \geq 1,$$

$$|p_{n+k} - \nu p_n p_k| \leq \begin{cases} 2, & 0 \leq \nu \leq 1; \\ 2|2\nu - 1|, & \text{otherwise,} \end{cases}$$

and

$$|p_1^3 - \nu p_3| \leq \begin{cases} 2|\nu - 4|, & \nu \leq 4/3; \\ 2\nu \sqrt{\frac{\nu}{\nu - 1}}, & 4/3 < \nu. \end{cases}$$

We derive the expression of the fourth Hankel determinant when $q = 4$ and $n = 1$ are put into (1.4) as follows :

$$H_4(1) = a_7 H_3(1) - a_6 T_1 + a_5 T_2 - a_4 T_3, \tag{2.18}$$

where

$$T_1 := a_6(a_3 - a_2^2) + a_3(a_2 a_5 - a_3 a_4) - a_4(a_5 - a_2 a_4), \tag{2.19}$$

$$T_2 := a_3(a_3 a_5 - a_4^2) - a_5(a_5 - a_2 a_4) + a_6(a_4 - a_2 a_3), \tag{2.20}$$

and

$$T_3 := a_4(a_3 a_5 - a_4^2) - a_5(a_2 a_5 - a_3 a_4) + a_6(a_4 - a_2 a_3). \tag{2.21}$$

Now, using Lemma 2.2, we first determine the bounds of T_1, T_2 , and T_3 .

By substituting the values of a_i 's ($i = 2, 3, \dots, 6$) in (2.19) using (2.2)-(2.4), we obtain

$$\begin{aligned} 5529600T_1 &= 581p_1^7 + 5040p_1^4 p_3 + 25920p_1^2 p_2 p_3 - 7068p_1^5 p_2 + 11040p_1^3 p_4 \\ &\quad - 115200p_3 p_4 + 7920p_1^3 p_2^2 - 69120p_2^2 p_3 + 74880p_1 p_2 p_4 - 25920p_1 p_2^3 \\ &\quad + 57600p_1 p_3^2 + 138240p_2 p_5 - 103680p_1^2 p_5 \end{aligned}$$

or

$$\begin{aligned} 5529600|T_1| \leq & |p_1^4(581p_1^3 + 5040p_3)| + |p_1^2p_2(25920p_3 - 7068p_1^3)| + |57600p_1p_2^2| \\ & + |p_2^2(7920p_1^3 - 69120p_3)| + |p_1p_2(74880p_4 - 25920p_2^2)| \\ & + |p_4(11040p_1^3 - 115200p_3)| + |p_5(138240p_2 - 103680p_1^2)|. \end{aligned}$$

Using Lemma 2.2 and the triangle inequality, we arrive at

$$\begin{aligned} |T_1| \leq & \frac{1848448 + 4976640\sqrt{\frac{15}{1571}} + 1843200\sqrt{\frac{15}{217}} + 442368\sqrt{\frac{30}{17}}}{5529600} \\ \approx & 0.616137. \end{aligned}$$

Now, we calculate the bound of T_2 in the similar way by substituting the values of a_i 's ($i = 2, 3, \dots, 6$) in (2.20) from (2.2)-(2.4), as follows:

$$\begin{aligned} 22118400T_2 = & 235p_1^8 + 8712p_1^5p_3 + 37440p_1^3p_2p_3 - 1156p_1^6p_2 - 63360p_1p_2^2p_3 \\ & - 14640p_1^4p_2^2 + 161280p_1p_3p_4 - 8400p_1^4p_4 + 368640p_3p_5 \\ & - 76800p_1^3p_5 - 8640p_1^2p_2^3 + 172800p_2^2p_4 - 345600p_4^2 - 40320p_1^2p_3^2 \\ & - 184320p_2p_3^2 + 178560p_1^2p_2p_4 - 184320p_1p_2p_5 \end{aligned}$$

or

$$\begin{aligned} 22118400|T_2| \leq & |p_1^5(235p_1^3 + 8712p_3)| + |p_1^3p_2(37440p_3 - 1156p_1^3)| + |8640p_1^2p_2^2| \\ & + |p_1p_2^2(63360p_3 + 14640p_1^3)| + |p_1p_4(161280p_3 - 8400p_1^3)| \\ & + |p_5(368640p_3 - 76800p_1^3)| + |p_4(172800p_2^2 - 345600p_4)| \\ & + |p_3^2(184320p_2 + 40320p_1^2)| + |p_1p_2(178560p_1p_4 - 184320p_5)|. \end{aligned}$$

Lemma 2.2 and the triangle inequality lead us to

$$\begin{aligned} |T_2| \leq & \frac{7821568 + 14376960\sqrt{\frac{65}{9071}} + 2949120\sqrt{\frac{6}{19}} + 737280\sqrt{\frac{42}{13}}}{22118400} \\ \approx & 0.543487. \end{aligned}$$

Next, we determine the bound of T_3 , by replacing the values of a_i 's ($i = 2, 3, \dots, 6$) from (2.2)-(2.4) in (2.21), as follows:

$$\begin{aligned} 597196800T_3 = & 6120p_1^8 + 143424p_1^5p_3 - 425p_1^9 - 9000p_1^6p_3 + 9000p_1^7p_2 + 172800p_1^4p_2p_3 \\ & + 302400p_1^3p_2^2 - 2764800p_3^3 + 1036800p_1^3p_2p_4 + 6220800p_2p_3p_4 \\ & - 17280p_1^4p_2^2 + 9953280p_3p_5 - 2073600p_1^3p_5 + 967680p_1^3p_2p_3 \\ & - 64512p_1^6p_2 - 1036800p_1p_2p_3^2 - 32400p_1^5p_2^2 - 777600p_1^2p_2^2p_3 \\ & + 1244160p_1p_3p_4 - 259200p_1^4p_4 - 97200p_1^5p_4 + 1555200p_1p_2^2p_4 \\ & - 4665600p_1p_4^2 - 414720p_1p_2^2p_3 - 172800p_1^3p_2^3 - 829440p_2p_3^2 \\ & - 829440p_1^2p_3^2 + 414720p_1^2p_2^3 - 622080p_1^2p_2p_4 - 4976640p_1p_2p_5 \end{aligned}$$

or

$$597196800|T_3| \leq |p_1^5(6120p_1^3 + 143424p_3)| + |p_1^6(425p_1^3 + 9000p_3)| + |17280p_1^4p_2^2|$$

$$\begin{aligned}
 &+ |p_1^4 p_2 (9000 p_1^3 + 172800 p_3)| + |p_3^2 (302400 p_1^3 - 2764800 p_3)| \\
 &+ |p_2 p_4 (1036800 p_1^3 + 6220800 p_3)| + |p_5 (9953280 p_3 - 2073600 p_1^3)| \\
 &+ |p_1^3 p_2 (967680 p_3 - 64512 p_1^3)| + |1036800 p_1 p_2 p_3^2| + |97200 p_1^5 p_4| \\
 &+ |p_1^2 p_2^2 (32400 p_1^3 + 777600 p_3)| + |p_1 p_4 (1244160 p_3 - 259200 p_1^3)| \\
 &+ |p_1 p_4 (1555200 p_2^2 - 4665600 p_4)| + |p_1^2 p_2 (414720 p_2^2 - 622080 p_4)| \\
 &+ |p_3^2 (829440 p_2 + 829440 p_1^2)| + |172800 p_1^3 p_2^3| \\
 &+ |p_1 p_2 (414720 p_2 p_3 + 4976640 p_5)|.
 \end{aligned}$$

By applying Lemma 2.2 and the triangle inequality,

$$\begin{aligned}
 |T_3| &\leq \frac{286061056 + 58982400 \sqrt{\frac{3}{19}} + 99532800 \sqrt{\frac{6}{19}} + 2211840 \sqrt{210}}{597196800} \\
 &\approx 0.665582.
 \end{aligned}$$

Remark 1. On the basis of the above calculations, the bounds of T_1 , T_2 and T_3 are 0.616137, 0.543487 and 0.665582, respectively.

To progress further, our next objective is to determine the bounds of the initial coefficients a_i where $i = 2, 3, 4, 5$. These bounds, as derived in [27], are summarized in the following remark.

Remark 2. For $f \in \mathcal{S}_e^*$, $|a_2| \leq 1$, $|a_3| \leq 3/4$, $|a_4| \leq 17/36$ and $|a_5| \leq 25/72$. Here the first three bounds are sharp.

Finding coefficient bounds for $n > 5$ becomes notably more challenging. In order to overcome this difficulty, we employ Lemma 2.2 to deduce the bounds for the sixth and seventh coefficients within the class of functions \mathcal{S}_e^* , as demonstrated in the subsequent theorem.

Theorem 2.2. Let $f \in \mathcal{S}_e^*$. Then $|a_6| \leq 587/1800 \approx 0.326111$ and $|a_7| \leq 1397/4320 \approx 0.32338$.

Proof. By suitably rearranging the terms given in (2.4), we have

$$57600 a_6 = 220 p_1^3 p_2 - 480 p_1^2 p_3 - 480 p_1 p_2^2 + 720 p_1 p_4 - 17 p_1^5 - 480 p_2 p_3 + 5760 p_5.$$

Using triangle inequality, it can be viewed as

$$\begin{aligned}
 57600 |a_6| &\leq |p_1^2 (220 p_1 p_2 - 480 p_3)| + |p_1 (720 p_4 - 480 p_2^2)| + |-17 p_1^5| \\
 &\quad + |5760 p_5 - 480 p_2 p_3|.
 \end{aligned} \tag{2.22}$$

Using Lemma 2.2, we arrive at the following inequality:

$$|a_6| \leq \frac{587}{1800} \approx 0.326111.$$

Similarly, considering (2.5), we have

$$\begin{aligned}
 8294400 a_7 &= 881 p_1^6 - 13260 p_1^4 p_2 + 48240 p_1^2 p_2^2 - 14400 p_2^3 + 29040 p_1^3 p_3 - 56160 p_1^2 p_4 \\
 &\quad + 69120 p_1 p_5 - 106560 p_1 p_2 p_3 - 57600 p_3^2 - 86400 p_2 p_4.
 \end{aligned}$$

Through the triangle inequality, it can also be seen as

$$8294400 |a_7| \leq |p_1^4 (881 p_1^2 - 13260 p_2)| + |p_2^2 (48240 p_1^2 - 14400 p_2)|$$

$$+ |p_1(69120p_5 - 106560p_2p_3)| + |p_1^2(29040p_1p_3 - 56160p_4)| \\ + |57600p_3^2| + |86400p_2p_4|.$$

Lemma 2.2 implies that $|a_7| \leq 1397/4320 \approx 0.32338$. \square

Theorem 2.3. *Let $f \in \mathcal{S}_e^*$, then*

$$|H_4(1)| \leq 0.29059.$$

The proof of the above theorem follows by substituting the values obtained from Theorem 2.1, Remark 1, Remark 2 and Theorem 2.2 in (2.18), therefore, it is skipped here.

3 Hankel Determinants for \mathcal{C}_e

3.1 Preliminaries

In this segment, we express the expressions of initial coefficients a_i ($i = 2, 3, \dots, 7$) involving Carathéodory coefficients. When $f \in \mathcal{C}_e$, we replace the L.H.S of (2.1) by $1 + zf''(z)/f'(z)$ and arrive at the following equation

$$1 + \frac{zf''(z)}{f'(z)} = e^{w(z)}.$$

Proceeding on the similar lines as done for the class \mathcal{S}_e^* , we obtain a_i ($i = 2, 3, \dots, 7$) in terms of p_j ($j = 1, 2, \dots, 5$), we obtain

$$a_2 = \frac{1}{4}p_1, \quad a_3 = \frac{1}{48}(p_1^2 + 4p_2), \quad a_4 = \frac{1}{1152}(-p_1^3 + 12p_1p_2 + 48p_3), \quad (3.1)$$

$$a_5 = \frac{1}{5760}(p_1^4 - 12p_1^2p_2 + 24p_1p_3 + 144p_4), \quad (3.2)$$

$$a_6 = \frac{1}{345600}(-17p_1^5 + 220p_1^3p_2 - 480p_1p_2^2 - 480p_1^2p_3 - 480p_2p_3 + 720p_1p_4 \\ + 5760p_5), \quad (3.3)$$

and

$$a_7 = \frac{1}{58060800}(881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 14400p_2^3 + 29040p_1^3p_3 - 106560p_1p_2p_3 \\ - 57600p_3^2 - 56160p_1^2p_4 - 86400p_2p_4 + 69120p_1p_5). \quad (3.4)$$

3.2 Sharp Third Hankel Determinant for \mathcal{C}_e

In this subsection, we establish the sharp bound of $|H_3(1)|$ for functions that belong to the class \mathcal{C}_e .

Theorem 3.1. *Let $f \in \mathcal{C}_e$, then*

$$|H_3(1)| \leq \frac{1}{144}. \quad (3.5)$$

This bound is sharp.

Proof. We follow the same steps which were used to prove Theorem 2.1. The values of a'_i s ($i = 2, 3, 4, 5$) from (3.1) and (3.2) are substituted into (1.5). Thus

$$H_3(1) = \frac{1}{6635520} \left(-173p^6 + 552p^4p_2 - 1872p^2p_2^2 - 3840p_2^3 + 2208p_3^3p_3 \right. \\ \left. + 8064pp_2p_3 - 11520p_3^2 - 6912p^2p_4 + 13824p_2p_4 \right).$$

Using (2.6)-(2.8) for simplification, we arrive at

$$H_3(1) = \frac{1}{6635520} \left(\alpha_1(p, \gamma) + \alpha_2(p, \gamma)\eta + \alpha_3(p, \gamma)\eta^2 + \psi(p, \gamma, \eta)\rho \right),$$

where $\gamma, \eta, \rho \in \mathbb{D}$,

$$\alpha_1(p, \gamma) := -5p^6 - 180\gamma^2p^2(4 - p^2)^2 + 1536\gamma^3(4 - p^2)^2 - 240\gamma^3p^2(4 - p^2)^2 \\ + 144\gamma^4p^2(4 - p^2)^2 + 12\gamma p^4(4 - p^2) - 120p^4\gamma^2(4 - p^2), \\ \alpha_2(p, \gamma) := (1 - |\gamma|^2)(4 - p^2)(240p^3 - 288p\gamma(4 - p^2) - 576p\gamma^2(4 - p^2)), \\ \alpha_3(p, \gamma) := (1 - |\gamma|^2)(4 - p^2)(-2880(4 - p^2) - 576|\gamma|^2(4 - p^2)), \\ \psi(p, \gamma, \eta) := 3456\gamma(1 - |\gamma|^2)(4 - p^2)^2(1 - |\eta|^2).$$

Since $|\rho| \leq 1$, also for the simplicity of the calculations, assume $x = |\gamma|$ and $y = |\eta|$,

$$|H_3(1)| \leq \frac{1}{6635520} \left(|\alpha_1(p, \gamma)| + |\alpha_2(p, \gamma)|y + |\alpha_3(p, \gamma)|y^2 + |\psi(p, \gamma, \eta)| \right) \leq N(p, x, y),$$

where

$$N(p, x, y) = \frac{1}{6635520} \left(N_1(p, x) + N_2(p, x)y + N_3(p, x)y^2 + N_4(p, x)(1 - y^2) \right), \quad (3.6)$$

with

$$N_1(p, x) := 5p^6 + 180x^2p^2(4 - p^2)^2 + 1536x^3(4 - p^2)^2 + 240x^3p^2(4 - p^2)^2 \\ + 144x^4p^2(4 - p^2)^2 + 12xp^4(4 - p^2) + 120p^4x^2(4 - p^2), \\ N_2(p, x) := (1 - x^2)(4 - p^2)(240p^3 + 288px(4 - p^2) + 576px^2(4 - p^2)), \\ N_3(p, x) := (1 - x^2)(4 - p^2)(2880(4 - p^2) + 576x^2(4 - p^2)), \\ N_4(p, x) := 3456x(1 - x^2)(4 - p^2)^2.$$

We must maximise $N(p, x, y)$ in the closed cuboid $V : [0, 2] \times [0, 1] \times [0, 1]$. By identifying the maximum values on the twelve edges, the interior of V , and the interiors of the six faces, we can prove this.

1. We start by taking into account, every interior point of V . Assume that $(p, x, y) \in (0, 2) \times (0, 1) \times (0, 1)$. We partially differentiate (3.6) with respect to p, x and y to locate the points of maxima in the interior of V . We obtain

$$\frac{\partial N}{\partial p} = \frac{(1 - x^2)}{138240} \left(96x(1 + 2x)y + 5p^4(-5 + 6x + 12x^2)y - 12p^2(-5 + 12x + 24x^2)y \right)$$

$$- 192p(5y^2 + x^2y^2 + 6x(1 - y^2)) + 48p^3(5y^2 + x^2y^2 + 6x(1 - y^2))\Big),$$

$$\begin{aligned} \frac{\partial N}{\partial x} = \frac{(4 - p^2)}{138240} & \left(12p(1 + 4x - 3x^2 - 8x^3)y + p^3(-3 - 17x + 9x^2 + 24x^3)y \right. \\ & - 48(4xy^2 + 2x^3y^2 + 3(-1 + y^2) + 9x^2(1 - y^2)) \\ & \left. + 12p^2(4xy^2 + 2x^3y^2 - 3(1 - y^2) + 9x^2(1 - y^2)) \right) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial N}{\partial y} = \frac{(1 - x^2)(4 - p^2)}{138240} & \left(24px(1 + 2x) - p^3(-5 + 6x + 12x^2) + 96(5 - 6x + x^2)y \right. \\ & \left. - 24p^2(5 - 6x + x^2)y \right). \end{aligned}$$

By solving $\partial N/\partial p = \partial N/\partial x = \partial N/\partial y = 0$, we get the common solutions (p, x, y) using the mathematical software, Mathematica as

$$(0, -0.57735, 0); \quad (-1.35285, 6.98337, 2.99877); \quad (1.35285, 6.98337, -2.99877);$$

and

$$(2.01517, 5.04402, -1.09029); \quad (-2.01517, 5.04402, 1.09029); \quad (-2.08219, 0.122286, -1.29086);$$

and

$$(0, 0.57735, 0); \quad (-1.92198, -0.0913794, 0.863409); \quad (1.92198, -0.0913794, -0.863409);$$

and

$$(-2, -1, -0.171635); \quad (2.08219, 0.122286, -1.29086).$$

However, none of them belong to the interior of cuboid V . Therefore, we conclude that N does not attain its maximum value inside V . Next, we proceed to find the critical points on each of the interior of faces of V .

2. Now, we study the interior of each of the six faces of the cuboid V .

On the face $p = 0$: We get $x, y \in (0, 1)$ and

$$N(0, x, y) = \frac{y^2(15 - 12x^2 - 3x^4) + 18x(1 - y^2) - 2x^3(5 - 9y^2)}{2160} =: c_1(x, y). \quad (3.7)$$

Since

$$\frac{\partial c_1}{\partial y} = \frac{y(1 - x)^2(x + 1)(5 - x)}{360}, \quad x, y \in (0, 1),$$

Since, $\partial c_1/\partial y$ becomes zero only when $x = \pm 1$ or at $x = 5$ or at $y = 0$. Clearly, all these points are not in $(0, 1)$. Thus, c_1 does not have any critical point in $(0, 1) \times (0, 1)$.

On the face $p = 2$: We obtain

$$N(2, x, y) := \frac{1}{20736}, \quad x, y \in (0, 1). \quad (3.8)$$

On the face $x = 0$: We have $p \in (0, 2)$, $y \in (0, 1)$ and

$$N(p, 0, y) = \frac{(p^3 + 96y - 24p^2y)^2}{1327104} =: c_2(p, y). \tag{3.9}$$

We solve $\partial c_2/\partial p = 0$ and $\partial c_2/\partial y = 0$ to locate the points of maxima. On solving $\partial c_2/\partial y = 0$, we obtain

$$y = -\frac{p^3}{24(4 - p^2)} =: y_p.$$

Upon calculations, we observe that such y_p does not belong to $(0, 1)$. Consequently, no such critical point of c_2 exists in $(0, 2) \times (0, 1)$.

On the face $x = 1$: We get $p \in (0, 2)$ and

$$N(p, 1, y) = \frac{24576 - 3264p^2 - 2448p^4 + 437p^6}{6635520} =: c_3(p). \tag{3.10}$$

After computing $\partial c_3/\partial p = 0$, we notice that c_3 has no critical point in $(0, 2)$.

On the face $y = 0$: We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$\begin{aligned} N(p, x, 0) = \frac{1}{6635520} & \left(6144x(9 - 5x^2) + 192p^2x(-144 + 15x + 100x^2 + 12x^3) \right. \\ & - 48p^4x(-73 + 20x + 80x^2 + 24x^3) \\ & \left. + p^6(5 - 12x + 60x^2 + 240x^3 + 144x^4) \right) =: c_4(p, x). \end{aligned}$$

Calculations through Mathematica lead to

$$\begin{aligned} \frac{\partial c_4}{\partial x} = \frac{1}{6635520} & \left(-61440x^2 + 6144(9 - 5x^2) + 192p^2x(15 + 200x + 36x^2) \right. \\ & - 48p^4x(20 + 160x + 72x^2) + 192p^2(-144 + 15x + 100x^2 \\ & + 12x^3) - 48p^4(-73 + 20x + 80x^2 + 24x^3) + p^6(-12 \\ & \left. + 120x + 720x^2 + 576x^3) \right) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial c_4}{\partial p} = \frac{1}{6635520} & \left(384px(-144 + 15x + 100x^2 + 12x^3) - 192p^3x(-73 + 20x \right. \\ & \left. + 80x^2 + 24x^3) + 6p^5(5 - 12x + 60x^2 + 240x^3 + 144x^4) \right) \end{aligned}$$

having real solutions (p, x) as follows:

$$(2.04471, 3.58205); \quad (-2.03352, -2.98401); \quad (1.47565, 1.6649);$$

and

$$(0, 0.774597); \quad (0, -0.774597); \quad (2.03352, -2.98401);$$

and

$$(2, 0.307071); \quad (-2.04471, 3.58205); \quad (-2, 0.307071);$$

and

$$(2, -0.407071); \quad (-2, -0.407071); \quad (-1.47565, 1.6649).$$

Thus, no solution exist for the system of equations, $\partial c_4/\partial x = 0$ and $\partial c_4/\partial p = 0$ in $(0, 2) \times (0, 1)$. On the face $y = 1$: We have $p \in (0, 2)$, $x \in (0, 1)$ and

$$N(p, x, 1) = \frac{1}{6635520} \left(5p^6 + (4 - p^2)(12p^4x + 120p^4x^2 + 180p^2(4 - p^2)x^2 + 1536(4 - p^2)x^3 + 240p^2(4 - p^2)x^3 + 144p^2(4 - p^2)x^4 + 576(4 - p^2)(5 + x^2)(1 - x^2) + 48(1 - x^2)(p^3(5 - 6x - 12x^2) + 24px(1 + 2x))) \right) =: c_5(p, x).$$

After numerical computations done through Mathematica, we note that the real solutions (p, x) of the system of equations

$$\frac{\partial c_5}{\partial x} = \frac{(4 - p^2)}{552960} \left(768x(-2 + 2x - x^2) + 24p^2x(21 - 6x + 16x^2) + 96p(1 + 4x - 3x^2 - 8x^3) - 8p^3(3 + 17x - 9x^2 - 24x^3) + p^4(1 - 10x - 60x^2 - 48x^3) \right)$$

and

$$\frac{\partial c_5}{\partial p} = \frac{1}{1105920} \left(768x(1 + 2x - x^2 - 2x^3) - 40p^4(5 - 6x - 17x^2 + 6x^3 + 12x^4) + 96p^2(5 - 12x - 29x^2 + 12x^3 + 24x^4) - 64p(120 - 111x^2 + 44x^3 - 36x^4) + 32p^3(60 + x - 68x^2 - 8x^3 - 36x^4) + p^5(5 - 12x + 60x^2 + 240x^3 + 144x^4) \right)$$

are

$$(2, -17.9049); \quad (23.831, 0); \quad (15.9898, -0.0325369);$$

and

$$(2, -8.75); \quad (8.16565, -2.24092); \quad (2.0765, 1.0018);$$

and

$$(-1.84536, -0.877352); \quad (-2, -0.775417); \quad (2.09099, 0.254719);$$

and

$$(0, 0); \quad (2.09338, 0); \quad (-1.92434, 0);$$

and

$$(-0.644804, 1.10265); \quad (-2, 0.725417).$$

Thus, we conclude that these two equations $\partial c_5/\partial x = 0$ and $\partial c_5/\partial p = 0$ also do not assume any solution in $(0, 2) \times (0, 1)$.

- Next, we check the maximum values of $N(p, x, y)$ obtained on the edges of the cuboid V . From (3.9), we have $N(p, 0, 0) = p^6/1327104 =: t_1(p)$. It is easy to observe that $t_1'(p) = 0$ for $p = 0$ in the interval $[0, 2]$. The maximum value of $t_1(p)$ is 0. Now, (3.9) reduces to

$N(p, 0, 1) = (96 - 24p^2 + p^3)^2/1327104 =: t_2(p)$ at $y = 1$. Since, $t'_2(p) < 0$ in $[0, 2]$, hence $p = 0$ is the point of maxima. Thus

$$N(p, 0, 1) \leq \frac{1}{144}, \quad p \in [0, 2].$$

Through computations, (3.9) shows that $N(0, 0, y)$ attains its maxima at $y = 1$. Hence

$$N(0, 0, y) \leq \frac{1}{144}, \quad y \in [0, 1].$$

Since, (3.10) is free from x , we have $N(p, 1, 1) = N(p, 1, 0) = (24576 - 3264p^2 - 2448p^4 + 437p^6)/6635520 =: t_3(p)$. Now, we observe that $t'_3(p) < 0$ in $[0, 2]$, consequently, $t_3(p)$ attains its maximum at $p = 0$. Hence

$$N(p, 1, 1) = N(p, 1, 0) \leq 0.0037037, \quad p \in [0, 2].$$

On substituting $p = 0$ in (3.10), we get, $N(0, 1, y) = 1/270$. The equation (3.8) does not contain any variable such as p, x and y . Therefore, the maxima of $N(p, x, y)$ on the edges is given by

$$N(2, 1, y) = N(2, 0, y) = N(2, x, 0) = N(2, x, 1) = \frac{1}{20736}, \quad x, y \in [0, 1].$$

Using (3.7), we obtain $N(0, x, 1) = (15 - 12x^2 + 8x^3 - 3x^4)/2160 =: t_4(x)$. Upon calculations, we see that t_4 is a decreasing function in $[0, 1]$ and its maximum value is achieved at $x = 0$. Hence

$$N(0, x, 1) \leq \frac{1}{144}, \quad x \in [0, 1].$$

On again using (3.7), we get $N(0, x, 0) = x(9 - 5x^2)/1080 =: t_5(x)$. On further calculations, we get $t'_5(x) = 0$ when $x = \sqrt{3/5}$, the point of maxima. Thus

$$N(0, x, 0) \leq 0.00430331, \quad x \in [0, 1].$$

From the above all possible cases, we observe that optimal value of $|H_3(1)|$ is $1/144$. Therefore, the inequality (3.5) holds. Now, consider the following function $f_2 \in \mathcal{C}_e$, defined by

$$f_2(z) = \int_0^z \left(\exp \left(\int_0^y \frac{e^{t^3} - 1}{t} dt \right) \right) dy = z + \frac{z^4}{12} + \frac{5z^7}{252} + \dots,$$

with $f_2(0) = f'_2(0) - 1 = 0$, plays the role of an extremal function for the bounds of $|H_3(1)|$ having values $a_3 = a_5 = 0$ and $a_4 = 1/12$. □

3.3 Fourth Hankel Determinant for \mathcal{C}_e

In this part of the section, we derive the bound of $|H_4(1)|$ including finding the bounds of sixth and seventh coefficients for functions in the class \mathcal{C}_e . By selecting $q = 4$ and $n = 1$ in (1.4), the expression of $H_4(1)$ can be obtained for functions in the class \mathcal{C}_e , which is given as follows:

$$H_4(1) = a_7H_3(1) - a_6U_1 + a_5U_2 - a_4U_3. \tag{3.11}$$

Here

$$U_1 := a_6(a_3 - a_2^2) + a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4), \tag{3.12}$$

$$U_2 := a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3), \quad (3.13)$$

and

$$U_3 := a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3). \quad (3.14)$$

We start by determining the bounds for U_1 , U_2 , and U_3 .

By substituting the values of a_i 's ($i = 2, 3, \dots, 6$) in (3.12) from (3.1)-(3.3), we obtain

$$\begin{aligned} 132710400U_1 &= 487p_1^7 - 6304p_1^5p_2 + 11440p_1^3p_2^2 - 24960p_1p_2^3 + 5280p_1^4p_3 \\ &\quad + 34560p_1p_3^2 + 19200p_1^2p_2p_3 - 53760p_2^2p_3 + 57600p_1p_2p_4 \\ &\quad - 138240p_3p_4 + 184320p_2p_5 - 92160p_1^2p_5 + 8640p_1^3p_4, \end{aligned}$$

can also be viewed as the following, due to the triangle inequality,

$$\begin{aligned} 132710400|U_1| &\leq |p_1^5(487p_1^2 - 6304p_2)| + |p_1p_2^2(11440p_1^2 - 24960p_2)| \\ &\quad + |p_1p_3(5280p_1^3 + 34560p_3)| + |p_2p_3(19200p_1^2 - 53760p_2)| \\ &\quad + |p_4(57600p_1p_2 - 138240p_3)| + |p_5(184320p_2 - 92160p_1^2)| \\ &\quad + |8640p_1^3p_4|. \end{aligned}$$

Using Lemma 2.2, we arrive at

$$|U_1| \leq \frac{4121}{345600} \approx 0.0119242.$$

We replace the values of a_i 's ($i = 2, 3, \dots, 6$) from (3.1)-(3.4) in (3.13) and proceed on the same lines to obtain the bound of U_2

$$\begin{aligned} 1592524800U_2 &= 463p_1^8 - 2732p_1^6p_2 - 23472p_1^4p_2^2 - 14400p_1^2p_2^3 + 14592p_1^5p_3 \\ &\quad - 108288p_1^2p_3^2 + 92928p_1^3p_2p_3 - 138240p_1p_2^2p_3 + 1105920p_3p_5 \\ &\quad - 25344p_1^4p_4 + 276480p_2^2p_4 - 995328p_4^2 + 373248p_1^2p_2p_4 \\ &\quad - 276480p_1p_2p_5 + 221184p_1p_3p_4 - 161280p_1^3p_5 - 322560p_2p_3^2, \end{aligned}$$

by implementing the triangle inequality,

$$\begin{aligned} 1592524800|U_2| &\leq |p_1^6(463p_1^2 - 2732p_2)| + |p_1^2p_2^2(-23472p_1^2 - 14400p_2)| \\ &\quad + |p_1^2p_3(14592p_1^3 - 108288p_3)| + |161280p_1^3p_5| \\ &\quad + |p_1^2p_4(373248p_2 - 25344p_1^2)| + |p_4(276480p_2^2 - 995328p_4)| \\ &\quad + |322560p_2p_3^2| + |p_1p_2p_3(92928p_1^2 - 138240p_2)| \\ &\quad + |221184p_1p_3p_4| + |p_5(1105920p_3 - 276480p_1p_2)|. \end{aligned}$$

By applying Lemma 2.2, we have

$$|U_2| \leq \frac{24947200 + 866304\sqrt{\frac{282}{61}}}{1592524800} \approx 0.0168348.$$

Again, substitute the values of a_i 's ($i = 2, 3, \dots, 6$) from (3.1)-(3.4) in (3.14) and proceed to calculate the bound of U_3 in the same manner.

$$38220595200U_3 = 11424p_1^8 - 128256p_1^6p_2 + 10812p_1^7p_2 - 503p_1^9 + 69120p_1^4p_2^2$$

$$\begin{aligned}
 &+ 552960p_1^2p_2^3 - 42192p_1^5p_2^2 - 181440p_1^3p_2^3 + 206208p_1^4p_2p_3 \\
 &- 11664p_1^6p_3 + 1889280p_1^3p_2p_3 - 1658880p_1p_2^2p_3 - 2211840p_1^2p_3^2 \\
 &- 2211840p_2p_3^2 + 283392p_1^3p_3^2 - 967680p_1p_2p_3^2 + 3317760p_1p_3p_4 \\
 &- 483840p_1^4p_4 + 1271808p_1^3p_2p_4 - 117504p_1^5p_4 + 1658880p_1p_2^2p_4 \\
 &- 5971968p_1p_4^2 + 6635520p_2p_3p_4 - 331776p_1^2p_3p_4 + 26542080p_3p_5 \\
 &- 6635520p_1p_2p_5 + 244224p_1^5p_3 - 794880p_1^2p_2^2p_3 - 2764800p_3^3 \\
 &- 829440p_1^2p_2p_4 - 3870720p_1^3p_5,
 \end{aligned}$$

can be visualized as the following with the help of the triangle inequality,

$$\begin{aligned}
 38220595200|U_3| \leq & |p_1^6(11424p_1^2 - 128256p_2)| + |p_1^7(10812p_2 - 503p_1^2)| \\
 & + |p_1^2p_2^2(69120p_1^2 + 552960p_2)| + |p_1^3p_2^2(42192p_1^2 + 181440p_2)| \\
 & + |p_1^4p_3(206208p_2 - 11664p_1^2)| + |p_1p_2p_3(1889280p_1^2 - 1658880p_2)| \\
 & + |p_3^2(2211840p_1^2 + 2211840p_2)| + |p_1p_3^2(283392p_1^2 - 967680p_2)| \\
 & + |p_1p_4(3317760p_3 - 483840p_1^3)| + |p_1^3p_4(1271808p_2 - 117504p_1^2)| \\
 & + |p_1p_4(1658880p_2^2 - 5971968p_4)| + |p_3p_4(6635520p_2 - 331776p_1^2)| \\
 & + |p_5(26542080p_3 - 6635520p_1p_2)| + |244224p_1^5p_3 - 794880p_1^2p_2^2p_3 \\
 & - 2764800p_3^3 - 829440p_1^2p_2p_4 - 3870720p_1^3p_5|.
 \end{aligned}$$

By applying Lemma 2.2, we get

$$|U_3| \leq \frac{560108544 + 106168320\sqrt{\frac{3}{41}}}{38220595200} \approx 0.015406.$$

Remark 3. The bounds of U_1 , U_2 and U_3 , based on the above calculations, are 0.0119242, 0.0168348, and 0.015406, respectively.

The bounds of a_i 's ($i = 2, 3, 4, 5$) for functions in the class \mathcal{C}_e are obtained in [27], presented below in the following remark:

Remark 4. For $f \in \mathcal{C}_e$, $|a_2| \leq 1/2$, $|a_3| \leq 1/4$, $|a_4| \leq 17/144$ and $|a_5| \leq 5/72$. The first three bounds are sharp.

Next, we calculate the bounds of the sixth and seventh coefficient of functions belonging to the class \mathcal{C}_e to establish our main result along the lines of Theorem 2.2.

Theorem 3.2. Let $f \in \mathcal{C}_e$. Then $|a_6| \leq 587/10800 \approx 0.0543519$ and $|a_7| \leq 0.0343723$.

Proof. A suitable rearrangement of the terms given in (3.3) provides us

$$345600a_6 = 5760p_5 - 480p_2p_3 + 720p_1p_4 - 480p_1p_2^2 - 17p_1^5 + 220p_1^3p_2 - 480p_1^2p_3.$$

Further, through the triangle inequality, it can be viewed as

$$\begin{aligned}
 345600|a_6| \leq & |5760p_5 - 480p_2p_3| + |p_1(720p_4 - 480p_2^2)| + |17p_1^5| \\
 & + |p_1^2(220p_1p_2 - 480p_3)|.
 \end{aligned}$$

Using Lemma 2.2, we arrive at

$$|a_6| \leq \frac{587}{10800} \approx 0.0543519.$$

Similarly, considering (3.4), we have

$$\begin{aligned} 58060800a_7 = & 881p_1^6 - 13260p_1^4p_2 + 48240p_1^2p_2^2 - 106560p_1p_2p_3 + 29040p_1^3p_3 \\ & - 57600p_3^2 + 69120p_1p_5 - 56160p_1^2p_4 - 86400p_2p_4 - 14400p_2^3. \end{aligned}$$

It can also be seen as with the aid of the triangle inequality,

$$\begin{aligned} 58060800|a_7| \leq & |p_1^4(881p_1^2 - 13260p_2)| + |p_1p_2(48240p_1p_2 - 106560p_3)| \\ & + |p_3(29040p_1^3 - 57600p_3)| + |p_1(69120p_5 - 56160p_1p_4)| \\ & + |p_2(86400p_4 + 14400p_2^2)|. \end{aligned} \tag{3.15}$$

Lemma 2.2 takes us at

$$|a_7| \leq \frac{2014080 + 921600\sqrt{\frac{15}{119}}}{58060800} \approx 0.0403246.$$

□

We obtain the following result by omitting the proof as it directly follows from Theorem 3.1, Remark 3, Remark 4, Theorem 3.2 and (3.11).

Theorem 3.3. *Let $f \in \mathcal{C}_e$, then*

$$|H_4(1)| \leq 0.00101775.$$

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