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An orthogonal class of p-Legendre polynomials on variable interval

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Abstract. The work incorporates a generalization of the Legendre polynomial by introducing a parameter $p > 0$ in its generating function. The coefficients thus generated, constitute a class of the polynomials which are termed as the p-Legendre polynomials. It is shown that this class turns out to be orthogonal with respect to the weight function: $(1-\sqrt{p}\ x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}-1}$ over the interval $(-\frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p}})$. Among the other properties derived, include the Rodrigues formula, normalization, recurrence relation and zeros. A graphic depiction for $p = 0.5, 1, 2,$ and 3 is shown. The p-Legendre polynomials are used to estimate a function using the least squares approach. The approximations are graphically depicted for $p = 0.7, 1, 2$.

Keywords. p-deformed polynomial, Legendre polynomial, orthogonality, Rodrigues formula, zeros

1 Introduction

The Circular Restricted 3 Body Problem (CR3BP) in astronomy is well known. It is pertaining to the spacecraft's motion in the presence of two bodies which are referred to as two primaries. One can take two primaries as Sun and Earth. In CR3BP, two primaries move in circular orbits around their barycentre whereas the third (infinitesimal) body moves in the gravitational field of the primaries (Figure 1). Now if m_1 and m_2 are masses of the two primaries and $m_2/(m_1+m_2) = \mu$, then normalizing the distance to be unity in rotating frame with unit angular velocity, the locations of the two primaries are found to be on the x-axis at the points $(-\mu, 0, 0)$ and $(1-\mu, 0, 0)$. The third (infinitesimal) body is taken to be located at some point (x, y, z) in the rotating frame. Now, if r_1 is the distance between the first primary and third body and r_2 is the distance between the second primary and third body, then [21, p. 8]

$$r_1^2 = (x+\mu)^2 + y^2 + z^2; \quad r_2^2 = (x-1+\mu)^2 + y^2 + z^2.$$

In order to expand the nonlinear term: $\frac{1-\mu}{r_1} + \frac{\mu}{r_2}$, the following formula is used [21, p. 146].

$$\frac{1}{\sqrt{(x-A)^2+(y-B)^2+(z-C)^2}} = \frac{1}{R}\sum_{n\geq 3} P_n\left(\frac{Ax+By+Cz}{rD}\right)\left(\frac{r}{R}\right)^n,$$

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in which $R^2 = A^2 + B^2 + C^2, r^2 = x^2 + y^2 + z^2$ and $P_n(*)$ is the Legendre polynomial.

Among many other physical phenomena, the Legendre polynomials are associated with one dimensional steady-state transport equation and neutron scattering functions for one-energy group (see [4] for the detailed account). Also, the Legendre polynomials are occurring as the coefficients in the power series expansion of an electric potential function [5, Ch.11, p. 552-561]. This polynomial also solves the Volterra integro-differential equations [25]. Legendre's spectral-collocation method has been used to solve stochastic delay differential equations and delay differentials [19]. Moreover, this polynomial is used in a spectral collocation method for solving the system of nonlinear Fredholm integral equations of second kind [20]. Besides this, the Legendre polynomial occurs in a Stochastic SIRS epidemic model [2] and in determining the mean values [13]. There is a good collection of multivariate orthogonal polynomials in [16].

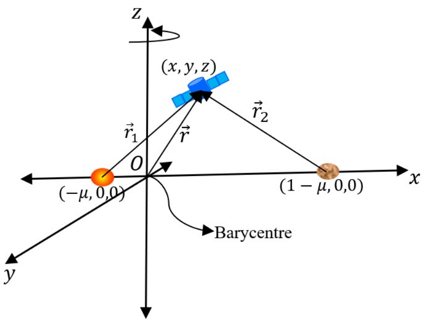

Figure 1: CR3BP

1.1 Notations and formulas

If $A(n, k)$ denotes the expression involving n, k and the parameter(s) if any, then [26, Lemma 10, p. 56]

$$\sum_{n\geq 0} \sum_{k\geq 0} A(n, k) = \sum_{n\geq 0} \sum_{0\leq k\leq n} A(n-k, k). \tag{1.1}$$

Let $p \in \mathbb{R}$ and $z \in \mathbb{C}$, then the Pochhammer p-symbol is denoted and defined as [12, Definition 1, p.181]

$$(z)_{n,p} = z(z+p)(z+2p)\cdots(z+(n-1)p). \tag{1.2}$$

The usual Pochhammer symbol: $(z)_n$ is the case $p = 1$. In fact, this Pochhammer p-symbol motivated Diaz and Periguan to define the p-Gamma function denoted and defined as follows [12, p.180].

Definition 1. For $z \in \mathbb{C}$ with $\Re(z) > 0$ and for $p > 0$,

$$\Gamma_p(z) = \int_0^\infty t^{z-1}\, e^{-\frac{t^p}{p}}\, dt. \tag{1.3}$$

The instance $p = 2$ provides the Gaussian integral $\Gamma_2(z)$ [12, p. 183]. As noted by Diaz and Periguan, the function $\Gamma_p(z)$ comes from the above Pochhammer p-symbol which occurs in the combinatorics of creation, annihilation operators [10, 11] and the perturbative computation of Feynman integrals [9]. From (1.2) and (1.3), the following formulas may be verified [28, Ch. 1, Eq. (1.3.8)].

$$(z)_{n,p} \quad = \quad \frac{\Gamma_p(z+np)}{\Gamma_p(z)}, \tag{1.4}$$

$$\begin{aligned}
(-np)_{k,p} &= p^k(-n)_k, & (1.5)\\
(z)_{m+n,p} &= (z)_{m,p}(z+mp)_{n,p}, & (1.6)\\
(z)_{n-k,p} &= \frac{(-1)^k(z)_{n,p}}{(p-z-np)_{k,p}}, & (1.7)\\
\Gamma_p(p) &= 1, & (1.8)\\
\Gamma_p(1) &= p^{\frac{1}{p}-1}\Gamma\left(\frac{1}{p}\right). & (1.9)
\end{aligned}$$

Interestingly, for $n = 1$, the formula (1.4) reduces to the p*-difference equation*:

$$\Gamma_p(z+p) = z\Gamma_p(z). \tag{1.10}$$

Following the the p-deformed Beta function in [12], we have

Definition 2. For $p > 0, a, b \in \mathbb{C}$ with $\Re(a, b) \neq 0, -p, -2p, \ldots,$

$$\begin{aligned}
B_p(a,b) &= \frac{1}{p}\int_0^1 t^{\frac{a}{p}-1}\,(1-t)^{\frac{b}{p}-1}\,dt\\
&= \frac{2}{p}\int_0^{\pi/2} \sin^{\frac{2a}{p}-1}\theta\cos^{\frac{2b}{p}-1}\theta\ d\theta,
\end{aligned} \tag{1.11}$$

when $t = \sin^2\theta$. Moreover, in parallel to the classical case, the relation [12, p. 187]:

$$B_p(a,b) = \frac{\Gamma_p(a)\,\Gamma_p(b)}{\Gamma_p(a+b)}, \tag{1.12}$$

holds true. Diaz and Pariguan also defined the p-hypergeometric function as follows [12, p. 188].

Definition 3. For $a = (a_1, a_2, \ldots, a_r) \in \mathbb{C}^r, k = (k_1, k_2, \ldots, k_r) \in (\mathbb{R}^+)^r, s = (s_1, s_2, \ldots, s_q) \in (\mathbb{R}^+)^q, b = (b_1, b_2, \ldots, b_q) \in \mathbb{C}^q$ with $b_i \in \mathbb{C}\backslash s_i\mathbb{Z}_{<0}$, the hypergeometric function $F(a, k, b, s)(x)$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, is given by the formal power series:

$$F(a,k,b,s)(x) = \sum_{n\geq 0} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\cdots(a_r)_{n,k_r}}{(b_1)_{n,s_1}(b_2)_{n,s_2}\cdots(b_q)_{n,s_q}}\frac{x^n}{n!}. \tag{1.13}$$

The ratio test permits the series to converge absolutely for all x if $r \leq q$. For $r = q+1$, the series converges absolutely if $|x| < \frac{s_1 s_2\cdots s_q}{k_1 k_2\cdots k_r}.$ The series diverges for $r > q+1$. For $y = F(a, k, b, s)(x)$ with $r \leq q+1$ and $x\frac{d}{dx} = D$, it satisfies the differential equation [12, p. 188]:

$$\begin{aligned}
&[D(s_1D+b_1-s_1)(s_2D+b_2-s_2)\cdots(s_qD+b_q-s_q)\\
&\quad -x(k_1D+a_1)(k_2D+a_2)\cdots(k_rD+a_r)]\,y = 0.
\end{aligned} \tag{1.14}$$

With regard to the standard generalized hypergeometric function notation and the choice $k_1 = k_2 = \cdots k_r = s_1 = s_2 = \cdots s_q = p$, we shall denote the series in (1.13) by

$${}_rF_q^p\left[\begin{array}{ccccc} a_1, & a_2, & \ldots, & a_r; & x\\ b_1, & b_2, & \ldots, & b_q & \end{array}\right].$$

The p-binomial series is an instance $r = 1, q = 0$ of (1.13) which is given by [12, Example 19, p. 189]

$$\sum_{n\geq 0}\frac{(a)_{n,p}}{n!}x^n = (1-px)^{-\frac{a}{p}},\quad a\in\mathbb{C}, p>0, |x|<\frac{1}{p}. \tag{1.15}$$

The Gauss summation formula [26, Theorem 18, p. 49]:

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \Re(c-a-b) > 0, $$

has the p-deformation [27, Cor. 6.2, p. 14]:

$$ {}_2F_1((a,b),(p,p),(c),p)(1/p) = {}_2F_1^p(a,b;c;1/p) = \frac{\Gamma_p(c)\Gamma_p(c-a-b)}{\Gamma_p(c-a)\Gamma_p(c-b)}, \tag{1.16} $$

where $p > 0$, $c \in \mathbb{C}\backslash p\mathbb{Z}_{\leq 0}$ and $\Re(c-a-b) > 0$.

It is noteworthy that the p-deformed Special Functions' theory does not always run parallel to the classical theory. In fact, the p-extension of a Special Function is a state-of-art. It is interesting to see that the interval of absolute convergence of the p-binomial series (1.15) is not fixed; in fact, it can be magnified or diminished by allowing p to be smaller or larger. This is not the case for the usual binomial series. Further, the function in (1.16) will be valid for any value of $p > 0$. Thus the p-function will be valid for the argument≥ 1 or < 1. The circle of convergence will change from $|z| = 1$ to $|z| = 1/p$. Not only $p -_2 F_1[*]$ functions, all those p-hypergeometric functions: ${}_{r+1}F_r[*]$, have variable regions of validity; for, the condition: $|z| < 1$ of absolute convergence will be replaced by the condition $|z| < 1/p$. Thus, the validity regions of such series can be enlarged. On the other hand, the construction of p-polynomials' class has the advantage that it gives rise to a large number of orthogonal polynomials (as p varies). Moreover, just like the p-Legendre polynomials' case, the orthogonality interval depends on the values of p; thereby the location of zeros will also go on vary. However, this is not the situation with all the p-polynomials (see for instance, [17]).

Since the introduction of the k-Gamma function and k-Hypergeometric function [12], this notion has been applied in several areas. A variety of integral representations are obtained for the k-gamma and k-digamma function in [1]. This notion is also employed in the Fractional Calculus; especially to the k-fractional integral and k-fractional derivative ([6], [15], [24]). Regarding the applications, the k-Gamma and k-Beta functions are extended along with the k-Beta distribution, and using these, the maximum likelihood estimators, central moments and some properties based on expectation are studied. Also, pertaining to the real life problems, the hazard rate function is computed, mean residue life function and entropy are determined in [7]. The k-Gamma and k-Beta functions have also been extended to the matrix argument and their properties namely, the functional relations, inequality, integral formula, and integral representations are derived; moreover, an application to statistics is also illustrated [18].
The present work uses 'p-deformation' since the letter 'k' is used as a summation index.

1.2 p-Deformed Legendre duplication formula

The Legendre duplication formula is well known [26, Eq. (2), p. 24]. Its p-version is obtained in the following theorem; but prior to this, we first prove in the following lemma which states that $\sqrt{\pi}$ also undergoes the p-deformation.

Lemma 1.1. *For $p > 0$, there holds the identity:* $\sqrt{\pi} = \sqrt{p}\,\Gamma_p\left(\frac{p}{2}\right)$.

Proof. In view of Definition 2,

$$ pB_p\left(\frac{p}{2},\frac{p}{2}\right) = \int_0^1 t^{-\frac{1}{2}}\,(1-t)^{-\frac{1}{2}}dt = \int_0^1 \frac{1}{\sqrt{\frac{1}{4}-\left(t-\frac{1}{2}\right)^2}}dt = \left[\sin^{-1}(2t-1)\right]_0^1 = \pi. $$

But from (1.8) and (1.12), $B_p\left(\frac{p}{2},\frac{p}{2}\right)=\Gamma_p^2\left(\frac{p}{2}\right)$, hence, $\Gamma_p^2\left(\frac{p}{2}\right)=\frac{\pi}{p}$. □

Remark 1. For $p=1$, the lemma recovers $\Gamma(\frac{1}{2})=\sqrt{\pi}$.

Theorem 1.1. *For $p>0$ and $z\in\mathbb{C}$ such that the p-Gamma functions involved are all defined, there holds the formula:*

$$\sqrt{\frac{\pi}{p}}\ \Gamma_p(2z)=2^{\frac{2z}{p}-1}\ \Gamma_p(z)\ \Gamma_p\left(z+\frac{p}{2}\right). \tag{1.17}$$

Proof. In Definition 2 and in (1.11) and (1.12), putting $a=b=z$ to get

$$\frac{\Gamma_p^2(z)}{2\ \Gamma_p(2z)}=\frac{1}{p\ 2^{\frac{2z}{p}-1}}\int_0^{\pi/2}\sin^{\frac{2z}{p}-1}(2\theta)\ d\theta=\frac{1}{p\ 2^{\frac{2z}{p}}}\int_0^{\pi}\sin^{\frac{2z}{p}-1}(\phi)\ d\phi,$$

where $2\theta=\phi$. Now from the theory of definite integrals, $f(x)=f(2a-x)$ implies

$$\int_0^{2a}f(x)\ dx=2\int_0^{a}f(x)\ dx,$$

hence,

$$\frac{\Gamma_p^2(z)}{2\ \Gamma_p(2z)}=\frac{2}{p\ 2^{\frac{2z}{p}}}\int_0^{\pi/2}\sin^{\frac{2z}{p}-1}(\phi)\ d\phi=\frac{1}{2^{\frac{2z}{p}}}\ B_p(z,\frac{p}{2})=\frac{1}{2^{\frac{2z}{p}}}\frac{\Gamma_p(z)\ \Gamma_p\left(\frac{p}{2}\right)}{\Gamma_p\left(z+\frac{p}{2}\right)}.$$

The theorem follows by simplifying the extreme sides expressions. □

Remark 2. For $p=1$, the p-duplication formula of this theorem reduces to the usual Legendre duplication formula [26, Eq. (2), p. 24].

2 p-Legendre Polynomial

We generalize the generating function $(1-2xt+t^2)^{-\frac{1}{2}}$ of the Legendre polynomial: $P_n(x)$ in the form $(1-p(2xt-t^2))^{-\frac{1}{2p}}$ for $p>0$, and expand it in powers of t as follows.

$$\begin{aligned}(1-p(2xt-t^2))^{-\frac{1}{2p}} &= \sum_{n\geq 0}\frac{\left(\frac{1}{2}\right)_{n,p}}{n!}\ (2xt-t^2)^n\\ &= \sum_{n\geq 0}\frac{\left(\frac{1}{2}\right)_{n,p}}{n!}\ t^n\sum_{0\leq k\leq n}\frac{n!}{(n-k)!k!}\ (-1)^k\ t^k\ (2x)^{n-k}\\ &= \sum_{n\geq 0}\ \sum_{0\leq 2k\leq n}\frac{(\frac{1}{2})_{n-k,p}}{(n-2k)!k!}(-1)^k(2x)^{n-2k}\ t^n.\end{aligned}$$

On the right hand side, the coefficients of the series in powers of t thus generated are denoted and given by

$$P_{n,p}(x)=\sum_{0\leq 2k\leq n}\frac{(-1)^k(\frac{1}{2})_{n-k,p}}{(n-2k)!k!}(2x)^{n-2k}$$

to which we call the p-Legendre polynomials constituting the class $\mathcal{P}$. Thus $\mathcal{P} = \{P_{n,p}(x); n \in \{0\} \cup \mathbb{N}, p > 0\}$ possess the generating function relation (GFR):

$$(1 - p(2xt - t^2))^{-\frac{1}{2p}} = \sum_{n\geq 0} P_{n,p}(x)\ t^n. \tag{2.1}$$

Remark 3. For $p = 1$, $P_{n,p}(x)$ immediately reduces to the classical form.

2.1 Hypergeometric Form

The hypergeometric form of a polynomial is useful in deriving certain properties. With the objective of deriving the differential equation as well as the Rodrigues formula, we obtain such a representation of $P_{n,p}(x)$. For this, we expand the generating function with the aid of the formulas (1.4), (1.5), (1.15), and assume the conditions $\left|\frac{2t(\sqrt{p}\ x-1)}{\sqrt{p}(1-\sqrt{p}\ t)^2}\right| < 1$ and $|t| < \sqrt{p}$. We then have

$$\begin{aligned}
(1 - p(2xt - t^2))^{-\frac{1}{2p}} &= [1 - 2pxt + pt^2]^{-\frac{1}{2p}} = [(1 - \sqrt{p}t)^2 - 2\sqrt{p}t(\sqrt{p}\ x - 1)]^{-\frac{1}{2p}} \\
&= (1 - \sqrt{p}\ t)^{-\frac{1}{p}} \left[1 - \frac{2\sqrt{p}\ t(\sqrt{p}\ x - 1)}{(1 - \sqrt{p}\ t)^2}\right]^{-\frac{1}{2p}} \\
&= (1 - \sqrt{p}\ t)^{-\frac{1}{p}} \sum_{k\geq 0} \frac{(\frac{1}{2})_{k,p}}{k!} \left(\frac{2t(\sqrt{p}\ x - 1)}{\sqrt{p}\ (1 - \sqrt{p}\ t)^2}\right)^k \\
&= \sum_{k\geq 0} \frac{(\frac{1}{2})_{k,p}}{p^{\frac{k}{2}} k!} (\sqrt{p}\ x - 1)^k (2t)^k (1 - \sqrt{p}\ t)^{-\frac{2kp+1}{p}} \\
&= \sum_{k\geq 0} \frac{(\frac{1}{2})_{k,p}}{p^{\frac{k}{2}} k!} (\sqrt{p}\ x - 1)^k (2t)^k \sum_{n\geq 0} \frac{(2kp+1)_{n,p}}{n!} \left(\frac{t}{\sqrt{p}}\right)^n \\
&= \sum_{n\geq 0} \sum_{k\geq 0} \frac{\Gamma_p(\frac{1}{2} + kp)}{p^{\frac{n+k}{2}} k!n!} \frac{\Gamma_p(2kp + 1 + np)}{\Gamma_p(\frac{1}{2})\Gamma_p(2kp + 1)} (\sqrt{p}\ x - 1)^k (2t)^k t^n.
\end{aligned}$$

In view of the p-Legendre duplication formula (1.17) with $z = kp + \frac{1}{2}$ and Lemme 1.1, we have

$$2^k \frac{\Gamma_p(\frac{1}{2} + kp)}{\Gamma_p(2kp + 1)} = 2^{1-k-\frac{1}{p}} \frac{\Gamma_p\left(\frac{p}{2}\right)}{\Gamma_p(kp + \frac{p+1}{2})} = 2^{1-k-\frac{1}{p}} \frac{\sqrt{\pi}}{\sqrt{p}\ \Gamma_p(kp + \frac{p+1}{2})}. \tag{2.2}$$

This, with the double sum (1.1) enable us to proceed for further simplification as follows.

$$\begin{aligned}
&(1 - p(2xt - t^2))^{-\frac{1}{2p}} \\
&= \frac{2^{1-\frac{1}{p}}\sqrt{\pi}}{\sqrt{p}\ \Gamma_p(\frac{1}{2})} \sum_{k\geq 0} \sum_{n\geq 0} \frac{\Gamma_p(2kp + np + 1)}{\Gamma_p(kp + \frac{1+p}{2})k!n!} \left(\frac{\sqrt{p}\ x - 1}{2}\right)^k \left(\frac{t}{\sqrt{p}}\right)^{n+k} \\
&= \frac{2^{1-\frac{1}{p}}\sqrt{\pi}}{\sqrt{p}\ \Gamma_p(\frac{1}{2})\Gamma_p(\frac{1+p}{2})} \sum_{n\geq 0} \frac{\Gamma_p(np + 1)t^n}{p^{\frac{n}{2}} n!} \sum_{0\leq k\leq n} \frac{(np + 1)_{k,p}}{(\frac{1+p}{2})_{k,p}} \frac{(-1)^k n!}{k!(n - k)!} \left(\frac{1 - \sqrt{p}\ x}{2}\right)^k \\
&= \frac{2^{1-\frac{1}{p}}\sqrt{\pi}}{\sqrt{p}\ \Gamma_p(\frac{1}{2})\Gamma_p(\frac{1+p}{2})} \sum_{n\geq 0} \frac{\Gamma_p(np + 1)}{p^{\frac{n}{2}} n!}\ {}_2F_1^p \left[\begin{matrix} -np, & np + 1; & \frac{1-\sqrt{p}\ x}{2p} \\ \frac{1+p}{2}; & & \end{matrix}\right] t^n.
\end{aligned}$$

Hence, from the generating function relation (2.1), the hypergeometric function form occurs in the form:

$$P_{n,p}(x) = \frac{2^{1-\frac{1}{p}}\sqrt{\pi}\ \Gamma_p(np+1)}{p^{\frac{n+1}{2}}\ n!\ \Gamma_p(\frac{1}{2})\Gamma_p(\frac{1+p}{2})}\ {}_2F_1^p\left[\begin{array}{ccc} -np, & np+1; & \frac{1-\sqrt{p}\ x}{2p} \\ & \frac{1+p}{2}; & \end{array}\right].$$

Finally, using the formulas (1.4) with $z=1$, and (1.17) with $z=1/2$, this reduces to the elegant form:

$$P_{n,p}(x) = \frac{(1)_{n,p}}{p^{n/2}n!}\ {}_2F_1^p\left[\begin{array}{ccc} -np, & np+1; & \frac{1-\sqrt{p}\ x}{2p} \\ & \frac{1+p}{2}; & \end{array}\right]. \tag{2.3}$$

Remark 4. For $p=1$, it reduces to the ${}_2F_1() - n, n+1; 1; \frac{1-x}{2})$ [26, Eq. (2), p. 64].

3 Differential Equation

The differential equation of a polynomial can be exploited for deriving the orthogonality property of the polynomial. Here particularizing the equation (1.14) appropriately, we obtain the equation satisfied by the polynomials $P_{n,p}(x) \in \mathcal{P}$. For that we choose $r=2, q=1, k_1=k_2=l_1=p, x\frac{d}{dx}=\mathfrak{D}$, then for $y = {}_2F_1^p\left[\begin{array}{ccc} a_1, & a_2; & z \\ & b_1; & \end{array}\right]$, the differential equation (1.14) gets reduced to

$$[\mathfrak{D}(p\mathfrak{D}+b_1-p) - z(p\mathfrak{D}+a_1)(p\mathfrak{D}+a_2)]y = 0.$$

Further simplification yields

$$[p(1-pz)z\mathfrak{D}^2 + (b_1-(a_2+a_1+p)pz)\mathfrak{D} - a_1a_2]y = 0. \tag{3.1}$$

In this, the substitutions $a_1=-np, a_2=np+1, b_1=\frac{1+p}{2}$ and $z=\frac{1-\sqrt{p}\ x}{2p}$, provide the equation satisfied by $y=P_{n,p}(x)$ of (2.3), in the form:

$$\left[4p\left(p-p^2\left(\frac{1-\sqrt{p}\ x}{2p}\right)\right)\left(\frac{1-\sqrt{p}\ x}{2p}\right)\frac{d^2}{dx^2} - 2\sqrt{p}\left(\frac{1+p}{2} - (p(np+1)+\right.\right.$$
$$\left.\left.\times p(-np)+p^2)\right)\left(\frac{1-\sqrt{p}\ x}{2p}\right)\frac{d}{dx} + np(np+1)\right]y = 0.$$

This finally simplifies to the elegant form:

$$\left[(1-\sqrt{p}\ x)(1+\sqrt{p}\ x)\frac{d^2}{dx^2} - (1+p)x\frac{d}{dx} + n(np+1)\right]y = 0. \tag{3.2}$$

We thereby obtain a family of ODEs (3.2) having the solutions $P_{n,p}(x)$ for a given p.

4 Rodrigues Formula

This formula expresses the polynomial as the n-th derivative of certain function. This formula is a state of art; it plays a typical role in introducing a polynomial [30], or in deriving an explicit form of a polynomial [26, Eq. (9), p. 162]. Our objective here is to apply this formula to normalize our p-polynomial. For that, we use the series transformation as stated in the following lemma [17, Lemma 2.1, p. 189].

Lemma 4.1. *For* $p>0$, *and* $|z|<\min.\{1,\frac{1}{p}\}$,

$$
{}_2F_1^p\left[\begin{array}{ccc} a, & b; & z \\ & c; & \end{array}\right]=(1-pz)^{-\frac{a}{p}}\,{}_2F_1^p\left[\begin{array}{ccc} a, & c-b; & \frac{-z}{1-pz} \\ & c; & \end{array}\right]. \tag{4.1}
$$

Using this, the Rodrigues formula is obtained in

Theorem 4.1. *For* $p>0,\ n\in\{0\}\cup\mathbb{N}$, *there holds the* n*-th derivative representation:*

$$
\begin{aligned}
P_{n,p}(x) &= \frac{(-1)^n(1)_{n,p}}{2^n\ n!\ (\frac{1+p}{2})_{n,p}}(1-\sqrt{p}\ x)^{1-\frac{p+1}{2p}}(1+\sqrt{p}\ x)^{1-\frac{p+1}{2p}} \\
&\quad\times D^n\left[(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}\right].
\end{aligned} \tag{4.2}
$$

Proof. The hypergeometric form (2.3) under the transformation (4.1), assumes the form:

$$
\begin{aligned}
P_{n,p}(x) &= \frac{p^{-\frac{n}{2}}\ (1)_{n,p}}{n!}\left(\frac{\sqrt{p}\ x+1}{2}\right)^n\sum_{0\le k\le n}\frac{(-np)_{k,p}(\frac{1+p}{2}-np-1)_{k,p}}{p^k(\frac{1+p}{2})_{k,p}k!} \\
&\quad\times(\sqrt{p}\ x+1)^{-k}(\sqrt{p}\ x-1)^k \\
&= \frac{p^{-\frac{n}{2}}\ (1)_{n,p}}{2^n\ n!}\sum_{0\le k\le n}\frac{(-n)_k(p-\frac{1+p}{2}-np)_{k,p}}{(\frac{1+p}{2})_{k,p}\ k!}\ (\sqrt{p}\ x-1)^k(\sqrt{p}\ x+1)^{n-k}.
\end{aligned}
$$

Using the formula (1.7) for $(p-\frac{1+p}{2}-np)_{k,p}$, gives

$$
\begin{aligned}
P_{n,p}(x) &= \frac{p^{-\frac{n}{2}}\ (1)_{n,p}}{2^n\ n!}\sum_{0\le k\le n}\frac{(-1)^k(-n)_k(\frac{1+p}{2})_{n,p}}{(\frac{1+p}{2})_{k,p}(\frac{1+p}{2})_{n-k,p}k!}\ (1+\sqrt{p}\ x)^{n-k}(\sqrt{p}\ x-1)^k \\
&= \frac{p^{-\frac{n}{2}}\ (1)_{n,p}\ (\frac{1+p}{2})_{n,p}}{2^n\ n!\ \Gamma_p(\frac{1+p}{2}+np)}\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k(\frac{1+p}{2})_{n,p}}{(\frac{1+p}{2})_{k,p}(\frac{1+p}{2})_{n-k,p}} \\
&\quad\times(1-\sqrt{p}\ x)^k(\sqrt{p}\ x+1)^{n-k}.
\end{aligned} \tag{4.3}
$$

Now, with $\frac{d}{dx}=D$,

$$
\begin{aligned}
D[(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+(n-1)}] &= -p^{-1/2}\left\{\frac{p+1}{2}+(n-1)p\right\}(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+(n-2)}, \\
D^2[(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+(n-1)}] &= p^{-1}\left\{\frac{p+1}{2}+(n-1)p\right\}\left\{\frac{p+1}{2}+(n-2)p\right\} \\
&\quad\times(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+(n-3)},
\end{aligned}
$$

in general,

$$
\begin{aligned}
&D^{n-k}[(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+(n-1)}] \\
&=(-\sqrt{p}\ )^{n-k}p^{k-n}\left\{\frac{p+1}{2}+(n-1)p\right\}\left\{\frac{p+1}{2}+(n-2)p\right\}\ \cdots\left\{\frac{p+1}{2}+kp\right\} \\
&\times\left\{\frac{p+1}{2}+(k-1)p\right\}\left\{\frac{p+1}{2}+(k-2)p\right\}\cdots\left\{\frac{p+1}{2}\right\}
\end{aligned}
$$

$$\times \frac{(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+(k-1)}}{\left\{\frac{p+1}{2}+(k-1)p\right\}\left\{\frac{p+1}{2}+(k-2)p\right\}\cdots\left\{\frac{p+1}{2}\right\}}$$
$$= (-1)^{n-k}p^{-(n-k)/2}\frac{(\frac{p+1}{2})_{n,p}}{(\frac{p+1}{2})_{k,p}}(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+(k-1)}. \tag{4.4}$$

Proceeding similarly, we arrive at

$$D^k[(1+\sqrt{p}\ x)^{\frac{p+1}{2p}+(n-1)}] \quad = \quad p^{-\frac{k}{2}}\frac{(\frac{p+1}{2})_{n,p}}{(\frac{p+1}{2})_{n-k,p}}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}+(n-k-1)}. \tag{4.5}$$

Using (4.4) and (4.5) in (4.3), gives

$$\begin{aligned} P_{n,p}(x) \quad &= \quad \frac{p^{-\frac{n}{2}}\ (1)_{n,p}}{2^n\ n!\ (\frac{1+p}{2})_{n,p}}\sum_{0\le k\le n}\binom{n}{k}(-1)^k\left[(-1)^{n-k}p^{\frac{n-k}{2}}(1-\sqrt{p}\ x)^{1-\frac{p+1}{2p}}\right.\\ &\quad \left.\times\ D^{n-k}\left\{(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}\right\}\right]\left[p^{\frac{k}{2}}(1+\sqrt{p}\ x)^{1-\frac{p+1}{2p}}\right.\\ &\quad \left.\times D^k\left\{(1+\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}\right\}\right]\\ &= \quad \frac{(-1)^n(1)_{n,p}}{2^n\ n!\ (\frac{1+p}{2})_{n,p}}(1-\sqrt{p}\ x)^{1-\frac{p+1}{2p}}(1+\sqrt{p}\ x)^{1-\frac{p+1}{2p}}\\ &\quad \times\sum_{0\le k\le n}\binom{n}{k}D^{n-k}\left\{(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}\right\}D^k\left\{(1+\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}\right\},\end{aligned}$$

whose simplification proves the theorem. □

Remark 5. The Rodrigues formula: $P_n(x) = (2^n n!)^{-1}\ D^n\{(x^2-1)^n\}$ follows at once from (4.2) when $p = 1$ ([26, Eq. (7), p. 162]).

5 Orthogonality

This property has several applications; for instance, in Lagrange interpolation, least square approximation, moment preserving spline approximation, Gauss quadrature for rational functions, discretization of the measure (for more details see [14, 23]), in mass transfer diffusion equation arising in food engineering [3]. Moreover, the range of orthogonality ensures the location of zeros of the polynomial. Interestingly, the class $\mathcal{P}$ turns out to be orthogonal over the variable interval as seen in the following theorem.

Theorem 5.1. *For $p > 0,\ n, m \in \{0\}\cup\mathbb{N}$ with $n \neq m$,*

$$\int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}}(1-\sqrt{p}\ x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}-1}y_n\ y_m\ dx = 0, \tag{5.1}$$

wherein $y_j = P_{j,p}(x)$.

Proof. We use the notation: $'\prime'$ to denote the derivative with respect to x. The differential equation (3.2) is multiplied by $(1-\sqrt{p}\ x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}-1}$ to get

$$(1-\sqrt{p}\ x)^{\frac{p+1}{2p}}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}}y'' - (1+p)x(1-\sqrt{p}\ x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}-1}y'$$

$$+(1-\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}-1}n(np+1)y=0. \tag{5.2}$$

Noticing that

$$\begin{aligned}
&\left[(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y'\right]' \\
&=(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y''+\left[(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}\left(\frac{p+1}{2p}\right)(-\sqrt{p}\,)(1-\sqrt{p}\,x)^{\frac{p+1}{2p}-1}\right. \\
&\left.+\left(\frac{p+1}{2p}\right)(\sqrt{p}\,)(1+\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}\right]y' \\
&=(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y''+(1-\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(p+1)xy',
\end{aligned}$$

hence the equation (5.2) assumes the form:

$$\left\{(1-\sqrt{p}x)^{\frac{p+1}{2p}}(1+\sqrt{p}x)^{\frac{p+1}{2p}}y'\right\}'+(1-\sqrt{p}x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}x)^{\frac{p+1}{2p}-1}n(np+1)y=0. \tag{5.3}$$

The general Strum-Liouville equation with the operator: $\frac{d}{dx}[s(x)\frac{d}{dx}]+t(x)$, is given by [29, Eq. (12.68), p. 237]

$$[s(x)u'(x)]'+t(x)u(x)+\lambda w(x)u(x)=0.$$

If $s(x)=(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}$, $u(x)=y=P_{n,p}(x)$, $t(x)=0$, $\lambda=n(np+1)$, $w(x)=(1+\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(1-\sqrt{p}\,x)^{\frac{p+1}{2p}-1}$, then this general equation reduces to the equation (5.3). The equation (5.3) gives rise to two equations; for $y=y_n=P_{n,p}(x)$ and for $y=y_m=P_{m,p}(x)$; as respectively stated below.

$$\begin{aligned}
&\left\{(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_n'\right\}'+(1-\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}-1} \\
&\times n(np+1)y_n=0,
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
&\left\{(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_m'\right\}'+(1-\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}-1} \\
&\times m(mp+1)y_m=0.
\end{aligned} \tag{5.5}$$

Now, (5.4) is multiplied by y_m and (5.5) is multiplied by y_n, and then one of the equations is subtracted from the other to get

$$\begin{aligned}
&y_m\left\{(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_n'\right\}'-y_n\left\{(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_m'\right\}' \\
&\qquad+(1-\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}-1}(n-m)(np+mp+1)y_ny_m=0.
\end{aligned} \tag{5.6}$$

Here the first two terms of this last equation can further be simplified by introducing the term $(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_n'y_m'$ as follows.

$$\begin{aligned}
&y_m\left\{(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_n'\right\}'-y_n\left\{(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_m'\right\}' \\
&=\left\{(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_n'y_m-(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}y_m'y_n\right\}' \\
&=\left\{(1-\sqrt{p}\,x)^{\frac{p+1}{2p}}(1+\sqrt{p}\,x)^{\frac{p+1}{2p}}(y_n'y_m-y_m'y_n)\right\}'.
\end{aligned}$$

Using this in (5.6), results in the elegant form:

$$\Big\{(1-\sqrt{p}\ x)^{\frac{p+1}{2p}}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}}(y_n'y_m-y_m'y_n)\Big\}'+(1-\sqrt{p}\ x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}-1}$$
$$\times(n-m)(np+mp+1)y_ny_m=0.$$

Now integrating the last equation with respect to x from a to b, gives

$$\begin{aligned}&(m-n)(np+mp+1)\int_a^b(1-\sqrt{p}\ x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}-1}y_ny_m\ dx\\ &=\Big[(1-\sqrt{p}\ x)^{\frac{p+1}{2p}}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}}(y_n'y_m-y_m'y_n)\Big]_a^b.\end{aligned}$$

For the choice $a=-\frac{1}{\sqrt{p}}$ and $b=\frac{1}{\sqrt{p}}$, the right hand side vanishes. Thus, the orthogonality of the class $\mathcal{P}$ follows for $n\neq m$. □

Remark 6. (1) It is noteworthy that for $p\in(0,1)$, the locations of the zeros get enlarged, whereas for $p>1$, the locations of zeros get diminished.
(2) For $p=1$, (5.1) reduces to the orthogonality of $P_n(x)$ [26, Eq. (7), p. 174].

6 Normalization

This property is pertaining to the evaluation of the orthogonality integral for the same degree of the polynomials. In this section, we normalize the sequence $\{P_{n,p}(x)\}$ by evaluating the integral in (5.1) for $n=m$. We have

Theorem 6.1. *For $y_n=P_{n,p}(x),\ p>0$,*

$$\int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}}\Psi^{(p,n)}(x)\ y_n^2\ dx=1, \tag{6.1}$$

in which the function:

$$\Psi^{(p,n)}(x)=p^{\frac{2-3p}{2p}}\ 2^{-\frac{1}{p}}\frac{n!(1+2np)\ \Gamma(\frac{1}{p})}{(1)_{n,p}\ \left(\Gamma_p(\frac{p+1}{2})\right)^2}\ (1-\sqrt{p}\ x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}-1}.$$

Proof. We use the notation $g(n)$ to denote the integral (5.1) for $n=m$, and use the Rodrigues formula (4.2) for y_n in the integrand, to get

$$\begin{aligned}g(n)&=\int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}}(1-\sqrt{p}\ x)^{\frac{p+1}{2p}-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}-1}y_n^2\ dx\\ &=\frac{(-1)^n(1)_{n,p}}{2^nn!\ (\frac{1+p}{2})_{n,p}}\int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}}\Big[D^n(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}\Big]y_n\ dx.\end{aligned}$$

After carrying out the method of integration by parts n-times, the integral simplifies to

$$g(n)=\frac{(1)_{n,p}}{2^n\ n!\ (\frac{1+p}{2})_{n,p}}\int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}}\Big[(1-\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}(1+\sqrt{p}\ x)^{\frac{p+1}{2p}+n-1}\Big](D^ny_n)\ dx. \tag{6.2}$$

Since $D^n(1-\sqrt{p}\ x)^n = (-1)^n p^{\frac{n}{2}} n!$, we get

$$D^n y_n = p^{-\frac{n}{2}} \frac{(-n)_n (1)_{n,p} (np+1)_{n,p}}{2^n (\frac{1+p}{2})_{n,p} n! n!} D^n (1-\sqrt{p}\ x)^n = \frac{p\ \Gamma_p(2np+1)}{2^n\ \Gamma_p(1)\ (\frac{1+p}{2})_{n,p}}. \tag{6.3}$$

Using (6.3) in (6.2), gives

$$\begin{aligned} g(n) &= \frac{(1)_{n,p}\Gamma_p(2np+1)}{2^{2n}\ n!\ \Gamma_p(1)\ \left((\frac{1+p}{2})_{n,p}\right)^2} \int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}} \left[(1-\sqrt{p}x)^{\frac{p+1}{2p}+n-1}(1+\sqrt{p}x)^{\frac{p+1}{2p}+n-1}\right] dx \\ &= \frac{(1)_{n,p}\Gamma_p(2np+1)}{2^{2n}\ n!\ \Gamma_p(1)\ \left((\frac{1+p}{2})_{n,p}\right)^2} I_n, \end{aligned}$$

say. For transforming the integral I_n to the Beta integral, we put $1+\sqrt{p}x = 2t$ to get

$$\begin{aligned} I_n &= \int_0^1 (2(1-t))^{\frac{p+1}{2p}+n-1} (2t)^{\frac{p+1}{2p}+n-1} \frac{2}{\sqrt{p}}\ dt = \frac{2^{\frac{1}{p}+2n}}{\sqrt{p}} \int_0^1 (1-t)^{\frac{p+1}{2p}+n-1}\ t^{\frac{p+1}{2p}+n-1}\ dt. \\ &= 2^{\frac{1}{p}+2n}\sqrt{p}\ \frac{\Gamma_p(\frac{p+1}{2}+np)\Gamma_p(\frac{p+1}{2}+np)}{\Gamma_p(p+1+2np)}. \end{aligned} \tag{6.4}$$

Thus from (1.9), (1.10) and Lemma 1.1, we obtain after little simplification,

$$g(n) = \frac{p^{\frac{3p-2}{2p}}\, 2^{\frac{1}{p}}\ (1)_{n,p}\ \left(\Gamma_p(\frac{p+1}{2})\right)^2}{n!\ (1+2np)\ \Gamma(\frac{1}{p})}. \tag{6.5}$$

The normalization thus, follows. □

Remark 7. For $p = 1$, this particularizes to $g(n) = \frac{2}{2n+1}$ of the Lagendre polynomial [26, Eq. (12), p. 175].

7 Recurrence Relations

The members $P_{n,p}(x)$ of $\mathcal{P}$ being orthogonal, they satisfy the three-term (or pure) recurrence relation (PRR). This will be derived with the help of the differential recurrence relations (DRRs).

7.1 Differential Recurrence Relations

With the objective of obtaining PRR, we derive the DRRs as follows. The GFR (2.1) when differentiated partially with respect to t and then multiplied by $2pt$, gives

$$2pt(1-p(2xt-t^2))^{\frac{-1-2p}{2p}}(x-t) = 2pt \sum_{n\geq 1} n P_{n,p}(x) t^{n-1}. \tag{7.1}$$

Again the GFR (2.1) is differentiated partially with respect to x and then multiplied by $(1-pt^2)$ to get

$$(1-pt^2)(1-p(2xt-t^2))^{\frac{-1-2p}{2p}} = (1-pt^2) \sum_{n\geq 1} P'_{n,p}(x) t^{n-1}. \tag{7.2}$$

Now, subtracting (7.1) from (7.2), we get

$$\begin{aligned} &(1-p(2xt-t^2))^{\frac{-1-2p}{2p}}\left[(1-pt^2)-2pt(x-t)\right] \\ &= \sum_{n\geq 1} P'_{n,p}(x)t^{n-1} - p\sum_{n\geq 1} P'_{n,p}(x)t^{n+1} - 2p\sum_{n\geq 1} nP_{n,p}(x)t^n. \end{aligned} \tag{7.3}$$

The left hand side function simplifies to the GFR $(1-p(2xt-t^2))^{\frac{-1}{2p}}$, hence in view of (2.1), we obtain the identity

$$\sum_{n\geq 0} P_{n,p}(x)t^n = \sum_{n\geq 1} P'_{n,p}(x)t^{n-1} - p\sum_{n\geq 1} P'_{n,p}(x)t^{n+1} - 2p\sum_{n\geq 1} nP_{n,p}(x)t^n,$$

which leads to the DRR:

$$(2np+1)P_{n,p}(x) = P'_{n+1,p}(x) - p\ P'_{n-1,p}(x). \tag{7.4}$$

Now, in order to derive another DRR, we need the following lemma of a general GFR.

Lemma 7.1. *If $G(2pxt-pt^2) = \sum_{n\geq 0}\mu_n(x)t^n$, then there holds the DRR:*

$$x\mu'_n(x) - n\mu_n(x) = \mu'_{n-1}(x), \quad n\geq 1. \tag{7.5}$$

Proof. The proof runs as follows. With the convention $G(2pxt-pt^2)=G$, the general GFR of the hypothesis is differentiated partially with respect to t, and again the same GFR is differentiated partially with respect to x to get

$$\frac{\partial G}{\partial t} = 2p(x-t)G', \quad \text{and} \quad \frac{\partial G}{\partial x} = 2ptG'.$$

These two derivatives are connected by the partial differential equation:

$$t\frac{\partial G}{\partial t} - (x-t)\frac{\partial G}{\partial x} = 0.$$

This equation with G is replaced by its power series of the hypothesis gives $\mu'_0(x)=0$ and for $n\geq 1$,

$$\sum_{n\geq 1} n\mu_n(x)t^n - x\sum_{n\geq 1}\mu'_n(x)t^n + \sum_{n\geq 1}\mu'_{n-1}(x)t^n = 0.$$

The lemma now follows by comparing the coefficients of the power series both sides. □

In (7.5), the choice $\mu_n(x) = P_{n,p}(x)$ readily provides the relation:

$$xP'_{n,p}(x) = nP_{n,p}(x) + P'_{n-1,p}(x). \tag{7.6}$$

The DRRs (7.4) and (7.6) when combined, yields

$$pxP'_{n,p}(x) = P'_{n+1,p}(x) - (np+1)P_{n,p}(x).$$

In this, replacing n by $n-1$, we get

$$pxP'_{n-1,p}(x) = P'_{n,p}(x) - (np-p+1)P_{n-1,p}(x). \tag{7.7}$$

Substituting $P'_{n-1,p}(x)$ from this DRR in (7.6), we find after some simplification,

$$(px^2-1)P'_{n,p}(x) = npxP_{n,p}(x) - (np-p+1)P_{n-1,p}(x). \tag{7.8}$$

7.2 Pure Recurrence Relation

We proceed to deduce it as follows. We first multiply the DRR (7.6) by (px^2-1) to get

$$x(px^2-1)P^{'}_{n,p}(x) = n(px^2-1)P_{n,p}(x) + (px^2-1)P^{'}_{n-1,p}(x). \tag{7.9}$$

In this, eliminating the derivative terms with the aid of (7.8), we get

$$\begin{aligned} &npx^2 P_{n,p}(x) - npx^2 P_{n,p}(x) + nP_{n,p}(x) = npxP_{n-1,p}(x) - pxP_{n-1,p}(x) \\ &+xP_{n-1,p}(x) + npxP_{n-1,p}(x) - pxP_{n-1,p}(x) - (np-2p+1)P_{n-2,p}(x), \end{aligned}$$

which on simplification, yields the desired recurrence (cf. [26, Eq. (2), p. 160] for $p=1$):

$$nP_{n,p}(x) = (2(n-1)p+1)xP_{n-1,p}(x) - [(n-2)p+1]P_{n-2,p}(x),\ n \geq 2.$$

8 Zeros

From the orthogonality property, it follows that the zeros of $P_{n,p}(x)$ lie within the interval $(-\frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p}})$. We use the MATLAB program for obtaining the zeros. As mentioned in [22, p. 39], the eigen values of a Companion matrix of a monic polynomial (as defined below) are precisely the zeros of that polynomial.

Definition 4. For a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, the companion matrix of order $n \times n$, denoted by $\mathcal{C}(f(x))$, is defined as [22, p. 39]

$$\mathcal{C}(f(x)) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

We consider the monic form:

$$\widetilde{P}_{n,p}(x) = \frac{n!}{2^n(\frac{1}{2})_{n,p}} P_{n,p}(x) = \sum_{0 \leq 2j \leq n} \gamma_{n,j}\ x^{n-2j},$$

of $P_{n,p}(x)$, where

$$\gamma_{n,j} = (-1)^j\ \frac{2^{-2j}\ n!\ (\frac{1}{2})_{n-j,p}}{(\frac{1}{2})_{n,p}\ (n-2j)!\ j!}.$$

With these $\gamma_{n,j}$'s, the companion matrix $\mathcal{C}(\widetilde{P}_{n,p}(x))$ will be same as the above matrix with last row elements: $a_j = \gamma_{n,j}$. The eigen values of this matrix are precisely the zeros of $\widetilde{P}_{n,p}(x)$ [22, Remark, p. 39].

8.1 MATLAB programming

We make an attempt to obtain the zeros as the eigen values of the class $\mathcal{P}$ by assigning the particular values to p and n on the MATLAB platform. The program follows.

```
clear all
```

```
clc
n=input('Enter the value of n:');
p=input('Enter the value of p:');
a=zeros(1,n+1);
for i=1:n+1
nume = (2)^(-2*(k-1)) * ((-1)^(k-1)) * fnp((1/2), (n - (k - 1)), p) * factorial(n);
deno = factorial(n - 2 * (k - 1)) * factorial(k - 1) * fnp((1/2), n, p);
z = ((n + 1) - (2 * (k - 1))) a(1, z) = nume/deno;
end
a
b=fliplr(a)
c=compan(b)
eig(c)
```

Example 1. For the special p-Legendre Polynomial $\widetilde{P}_{4,2}(x)$, the following is the output.
OUTPUT:
Enter the value of n: 4
Enter the value of p: 2
a = [0.0256 0 − 0.4615 0 1.0000]
b = [1.0000 0 − 0.4615 0 0.0256]

$$c = \begin{bmatrix} 0 & 0.4615 & 0 & -0.0256 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

Eigenvalues = 0.6300, -0.6300, -0.2542, 0.2542.

Thus, the zeros of $\widetilde{P}_{4,2}(x)$ are $\pm 0.6300, \pm 0.2542$ in the interval $(-0.7071, 0.7071)$. Similarly, for $n = p = 3$, this program provides the zeros: $0, \pm 0.4804$ of $\widetilde{P}_{3,3}(x)$ in $(-0.57735, 0.57735)$. Likewise, for $n = 4, p = 0.25$, we obtain the zeros: $\pm 1.4548, \pm 0.5324$ of $\widetilde{P}_{4,0.25}(x)$ in $(-2, 2)$.

9 Application: Least square approximation

The class $\mathcal{P}$ of p-Legendre polynomials can be used to approximate a continuous function $F(x)$ by using the Least square approximation method [8, Sec. 8.2, p. 510, Theorem 8.6, p. 515].

Theorem 9.1. *If $\{\mu_j(x); j \in \{0\} \cup \mathbb{N}\}$ is an orthogonal set of polynomials with respect to the weight function $W(x) > 0$ on an interval $[a, b]$, then the least square approximation to a continuous function $F(x)$ on this interval, is given by*

$$\epsilon\left(c_0, c_1, \ldots, c_n\right) = \int_a^b W(x) \left[F(x) - \sum_{0 \le i \le n} c_i \mu_i(x)\right]^2 dx = minimum, \tag{9.1}$$

where the coefficients

$$c_i = \frac{\int_a^b W(x) F(x) \mu_i(x) dx}{\int_a^b W(x) [\mu_i(x)]^2 \, dx}. \tag{9.2}$$

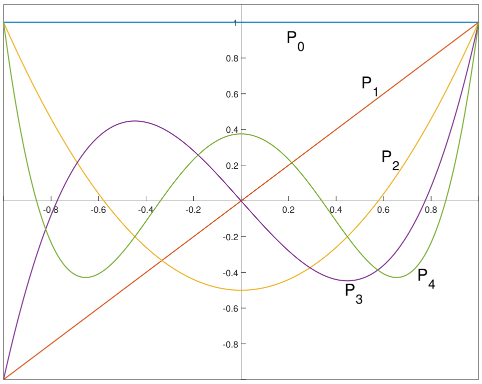

Figure 2: Legendre Polynomial($p = 1$)

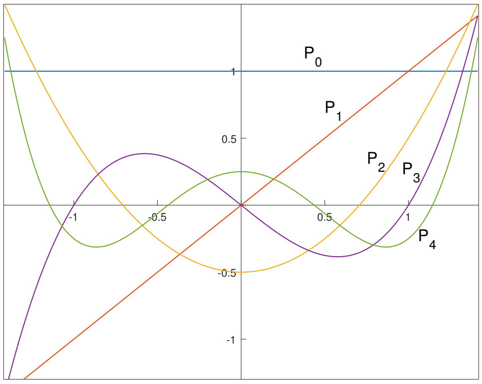

Figure 3: Legendre Polynomial for $p = 0.5$

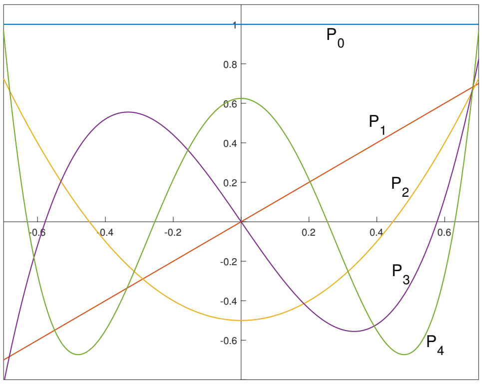

Figure 4: Legendre Polynomial for $p = 2$

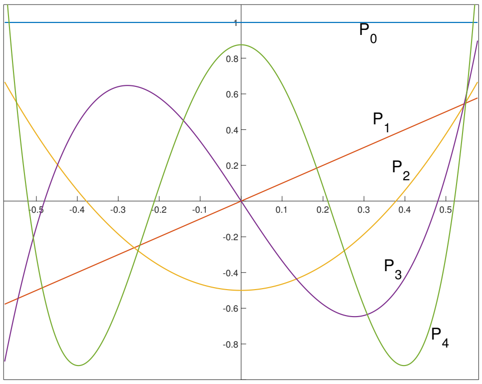

Figure 5: Legendre Polynomial for $p = 3$

These coefficients are determined form (9.1) by solving the PDEs $\partial\epsilon/\partial c_i = 0$ which are necessary conditions to minimize $\epsilon\,(c_0, c_1, \ldots, c_n)$ [8, p. 510]. Following this theorem, we have for the polynomials $P_{n,p}(x)$,

$$\epsilon\,(c_0, c_1, \ldots, c_n) = \int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}} (1 - px^2)^{\frac{p+1}{2p}-1} \left[F(x) - \sum_{0\le i\le n} c_i P_{i,p}(x)\right]^2 dx = minimum, \qquad (9.3)$$

in which

$$c_i = \frac{\int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}} (1 - px^2)^{\frac{p+1}{2p}-1} F(x) P_{i,p}(x)\; dx}{\int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}} (1 - px^2)^{\frac{p+1}{2p}-1} [P_{i,p}(x)]^2\; dx}. \qquad (9.4)$$

We illustrate this theorem by choosing $n = 0, 1, 2$ and $F(x) = x^3$. We record the particular polynomials $P_{0,p}(x) = 1, P_{1,p}(x) = x, P_{2,p}(x) = \frac{1}{2}[(1 + 2p)x^2 - 1]$, and $P_{3,p}(x) = \frac{1}{6}(1 + 2p)(1 + 4p)x^3 - \left(\frac{1}{2} + p\right) x$.

Example 2. Using the p-Legendre polynomials $P_{n,p}(x)$, the function $F(x) = x^3$ can be approximated in the sense of least square on the interval $(-1/\sqrt{p}, 1/\sqrt{p})$ as follows.

We consider $x^3 \approx c_0 P_{0,p}(x) + c_1 P_{1,p}(x) + c_2 P_{2,p}(x)$, then

$$\epsilon(c_0, c_1, c_2) = \int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}} (1-px^2)^{\frac{p+1}{2p}-1} \left[x^3 - c_0 P_{0,p}(x) + c_1 P_{1,p}(x) + c_2 P_{2,p}(x)\right]^2 dx.$$

Now, with $g(n)$ of (6.5), we have from (9.4),

$$c_0 = [g(0)]^{-1} \int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}} (1-px^2)^{\frac{p+1}{2p}-1} x^3 P_{0,p}(x)\ dx,$$
$$c_1 = [g(1)]^{-1} \int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}} (1-px^2)^{\frac{p+1}{2p}-1} x^3 P_{1,p}(x)\ dx,$$
$$c_2 = [g(2)]^{-1} \int_{-\frac{1}{\sqrt{p}}}^{\frac{1}{\sqrt{p}}} (1-px^2)^{\frac{p+1}{2p}-1} x^3 P_{2,p}(x)\ dx.$$

Here, the coefficients $c_0 = 0 = c_2$ since the integrand of their respective integrals being odd functions of x. The evaluation of the integral of coefficient c_1 may be carried out by means of the substitution $px^2 = u$, say. In order to avoid the mixing up of p-Gamma functions and usual Gamma functions in evaluation, we adhere to the Definition 2 and the relation (1.12). Thus, we have

$$\begin{aligned} c_1 &= \frac{1}{p}\frac{\int_0^1 (1-u)^{\frac{p+1}{2p}-1} u^{\frac{3}{2}} du}{\int_0^1 (1-u)^{\frac{p+1}{2p}-1} u^{\frac{1}{2}} du} = \frac{1}{p}\frac{\beta_p(\frac{p+1}{2}, \frac{5p}{2})}{\beta_p(\frac{p+1}{2}, \frac{3p}{2})} = \frac{1}{p}\frac{\Gamma_p(\frac{p+1}{2})\Gamma_p(\frac{5p}{2})}{\Gamma_p(\frac{6p+1}{2})} \cdot \frac{\Gamma_p(\frac{4p+1}{2})}{\Gamma_p(\frac{p+1}{2})\Gamma_p(\frac{3p}{2})} \\ &= \frac{1}{p}\frac{(\frac{3p}{2})\Gamma_p(\frac{3p}{2})}{(\frac{4p+1}{2})\Gamma_p(\frac{4p+1}{2})} \cdot \frac{\Gamma_p(\frac{4p+1}{2})}{\Gamma_p(\frac{3p}{2})} = \frac{3}{4p+1}. \end{aligned}$$

Consequently,

$$x^3 \approx \frac{3}{4p+1} P_{1,p}(x) = \frac{3}{4p+1} x.$$

For $p = 0.7$, $x^3 \approx (0.78947)x$; for $p = 1$, $x^3 \approx (0.6)x$; and for $p = 2$, $x^3 \approx (0.3333)x$.

The graphs in figures 6-8 are in the order of improvement of the approximation.

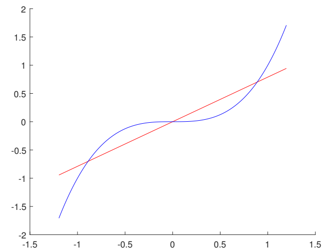

Figure 6: $x^3 \approx c_1 P_{1,0.7}(x)$

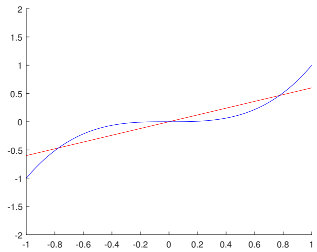

Figure 7: $x^3 \approx c_1 P_{1,1}(x)$

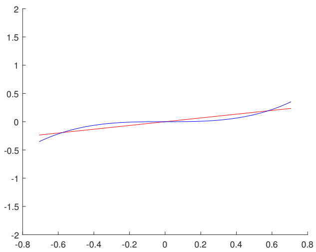

Figure 8: $x^3 \approx c_1 P_{1,2}(x)$

10 Conclusion

(I) The graphs show the behavior of the polynomial for different values of the parameter p. With regard to the figure 1 of graphs of the classical Legendre polynomial, we have the following observations. As the values of p increase, the function value $y = P_{n,p}(x)$ also increase. At the same time the span of graph decrease. To see this, consider the graph in Figure-4. The graph of $P_4 = P_{4,2}(x)$ intersects the x-axis little away from the values $x = \pm 0.2$; whereas in Figure-5, intersection of same graph with x-axis occurs relatively very near to $x = \pm 0.2$. The contrary behave for the values $p < 1$ (Fig. 3).

(II) Graphs in figures 6 through 8 exhibit the approximation of the function $f(x) = x^3$ by the polynomial $P_{1,p}(x)$ with 0.7, 1, and 2. The approximations of the two new classes: $\{P_{n,p}(x); 0 < p < 1\}$ and $\{P_{n,p}(x); p > 1\}$ are illustrated with specific choices of values of p (Fig. 6,8). It can be seen from the flow of the figures $6 \to 7 \to 8$, that the nice approximation is achieved as the values of p increase.

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References

[1] S. Ahmed, Some integrals involving k gamma and k digamma function, Journal of the Egyptian Mathematical Society, **28** (2020), Article number: 39.

[2] I. Ali and S. U. Khan, Threshold of Stochastic SIRS Epidemic Model from Infectious to Susceptible Class with Saturated Incidence Rate Using Spectral Method, Symmetry, **14** (2022), 1–15.

[3] I. Ali and M. T. Saleem, Applications of Orthogonal Polynomials in Simulations of Mass Transfer Diffusion Equation Arising in Food Engineering, Symmetry, 527 (2023), 1–15. doi: https://doi.org/10.3390/sym15020527

[4] F. Anli and S. Gungor, Some useful properties of Legendre polynomials and its applications to neutron transport equation in slab geometry, Applied Mathematical Modelling, **31** Issue 4 (2007), 727–733.

[5] G. B. Arfken and H. J. Weber, Mathematical methods for Physicists, 6^{th} edition, Elsevier Academic Press, London, 2005.

[6] M. Aydin and N. I. Mahmudov, Study of the φ-generalized type k-fractional integrals or derivatives and some of their properties, Turk J. Math., **46** (2022), 1384 – 1386. doi:10.3906/mat-2110-51

[7] I. B. Bapna and R. S. Prajapat, Some applications of extension k-Gamma, k-Beta functions and k-Beta distribution, J. of Ramanujan Society of Mathematics and Mathematical Sciences, **7**, No. 2 (2020), 83–98.

[8] R. L. Burden and J. D. Faires, Numerical Analysis, Brooks/Cole, Cengage Learning, USA, 2011.

[9] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, E. Witten, Quantum fields and strings: A course for mathematicians, American Mathematical Society, 1999.

[10] R. Diaz, E. Pariguan, Symmetric quantum Weyl algebras, Annales Mathematiques Blaise Pascal, **11** (2004), 187–203.

[11] R. Diaz, E. Pariguan, Quantum symmetric functions, Communications in Algebra, **6**(33)(2005), 1947–1978.

[12] R. Diaz and E. Pariguan, On hypergeometric function and Pochhammer k-symbol, Divulgaciones Mathematicas, **15** (2007), 179–192.

[13] D. Dryanov and Q. I. Rahman, On Certain Mean Values of Polynomials on the Unit Interval, Journal of Approximation Theory, **101** (1999), 92 – 120.

[14] W. Gautschi, Orthogonal polynomials: Applications and Computation, Acta Numerica, (1996), 45–119.

[15] A. Gupta and C.L. Parihar, Siago's K-Fractional Calculus Operators, Malaya J. Mat., **5** (3) (2017), 494–504.

[16] M. E. H. Ismail and R. Zhang, A review of multivariate orthogonal polynomials, Journal of the Egyptian Mathematical Society, **25** (2017), 91–110.

[17] N. R. Joshi and B. I. Dave, Orthogonality and other properties of a p-deformed Jacobi polynomial, The Mathematics Student, **92**, Nos. 3-4, (2023), 187–202.

[18] G. S. Khammash, P. Agarwal and J. Choi, Extended k-Gamma and k-Beta Functions of Matrix Arguments, Mathematics **8** (2020), Article No. 1715; doi:10.3390

[19] S. U. Khan and I. Ali, Application of Legendre spectral-collocation method to delay differential and stochastic delay differential equation, AIP Advances, **8** No. 3 (2018), 1–10.

[20] S. U. Khan and I. Ali, Convergence and error analysis of a spectral collocation method for solving system of nonlinear Fredholm integral equations of second kind, Computational and Applied Mathematics, **38** 125, (2019).

[21] W. S. Koon, M. W. Lo, J. E. Marsden and S. D. Ross, Dynamical Systems, the Three-Body Problem and Space Mission Design, 2011.

[22] M. Mignotte and D. Ştefănescu, Polynomials: An Algorithmic Approach, Springer, 1999.

[23] G. V. Milovanovic, Inner product spaces and applications: Orthogonal polynomial systems and some applications, Longman Publisher, 1997.

[24] N. Nazeer, M. I. Asjad, M. K. Azam, and A. Akgül, Study of Results of Katugampola Fractional Derivative and Chebyshev Inequailities, Int. J. Appl. Comput. Math., **8** (2022), 225. doi: 10.1007/s40819-022-01426-x

[25] M. O. Olayiwola, A.F. Adebisi and Y.S. Arowolo, Application of Legendre Polynomial Basis Function on the Solution of Volterra Integro-Differential Equations Using Collocation Method, Çankaya University Journal of Science and Engineering, **17** (2020), 041–051.

[26] E. D. Rainville, Special Functions, Macmillan Co., New York, 1960.

[27] R. V. Savalia and B. I. Dave, A General Inversion pair and p-deformation of Askey Scheme, The Journal of the Indian Math. Soc., **86** (2019), 1–19.

[28] R. V. Savalia, A system of p-polynomials and its q-analogue, The thesis submitted to the Maharaja Sayajirao University of Baroda, 2019.

[29] James B. Seaborn, Hypergeometric Functions and Their Applications, First edition, Springer Science+Business Media, LLC, 1991.

[30] H. M. Srivastava and J. P. Singhal, A class of polynomials defined by generalized Rodrigues' formula, Annali di Matematica Pura ed Applicata, **90** (1971), 75—85.

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