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The adjoint map of Euclidean plane curves and curvature problems

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Abstract. The adjoint map of a pair of naturally parametrized curves in the Euclidean plane is studied from the point of view of the curvature. A main interest is when the given curve and its adjoint curve share the same natural parameter and the same curvature. For the general linear second order differential equation we introduce a function expressing the deformation of curvatures induced by the adjoint map.

Keywords. Parametrized plane curve, Wronskian, adjoint curves, curvature.

1 Introduction

Fix the smooth regular curve $r : I \subseteq \mathbb{R} \to \mathbb{E}^2 = (\mathbb{R}^2, < \cdot, \cdot >_{can})$ having the Wronskians $W(r) > 0$ (hence r is not a line through the origin O of $\mathbb{R}^2$) and $W(r') \neq 0$. Expressing the given curve as $r(\cdot) = (x(\cdot), y(\cdot))$ its component functions x, y are solutions of the Wronskian linear differential equation:

$$\begin{cases} W(x, y, u = u(\cdot)) := \begin{vmatrix} x & y & u \\ x' & y' & u' \\ x'' & y'' & u'' \end{vmatrix} = 0 \to \mathcal{E}^2 : u''(t) + p(t)u'(t) + q(t)u(t) = 0, \\ p := -\frac{[W(r)]'}{W(r)}, \quad q := \frac{W(r')}{W(r)}, \quad \mathcal{E}^2 : \frac{d}{dt}\left(\frac{u'}{W(r)}\right) + \frac{W(r')u}{(W(r))^2} = 0. \end{cases} \tag{1.1}$$

It is well-known that the general solution of (1.1) is provided by two real constants C_1, C_2 through the formula:

$$u(t) = C_1 x(t) + C_2 y(t), \quad C_1 = \frac{W(u, y)}{W(r)}, \quad C_2 = \frac{W(x, u)}{W(r)}. \tag{1.2}$$

For further use, let $P = P(t)$ be the anti-derivative of the first coefficient function $p = p(t)$ and $k = k(t)$ the usual curvature of r; we suppose that r has no inflexion points, so $k > 0$ or $k < 0$ on I. A main hypothesis of this short note is that t is a natural parameter for r; then $I = (0, L(r))$

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with $L(r)$ the length of r and the module $|W(r)|$ is the Euclidean distance form the origin O to the tangent line of the curve. The two functions above are:

$$P = -\ln W(r), \quad k = W(r') = q\exp(-P) \to q \neq 0. \tag{1.3}$$

The second order ordinary differential equation (SODE) $\mathcal{E}^2$ expresses u as solution of the differential operator:

$$\mathcal{D} := \frac{d^2}{dt^2} + p\frac{d}{dt} + q = \sum_{i=0}^{2} \mu_i \frac{d^i}{dt^i} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \tag{1.4}$$

and we recall that any k-differential operator $\mathcal{D} := \sum_{i=0}^{k} \mu_i \frac{d^i}{dt^i}$ has an *adjoint operator* ([3], [6, p. 218]):

$$\mathcal{D}_a := \sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i} (\mu_i \cdot). \tag{1.5}$$

Fix now *the adjoint curve* $r_a(t) = (x_a(t), y_a(t))$ which corresponds to *the adjoint SODE* $\mathcal{E}^2_a$ provided by $\mathcal{D}_a$. Our aim is to study the corresponding curvature transformation $k \to k_a$ due to the fundamental theorem of plane curves ([1, p. 52]), which states the main role of this differential invariant in the geometry of r. Expressing the initial curve in generalized polar coordinates we characterize the curvature-preserving adjoint map and two examples are discussed. Since finding non-selfdual examples of adjoint curves with the same curvature is a difficult problem we introduce a curvature-deformation function of a general linear SODE; again two examples are considered.

2 The adjoint map and curvature problems

We fix now the expression of the initial curve:

$$r(t) := \rho(t)\exp(i\omega(t)), \rho > 0, \quad r'(t) = \rho'(t)\exp(i\omega(t)) + \rho(t)\omega'(t)\exp\left(i\omega(t) + \frac{\pi}{2}\right) \tag{2.1}$$

and hence since t is a natural parameter we have:

$$(\rho')^2 + (\rho\omega')^2 = 1. \tag{2.2}$$

The characterization of the curvature-preserving transformations $r \to r_a$ is provided by:

Theorem 2.1. *Suppose that t is also a natural parameter for the adjoint curve r_a. Then the adjoint map $r \to r_a$ is curvature-preserving if and only if:*

$$\frac{[\ln(\rho\sqrt{1-(\rho')^2})]''}{\rho^2(1-(\rho')^2) - 1} = \frac{[\ln(\rho\sqrt{1-(\rho')^2})]'}{\rho\rho'}. \tag{2.3}$$

Proof. Since t is a natural parameter for r we can express q as function of ρ. Indeed the system (1.1) means:

$$\begin{cases} x'' + px' + qx = 0 \\ y'' + py' + qy = 0. \end{cases} \tag{2.4}$$

Multiplying the first equation with x', the second equation with y' and adding the resulting relations we obtain:

$$q = -\frac{2p}{(\rho^2)'} = -\frac{p}{\rho\rho'}. \tag{2.5}$$

The adjoint SODE of $\mathcal{E}^2$ is:

$$\mathcal{E}^2_a : u''_a(t) - p(t)u'_a(t) + (q(t) - p'(t))u_a(t) = 0 \tag{2.6}$$

and the hypothesis gives the adjoint curvature:

$$k_a = (q - p')\exp(P). \tag{2.7}$$

The equality $k = k_a$ (which implies $q \neq p'$) combined with (2.5) reads as:

$$q = \frac{P''}{1 - \exp(-2P)} = -\frac{P'}{\rho\rho'}. \tag{2.8}$$

From (2.1) we obtain the Wronskian of the initial curve:

$$W(r) = \rho^2\omega' \tag{2.9}$$

which, via (2.2), means:

$$W(r) = \rho\sqrt{1 - (\rho')^2} \tag{2.10}$$

and then:

$$P = -\ln(\rho\sqrt{1 - (\rho')^2}). \tag{2.11}$$

Replacing this last formula into (2.8) yields the claimed relation (2.3). □

Example 2.2 The degenerate case of the characterization (2.3) is provided by the constancy of P, equivalently, from (1.2), the constancy of $W(r)$. From (2.9) we have:

$$\rho^2\omega' = constant \tag{2.12}$$

which is exactly the Kepler second law ([2, p. 235]). This means that r is a central conic and it is well-known that the operator (1.3) is self-adjoint (i.e. $p = 0$) if and only if r is projectively equivalent to a conic ([6, p. 220]). In particular, if $\rho = constant = R > 0$ then the condition (2.2) gives $\omega(t) = \frac{t}{R}$ i.e. we have the well-known natural parametrization of the circle $\mathcal{C}(O, R)$. □

Example 2.3 We can start now directly with a naturally parametrized curve. It is the Cornu spiral ([1, p. 54]) $r : (0, +\infty) \to \mathbb{E}^2$:

$$r(t) = \left(\int_0^t \cos(u + \frac{u^2}{2})du, \int_0^t \sin(u + \frac{u^2}{2})du\right), \quad k(t) = t + 1 > 1 \tag{2.13}$$

for which we obtain the coefficient functions:

$$\begin{cases} P(t) = -\ln\left[\sin(t + \frac{t^2}{2})\int_0^t \cos(u + \frac{u^2}{2})du - \cos(t + \frac{t^2}{2})\int_0^t \sin(u + \frac{u^2}{2})du\right], \\ q(t) = (t + 1)\left[\sin(t + \frac{t^2}{2})\int_0^t \cos(u + \frac{u^2}{2})du - \cos(t + \frac{t^2}{2})\int_0^t \sin(u + \frac{u^2}{2})du\right]^{-1}. \end{cases} \tag{2.14}$$

Due to the very complicated computations we use the Wolfram Alpha to find the SODE satisfies by the components of r. Unfortunately, this free software provides not a SODE but a third order differential equation:

$$(1 + t)U''' - U'' + (1 + t)^3U' = 0. \tag{2.15}$$

In fact, for a naturally parametrized plane curve it is well-known to satisfy the third order differential equation:

$$\mathcal{E}^3 : kU''' - k'U'' + k^3U' = 0 \tag{2.16}$$

so, the relation (2.15) is exactly a recognition of this fact. The adjoint equation to (2.16) is:

$$\mathcal{E}_a^3 : U_a''' + \frac{k'}{k}U_a'' + \left[2\left(\frac{k'}{k}\right)' + k^2\right]U_d' + \left[\left(\frac{k'}{k}\right)'' + 2kk'\right]U_a = 0. \tag{2.17}$$

□

Example 2.4 Trying to connect the SODE (1.1) with the third order differential equation (2.16) we derive $\mathcal{E}^2$ in order to obtain $d\mathcal{E}^2 = \mathcal{E}^3$; then the equality of the coefficients with (2.16) means:

$$\begin{cases} p = -\frac{k'}{k}, \\ p' + q = k^2, \\ q' = 0. \end{cases} \tag{2.18}$$

Therefore, both coefficients p, q are given as functions of k, which is supposed to be strictly positive:

$$q = k^2 + \left(\frac{k'}{k}\right)' = constant = C, \quad P = \ln\frac{1}{k}. \tag{2.19}$$

The choice $C = 1$ is inspired by the hypothesis of natural parameter for t; then the above second order non-linear differential equation in the unknown k has an implicit solution, provided by Wolfram Alpha:

$$\mathcal{C} + t = \int_1^{k(t)} \frac{du}{\sqrt{\mathcal{C}u^2 - u^4 + 2u^2\ln u}}, \quad \mathcal{C} \in \mathbb{R}. \tag{2.20}$$

We study now the same problem for the adjoint equations i.e. when $d\mathcal{E}_a^2 = \mathcal{E}_a^3$. The derivative of $\mathcal{E}_a^2$ is:

$$u_a''' - pu_a'' + (q - 2p')u_a' + (q' - p'')u_a = 0 \tag{2.21}$$

and this equation coincides with (2.17) if and only if:

$$p = -\frac{k'}{k}, \quad q = k^2. \tag{2.22}$$

It follows then that (1.1) is similar to (2.16) but as second order differential equation and not as a third order one. Comparing (2.22) with the initial expression (1.1) of the coefficient functions p, q it results:

$$W(r) = \mathcal{C}k, \quad W(r') = \mathcal{C}k^3, \quad \mathcal{C} \in \mathbb{R}^*. \tag{2.23}$$

The supplementary hypothesis of natural parametrization for r reduces the equations above to $k^2 = \frac{1}{\mathcal{C}}$ and then r is a circle and (1.1) is a self-adjoint SODE. □

Let us recall now some known facts concerning the transformation of parameter in a linear SODE $\mathcal{E} : u'' + p' + qu = 0$. Let $\tilde{t} = \tilde{t}(t)$ such a change of parameter; then the new linear SODE is:

$$u_{\tilde{t}\tilde{t}} + \tilde{p}u_{\tilde{t}} + \tilde{q}u = 0, \quad p = \tilde{t}'\tilde{p} - \frac{\tilde{t}''}{\tilde{t}'}, \quad q = (\tilde{t}')^2\tilde{q}. \tag{2.24}$$

It follows two *relative invariants*:

$$\sqrt{q}dt = \sqrt{\tilde{q}}d\tilde{t}, \quad \left(2p + \frac{q'}{q}\right)dt = \left(2\tilde{p} + \frac{\tilde{q}_{\tilde{t}}}{\tilde{q}}\right)dt \tag{2.25}$$

and therefore we have *the absolute invariant*:

$$I(\mathcal{E}) := \frac{2p + \frac{q'}{q}}{\sqrt{q}} = \frac{2pq + q'}{q^{\frac{3}{2}}}. \tag{2.26}$$

Hence two linear SODEs, $\mathcal{E}$ and $\tilde{\mathcal{E}}$, are equivalent through a transformation $\tilde{t} = \tilde{t}(t)$ if and only if $I(\mathcal{E}) = I(\tilde{\mathcal{E}})$. For the two parts of the example 2.4 above we have:

$$I\left(\mathcal{E}\left(p = -\frac{k'}{k}, q = 1\right)\right) = 2p = -2\frac{k'}{k}, \quad I\left(\mathcal{E}\left(p = -\frac{k'}{k}, q = k^2\right)\right) = 0. \tag{2.27}$$

It is worth to remark that in both cases studied in the example 2.4 the first coefficient function is $p = -\frac{k'}{k}$. In this case we can perform a change of parameter which reduces the SODE $\mathcal{E}^2$ to a simpler form. Namely, let $\tau = \tau(t)$ be *the structural angle function* of r i.e. $\tau(t) = \int_{t_0}^{t} k(u)du$; then $r'(t) = \exp(i\tau(t))$. For this new parameter we have:

$$r' = k\frac{dr}{d\tau}, \quad r'' = k^2\frac{d^2r}{d\tau^2} + k'\frac{dr}{d\tau} \tag{2.28}$$

and then the SODE $\mathcal{E}^2 : r'' - \frac{k'}{k}r' + q(t)r(t) = 0$ reduces to:

$$k^2(t(\tau))\frac{d^2r}{d\tau^2}(\tau) + q(t(\tau))r(t(\tau)) = 0. \tag{2.29}$$

We finish this section with an example showing again the complexity of finding $\mathcal{E}^2$ even for a simple curve.

Example 2.5 The well-known catenary curve is ([4]):

$$r(t) = (\ln(t + \sqrt{1+t^2}), \sqrt{1+t^2}) \tag{2.30}$$

and then:

$$\begin{cases} k(t) = \frac{1}{1+t^2} \in (0,1), \tau(t) = \arctan t \to r'(\tau) = \exp(i\tau), \\ W(r)(t) = \frac{t\ln(t+\sqrt{1+t^2})}{\sqrt{1+t^2}} - 1, \\ W(r)(\tau) = \sin\tau \ln\frac{1+\sin\tau}{\cos\tau} - 1 = \sin\tau\ln\frac{\cos\frac{\tau}{2}+\sin\frac{\tau}{2}}{\cos\frac{\tau}{2}-\sin\frac{\tau}{2}} - 1. \end{cases} \tag{2.31}$$

It results:

$$\begin{cases} p(t) = -\frac{1}{t}\left[\frac{1}{1+t^2} + \frac{\sqrt{1+t^2}}{t\ln(t+\sqrt{1+t^2})-\sqrt{1+t^2}}\right], \\ q(t) = \frac{1}{\sqrt{1+t^2}[t\ln(t+\sqrt{1+t^2})-\sqrt{1+t^2}]}. \end{cases} \tag{2.32}$$

Obviously, considered separately the functions x, y satisfy more simple SODE. For y we have $y' = \frac{t}{1+t^2}y$ which by derivation means:

$$y'' - \frac{t}{1+t^2}y' + \frac{t^2-1}{(1+t^2)^2}y = 0. \tag{2.33}$$

The adjoint SODE of this last SODE is $y_a'' + \frac{t}{1+t^2}y_a' = 0$ with the solution $y_a(t) = x(t) = \ln(t+\sqrt{1+t^2})$. $\square$

3 The curvature-deformation function of a linear SODE

Motivated by the complexity to find curvature-preserving adjoint maps, in this section we introduce a measure of the difference of curvatures. It is worth to point out that this notion works directly for a SODE (1.1), irrespective if it represents or not a given plane curve.

Definition 3.1 The *curvature-deformation function* of the linear SODE $\mathcal{E}$ expressed in (1.1) is the smooth function $CD(\mathcal{E}) : I \to \mathbb{R}$:

$$CD(\mathcal{E}) := q\exp(-P) - (q - p')\exp(P), \quad P = \int p. \tag{3.1}$$

It follows directly:

$$CD(\mathcal{E}) = P''\exp(P) - 2q\sinh(P) \tag{3.2}$$

and then given a pair of smooth functions $(p, \mathcal{F})$ there exists a unique q such that $CD(\mathcal{E}) = \mathcal{F}$, namely:

$$q = \frac{P''\exp(P) - \mathcal{F}}{2\sinh(P)}. \tag{3.3}$$

Example 3.2 Fix the constants α, β, γ and *the Gauss hypergeometric equation* with $I = (1, +\infty)$:

$$\mathcal{E}(\alpha, \beta, \gamma) : t(t-1)u'' + [(\alpha + \beta + 1)t - \gamma]u' + \alpha\beta u = 0. \tag{3.4}$$

For $\alpha + \beta = \gamma = 1$ we obtain $P(t) = \ln(t^2 - t)$ and finally:

$$CD(\mathcal{E})(t) = \alpha\beta\left[\frac{1}{(t^2 - t)^2} - 1\right] - 2 - \frac{1}{t^2 - t}. \tag{3.5}$$

The adjoint SODE to the particular hypergeometric SODE ($\alpha + \beta = \gamma = 1$) is:

$$\mathcal{E}_a(\alpha + \beta = 1 = \gamma) : t(t-1)u_a'' - (2t-1)u_a' + \left(\alpha\beta + 2 + \frac{1}{t^2 - t}\right)u_a = 0. \tag{3.6}$$

The absolute invariant of our particular hypergeometric SODE is (supposing $\alpha\beta > 0$):

$$I(\alpha + \beta = 1 = \gamma) = \frac{(2t-1)(t^2 - t)}{\sqrt{\alpha\beta}} > 0. \tag{3.7}$$

□

Example 3.3 Let $\mathcal{E} : u'' = \Lambda(u', u, t)$ be a general SODE. In the paper [5] is considered a Wunschmann-type condition for it:

$$\Lambda_{ut} + \Lambda_{uu}u' + \Lambda_{uu'}\Lambda = 2\Lambda_u\Lambda_{u'}. \tag{3.8}$$

This condition means that on the two-dimensional manifold of solutions of $\mathcal{E}$ (conform (1.2)) there exists a diagonal Riemannian (or semi-Riemannian) metric satisfying the Hamilton-Jacobi equation. If $\mathcal{E}$ is a linear one:

$$\Lambda(u', u, t) := -p(t)u' - q(t)u \tag{3.9}$$

then the relation (3.8) means:

$$-q' = 2pq \to I(\mathcal{E}) = 0 \tag{3.10}$$

and then supposing $q > 0$ it results $P = \ln\frac{1}{\sqrt{q}}$. We compute the curvature-deformation function in terms of q:

$$CD(\mathcal{E}) = q^{\frac{3}{2}} - q^{\frac{1}{2}} - \frac{1}{2q}\left(\frac{q'}{q}\right)'. \tag{3.11}$$

Then the case $q = 1$ discussed in the previous section gives a vanishing $CD(\mathcal{E})$. □

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