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Exploring helices in Minkowski 3-space $\mathbb{E}_1^3$

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Abstract. This study focuses on the geometric exploration of helices in Minkowski 3-space. For this purpose, we study the problem of constructing a spacelike general helix, slant helix, or Darboux helix with a timelike principal from a given plane curve, fixed vector, and constant angle. We obtain a parametric representation for the helices whose projection is onto the plane P perpendicular to the fixed vector U share the same fixed vector. In addition, we furnish various illustrative examples showcasing the geometric characteristics of these helices.

Keywords. General helix, slant helix, Darboux helix, position vector, Minkowski 3-space

1 Introduction

The study of spatial curves, particularly helices, holds a pivotal and captivating place within the realm of mathematics and natural sciences. Helices have garnered extensive attention because of their versatile applications in various fields, ranging from the structural biology of DNA to the nanoscale world of carbon nanotubes, as well as mechanical components such as screws and springs. Beyond their tangible applications, helix curves and helical structures have proven indispensable in scientific disciplines, including fractal geometry, computer-aided design, and computer graphics. These applications extend to tool path descriptions, kinematic motion simulations, and even highway design, as exemplified in previous studies [5], [8], [9], and [24].

In the realm of differential geometry, a *general helix* is defined as a geometric curve with the distinctive property that its tangent vector field maintains a consistent angle with a fixed straight line, which serves as the axis of the general helix, in Euclidean 3-space $\mathbb{E}^3$. This characteristic was first explained by M.A. Lancretin in 1802 and subsequently validated by B. de Saint Venant in 1845 (additional information can be found in the reference [14] and [15]). It asserts that a curve can be classified as a general helix if and only if the ratio (τ/κ) remains constant along the curve, where κ and τ represent the curvature and torsion of the curve, respectively. Furthermore, it is worth noting that a curve is designated as a *circular helix* when both curvatures, κ and τ, are non-zero constants.

In Euclidean 3-space, a new type of helix called a *slant helix* is defined as a curve with a non-vanishing curvature whose principal normal vector N makes a constant angle with a fixed

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direction. This concept was first articulated by Izumiya and Takeuchi, who also outlined various properties of the slant helix (see [13]). In addition, in [20], Şenol defined special helices referred to as *Darboux helices*. These helices are characterized by the property that their Darboux vector makes a constant angle with a fixed straight line. Şenol provided a comprehensive characterization of the Darboux helix. Motivated by what happens in the Euclidean ambient space, several authors have undertaken similar investigations in the realm of Lorentzian spaces, including Lorentzian space forms [1, 6, 11, 16, 18, 23]

Determining the position vector for various curves is a significant topic of study. In Euclidean space $\mathbb{E}^3$, Ali recently derived the parametric representation of general and slant helices ([3, 4]). In Minkowski 3-space, Ali also conducted research on the position vectors of spacelike general helices ([2]). Also, Uçum *et al.* et al. delved into the characteristics of spacelike helices, timelike helices, pseudo-null helices, and null Cartan helices with spacelike slope axes. Their paper not only establishes conditions for curves to possess spacelike slope axes but also offers parametric equations for these curve types ([21]). Furthermore, general helices with timelike or lightlike slope axes in Minkowski 3-space were studied by the authors in [7, 22]

In this study, we geometrically approach helices in Minkowski $3-$space using a spacelike curve with a timelike principal normal. For this purpose, we study the problem of constructing a general helix, slant helix, or Darboux helix from a given plane curve, fixed vector, and constant angle. We obtain a parametric representation of the helices. Moreover, we provide various examples of these helices and illustrate their figures using *Mathematica*©.

2 Basic concepts

The Minkowski space, denoted as $\mathbb{R}^3_1$, is define it as a real vector space $\mathbb{R}^3$ equipped with the standard indefinite flat metric given by:

$$\langle x, y \rangle_L = -x_1y_1 + x_2y_2 + x_3y_3,$$

where x and y are vectors in $\mathbb{R}^3_1$. Because of this indefinite metric, vectors in $\mathbb{R}^3_1$ can be classified as "spacelike," "timelike," or "null" (also known as "lightlike") depending on whether $\langle x, x \rangle_L$ is greater than 0, less than 0, or equal to 0. Notably, the zero vector is considered spacelike. The norm (or length) of vector $x \in \mathbb{R}^3_1$ is defined as $\|x\| = \sqrt{|\langle x, x \rangle_L|}$. The two vectors x, $y \in \mathbb{R}^3_1$ are orthogonal if and only if $\langle x, y \rangle_L = 0$, as mentioned in [17].

Next, the "vector product" of two vectors u and v in $\mathbb{R}^3_1$ is defined as follows ([23]):

$$u \times_L v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Now, let us summarize Lemmas 2.1 and 2.2, which were cited from [17] and [19]:

Lemma 2.1. *In Minkowski 3-space $\mathbb{R}^3_1$, the following properties hold:*
i) Timelike vectors are never orthogonal.
ii) Null vectors are orthogonal if and only if they are linearly dependent.
iii) A timelike vector is never orthogonal to a null vector.

Lemma 2.2. *(i) For two timelike vectors u and v in $\mathbb{R}^3_1$, there exists a unique real number $\theta \geqslant 0$such that*

$$\langle u, v \rangle_L = \|u\| \, \|v\| \cosh \theta,$$

where θ is called the "Lorentzian timelike angle" between u and v.
(ii) For two spacelike vectors u and v in $\mathbb{R}^3_1$ that span a spacelike vector subspace, there exists a unique real number $\theta \in [0,\pi]$ such that

$$\langle u,v\rangle_L = \|u\|\,\|v\|\cos\theta,$$

where θ is called the "Lorentzian spacelike angle" between u and v.
(iii) For two spacelike vectors u and v in $\mathbb{R}^3_1$ that span a timelike vector subspace, there exists a unique real number $\theta \in [0,\pi]$ such that

$$\langle u,v\rangle_L = \|u\|\,\|v\|\cosh\theta,$$

where θ is called the "Lorentzian timelike angle" between u and v.
(iv) For a spacelike vector u and a timelike vector v in $\mathbb{R}^3_1$, there exists a unique real number $\theta \geqslant 0$ such that

$$\langle u,v\rangle_L = \|u\|\,\|v\|\sinh\theta.$$

where θ is called the "Lorentzian timelike angle" between u and v.

Now, we consider curves in the Minkowski 3-space $\mathbb{R}^3_1$. A curve $\alpha : I \to \mathbb{R}^3_1$ can be categorized as "spacelike", "timelike", or "null" ("lightlike") depending on the nature of its velocity vectors, as described in [17]. If $\langle\alpha',\alpha'\rangle_L = \pm 1$ for all values of s, the non-null curve α is referred to as a "unit speed" curve or one parameterized by the arc length function s.

We introduce the Frenet frame denoted as $\{T,N,B\}$, which moves along the curve α. Here, T is the tangent vector field, N is the principal normal vector field, and B is the binormal vector field of α. Depending on the causality of the curve α, we have the following Frenet formulae ([17, 14, 23]):

For a "non-null curve with the non-null principal normal" in $\mathbb{R}^3_1$, the Frenet formulae are:

$$T' = \epsilon_N \kappa N, \quad N' = -\epsilon_T \kappa T - \epsilon_T \epsilon_N \tau B, \quad B' = -\epsilon_N \tau N, \tag{2.1}$$

where

$$\begin{gathered}\langle T,T\rangle_L = \epsilon_T = \pm 1, \quad \langle N,N\rangle_L = \epsilon_N = \pm 1, \quad \langle B,B\rangle_L = -\epsilon_T\epsilon_N, \\ \langle T,N\rangle_L = \langle T,B\rangle_L = \langle N,B\rangle_L = 0,\end{gathered} \tag{2.2}$$

and

$$T \times_L N = -\epsilon_T \epsilon_N B, \quad N \times_L B = \epsilon_T T, \quad B \times_L T = \epsilon_N N. \tag{2.3}$$

Here, the functions $\kappa = \kappa(s)$ and $\tau = \tau(s)$ represent the curvature and torsion of the curve α, respectively.

For a "pseudo null curve" in $\mathbb{R}^3_1$ (a spacelike curve with a non-null principal normal), the Frenet formulae are:

$$T' = \kappa N, \quad N' = \tau N, \quad B' = -\kappa T - \tau B, \tag{2.4}$$

where

$$\begin{gathered}\langle T,T\rangle_L = 1, \quad \langle N,N\rangle_L = \langle B,B\rangle_L = 0, \\ \langle T,N\rangle_L = \langle T,B\rangle_L = 0, \quad \langle N,B\rangle_L = 1,\end{gathered} \tag{2.5}$$

and

$$T \times_L N = N, \quad N \times_L B = T, \quad B \times_L T = B. \tag{2.6}$$

In this case, the curvature κ takes values of either 0 for a straight line or 1 for all other cases

For a "null (Cartan) curve" in $\mathbb{R}_1^3$, the Frenet formulae are defined with respect to a parameterization using a "pseudo-arc" s, i.e., $\langle \alpha''(s), \alpha''(s)\rangle_L = 1$:

$$T' = \kappa N, \quad N' = \tau T - \kappa B, \quad B' = -\tau N, \tag{2.7}$$

where

$$\begin{aligned} \langle T,T\rangle_L = \langle B,B\rangle_L = 0, \quad \langle N,N\rangle_L = 1, \\ \langle T,N\rangle_L = \langle N,B\rangle_L = 0, \quad \langle T,B\rangle_L = 1, \end{aligned} \tag{2.8}$$

and

$$T \times_L N = T, \quad N \times_L B = B, \quad B \times_L T = N. \tag{2.9}$$

In this case, the curvature κ takes values of either 0 for a straight line or 1 for all other cases.

Finally, we deals with curves γ parametrized by the arc-length s_γ in the Lorentzian plane $\mathbb{R}_1^2$. The Frenet frame $\{T_\gamma, N_\gamma\}$ along the curve γ is introduced, where $T = \dot{\gamma} = \frac{d\gamma}{ds_\gamma}$ and $N = \ddot{\gamma}$ are the tangent and principal normal vectors, respectively. The curvature of the curve is denoted as k_γ, and the slope angle is ϕ. The Frenet formulae are as follows:

$$\dot{T}_\gamma = k_\gamma N_\gamma, \quad \dot{N}_\gamma = k_\gamma T_\gamma, \tag{2.10}$$

where

$$\langle T_\gamma, T_\gamma\rangle_L = \langle N_\gamma, N_\gamma\rangle_L = \epsilon = \pm 1, \quad \langle T_\gamma, N_\gamma\rangle_L = 0,$$

and

$$\frac{d\phi}{ds_\gamma} = k_\gamma. \tag{2.11}$$

Remarkably, the representation of the Frenet formulae remains consistent regardless of whether the curve is spacelike or timelike, as discussed in [10].

3 Spacelike helices with timelike principal normal in Minkowski 3-space $\mathbb{R}_1^3$

In this section, we study how to find the position vector of a helix, slant helix, or Darboux helix using a plane curve, fixed vector, and a constant angle. Additionally, we provide insights into the curvature of these helices and present illustrative examples.

Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{R}_1^3$ denote a unit speed spacelike curve with the timelike principal normal N in $\mathbb{R}_1^3$. In the following, we explore three scenarios: α as a general helix, a slant helix, or a Darboux helix.

3.1 General helices

Consider $\alpha(s)$ as a general helix with a unit fixed axis U in $\mathbb{R}_1^3$, satisfying the following condition:

$$\langle T(s), U\rangle_L = c, \tag{3.1}$$

and the fixed axis U of $\alpha(s)$ can then be expressed as:

$$U = cT(s) + u_2(s) N(s) + u_3(s) B(s), \tag{3.2}$$

where u_2 and u_3 are differentiable functions of s, and $c \in \mathbb{R} \setminus \{0\}$.

Differentiating (3.1) with respect to s, and using (2.1) for $\epsilon_T = -\epsilon_N = 1$, we obtain:

$$\langle N(s), U\rangle_L = u_2(s) = 0 \tag{3.3}$$

Substituting (3.3) into (3.2), we get:

$$U = cT(s) + u_3(s) B(s). \tag{3.4}$$

Differentiating (3.4) with respect to s and using (2.1), we have the following system:

$$\begin{cases} u_3' = 0, \\ -c\kappa + u_3\tau = 0. \end{cases}$$

This system implies that $u_3(s) = constant \neq 0$ and $\frac{\tau}{\kappa} = \frac{c}{u_3} = constant \neq 0$.

Now, we project the curve α onto the plane P through the point $\alpha(0)$ and perpendicular to the unit vector U.

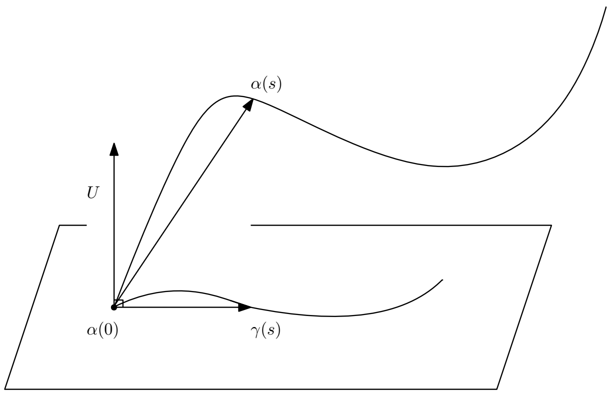

Figure 1: The curve α and its projection γ onto the plane P through the point $\alpha(0)$ and perpendicular to the unit vector U

From Figure 1, we observe that the position vector of α relative to the chosen origin $\alpha(0)$ can be expressed as follows:

$$\alpha(s) = \gamma(s) + \delta_U(s) U. \tag{3.5}$$

Here, γ is the projection of α onto the plane P, and δ_U is a function of s. Notably, since γ lies in the plane P with unit normal vector U, the vectors γ' and γ'' are orthogonal to the vector U. Consequently, based on Lemma 2.1, we conclude that U is a spacelike vector.
Differentiating relation (3.5) with respect to s, we obtain:

$$T = \gamma' + \frac{d\delta_U}{ds} U.$$

Taking the scalar product with U, we derive:

$$\langle T, U\rangle_L = \langle \gamma', U\rangle_L + \langle U, U\rangle_L \frac{d\delta_U}{ds}. \tag{3.6}$$

As $\langle \gamma', U\rangle_L = 0$ for all s, the last equation simplifies to:

$$\frac{d\delta_U}{ds} = \langle T, U\rangle_L. \tag{3.7}$$

Further, from part (ii) of Lemma 2.2, we have:

$$\langle T(s), U\rangle_L = \cos\theta = const.,$$

and

$$U = \cos\theta\, T(s) + \sin\theta\, B(s),$$

where $\theta = \theta(s) \in (0,\pi)$ represents the constant angle between T and U. Additionally, the curvatures κ and τ of the curve α satisfy the following relation:

$$\frac{\tau}{\kappa} = \cot\theta. \tag{3.8}$$

From the relations (3.6) and (3.7), we conclude that:

$$\frac{d\delta_U}{ds} = \cos\theta,$$

which remains constant for all $s \in I$. Integrating the above expression with the initial condition $\alpha(0)$ yields:

$$\delta_U(s) = s\cos\theta, \tag{3.9}$$

is a real-valued function of s.

Substituting relation (3.9) into relation (3.5), we find:

$$\alpha(s) = \gamma(s) + s\cos\theta U. \tag{3.10}$$

This implies that α can be expressed using the given γ, U and θ.

Moreover, the parametrization (3.10) of γ cannot be a unit speed parametrization of γ. Consequently, differentiating relation (3.10) with respect to s, we obtain:

$$\frac{d\gamma}{ds} = \gamma'(s) = T - \cos\theta U,$$

where

$$\|\gamma'\| = \sin\theta.$$

As a result, γ is a spacelike curve in the plane P, and the arc-length function s_γ of γ is:

$$s_\gamma = \int_0^s \|\gamma'(s)\|\, ds = s\sin\theta. \tag{3.11}$$

We now have the unit speed tangent vector T_γ of γ as:

$$T_\gamma = \frac{\gamma'}{\|\gamma'\|} = \frac{1}{\sin\theta}T - \cot\theta U. \tag{3.12}$$

Differentiating the last relation with respect to s_γ and using (2.1), (3.11), and (3.12), we have:

$$\frac{dT_\gamma}{ds_\gamma} = \frac{dT_\gamma}{ds}\frac{ds}{ds_\gamma} = -\frac{\kappa}{\sin^2\theta}N,$$

and since $\dot{T}_\gamma = k_\gamma N_\gamma$,

$$-\frac{\kappa}{\sin^2\theta}N = k_\gamma N_\gamma.$$

It is evident that the normal vector N_γ of γ is parallel to the vector N, and consequently:

$$\kappa(s) = -k_\gamma \sin^2\theta.$$

From the relation (3.8), we can deduce the torsion of α as follows:

$$\tau(s) = -k_\gamma \sin\theta\cos\theta.$$

Therefore, we can state the following theorem and corollaries:

Theorem 3.1. *Let γ be a spacelike curve parameterized by the arc-length s_γ lies in a plane with the spacelike normal vector U, and let $\theta \in (0,\pi)$ be a constant angle. Then, there is a unit speed spacelike general helix α with a timelike principal normal in $\mathbb{R}_1^3$ such that $\langle T, U\rangle = \cos\theta = const.$ for all s, and its position vector is given by:*

$$\alpha(s) = \gamma(s) + s\cos\theta U,$$

and its arc-length parameter s is given by:

$$s = \frac{s_\gamma}{\sin\theta}.$$

Corollary 3.2. *The curvatures κ and $\tau \neq 0$ of the general helix α in $\mathbb{R}_1^3$ are given by:*

$$\kappa(s) = -k_\gamma \sin^2\theta,$$

and

$$\tau(s) = -k_\gamma \sin\theta\cos\theta,$$

where k_γ is the curvature of γ.

Corollary 3.3. *The general helix α is a circular helix if and only if k_γ is constant.*

Example 1. Consider a plane spacelike curve

$$\gamma(s_\gamma) = \left(\int \sinh\left(\ln\left(s_\gamma+3\right)\right)\, ds_\gamma, \int \cosh\left(\ln\left(s_\gamma+3\right)\right)\, ds_\gamma, 0\right).$$

This curve lies in the plane $z=0$ with unit normal $U=(0,0,1)$. Let us choose $\theta = \frac{\pi}{4}$. Assuming that the curve γ is a projection of the curve α onto the plane $z=0$, we apply Theorem 3.1 to determine the arc-length function s_γ of γ:

$$s_\gamma = s\sin\theta = \frac{\sqrt{2}s}{2},$$

and the position vector of α is found as follows (see Figure 2):

$$\alpha(s) = \left(\int \frac{1}{\sqrt{2}}\sinh\left(\ln\left(\frac{\sqrt{2}s+6}{2}\right)\right)\, ds, \int \frac{1}{\sqrt{2}}\cosh\left(\ln\left(\frac{\sqrt{2}s+6}{2}\right)\right)\, ds, \frac{\sqrt{2}s}{2}\right).$$

Then, the Frenet apparatus of α is obtained as:

$$\begin{aligned}
T &= \left(\tfrac{1}{\sqrt{2}}\sinh\left(\ln\left(\tfrac{\sqrt{2}s+6}{2}\right)\right), \tfrac{1}{\sqrt{2}}\cosh\left(\ln\left(\tfrac{\sqrt{2}s+6}{2}\right)\right), \tfrac{1}{\sqrt{2}}\right),\\
N &= \tfrac{\alpha''}{\|\alpha''\|} = \left(\cosh\left(\ln\left(\tfrac{\sqrt{2}s+6}{2}\right)\right), \sinh\left(\ln\left(\tfrac{\sqrt{2}s+6}{2}\right)\right), 0\right),\\
B &= T\times N = \left(\tfrac{1}{\sqrt{2}}\sinh\left(\ln\left(\tfrac{\sqrt{2}s+6}{2}\right)\right), \tfrac{1}{\sqrt{2}}\cosh\left(\ln\left(\tfrac{\sqrt{2}s+6}{2}\right)\right), -\tfrac{1}{\sqrt{2}}\right),\\
\kappa &= \tfrac{1}{\sqrt{2}s+6},\\
\tau &= \tfrac{1}{\sqrt{2}s+6}.
\end{aligned}$$

Also, from Corollaries 3.2 and 3.3, we can conclude that the curvature of γ is $k_\gamma = -\frac{1}{s_\gamma+3} \neq 0$, and hence, the curve α is a general spacelike helix with the timelike principal normal, but it is not a circular helix.

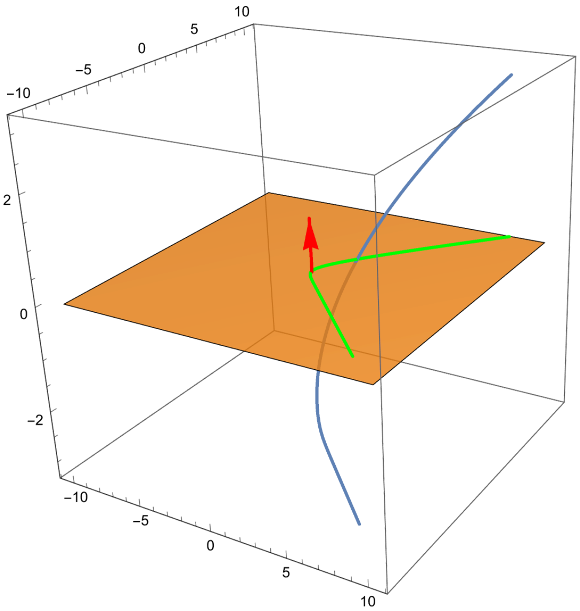

Figure 2: The spacelike general helix α is blue, the plane curve γ is green, the plane $z = 0$ is brown, and the axis U is red.

3.2 Slant helices

Consider a unit speed spacelike slant curve denoted by α with the timelike principal normal N in $\mathbb{R}^3_1$, such that

$$\langle N(s), V\rangle_L = c, \tag{3.13}$$

where V represents the axis of α, and $c \in \mathbb{R}_0 \backslash \{0\}$.
Now, let us define another curve β as the integral curve of the principal normal N of α, given by:

$$\beta = \int N \, ds.$$

As a result, $\beta' = N$, and $\langle \beta', \beta'\rangle_L = -1$, making β a timelike curve with the same arc length as α. Additionally, we can deduce that β is a timelike general helix with the axis V, and its Frenet apparatus can be expressed as:

$$\begin{aligned}
&T_\beta = \beta' = N,\\
&N_\beta = \tfrac{\beta''}{\|\beta''\|} = \tfrac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa T + \tau B\right),\\
&B_\beta = T_\beta \times_L N_\beta = \tfrac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau T + \kappa B\right),\\
&\kappa_\beta = \sqrt{\kappa^2+\tau^2},\\
&\tau_\beta = \tfrac{\kappa^2}{\kappa^2+\tau^2}\left(\tfrac{\tau}{\kappa}\right)'.
\end{aligned} \tag{3.14}$$

Furthermore, the axis$V$ can be represented as

$$V = -cT_\beta(s) + v_2(s) N_\beta(s) + v_3(s) B_\beta(s), \tag{3.15}$$

where v_2 and v_3 are differentiable functions of s, and $c \in \mathbb{R} \backslash \{0\}$. Given that $\langle T_\beta(s), V\rangle_L = c$ is a constant, we can derive:

$$v_2(s) = 0 \quad \text{and} \quad -c\kappa_\beta - v_3\tau_\beta = 0, \tag{3.16}$$

where $v_3(s) = constant \neq 0$ for all s.

Let us consider the projection of the general helix β onto a plane $\bar{P}$ through point $\beta(0)$ and perpendicular to vector V. In this case, there exists a curve $\bar{\gamma}$, parameterized by the arc-length $s_{\bar{\gamma}}$, lying in the plane $\bar{P}$ and having V as its normal vector. This can be expressed as

$$\beta(s) = \bar{\gamma}(s) + \delta_V V, \tag{3.17}$$

where

$$\frac{d\delta_V}{ds} = \langle T_\beta, V\rangle,$$

where δ_V is a function of s. Here, it is important to note that the vector V is spacelike since $\bar{\gamma}'$ and $\bar{\gamma}''$ are non-null vectors. By considering these relationships and using (3.15) and (3.16), the vector V can be represented as:

$$V = -\sinh\varphi\ T_\beta(s) + \cosh\varphi\ B_\beta(s),$$

and the curvatures κ_β and τ_β of the curve β satisfy the relation:

$$\frac{\tau_\beta}{\kappa_\beta} = -\tanh\varphi. \tag{3.18}$$

where $\varphi = \varphi(s) \geq 0$ is the constant angle between T_β and V.

Differentiating relation (3.17) with respect to s, we obtain:

$$\frac{d\bar{\gamma}}{ds} = \bar{\gamma}'(s) = T_\beta - \sinh\varphi V,$$

with $\|\gamma'\| = \cosh\varphi$. Therefore, $\bar{\gamma}$ is a timelike curve in the plane $\bar{P}$, and the arc-length function $s_{\bar{\gamma}}$ of $\bar{\gamma}$ is:

$$s_{\bar{\gamma}} = \int_0^s \|\bar{\gamma}'(s)\|\, ds = s\cosh\varphi. \tag{3.19}$$

We can express the unit speed tangent vector $T_{\bar{\gamma}}$ of $\bar{\gamma}$ as:

$$T_{\bar{\gamma}} = \frac{1}{\cosh\varphi} T_\beta - \tanh\varphi V.$$

Differentiating the above relation with respect to $s_{\bar{\gamma}}$ and using (2.1) and (3.19), we have

$$\frac{dT_{\bar{\gamma}}}{ds_{\bar{\gamma}}} = \frac{dT_{\bar{\gamma}}}{ds}\frac{ds}{ds_{\bar{\gamma}}} = \frac{\kappa_\beta}{\cosh^2\varphi} N_\beta,$$

and, since $\dot{T}_{\bar{\gamma}} = k_{\bar{\gamma}} N_{\bar{\gamma}}$, it follows that

$$\frac{\kappa_\beta}{\cosh^2\varphi} N_\beta = k_{\bar{\gamma}} N_{\bar{\gamma}}.$$

From this, we can conclude that the normal vector $N_{\bar{\gamma}}$ of $\bar{\gamma}$ is parallel to the normal vector N_β of β. Based on the above relationships, we can derive expressions for the curvature and torsion of β:

$$\kappa_\beta(s) = k_{\bar{\gamma}}\cosh^2\varphi, \tag{3.20}$$

and

$$\tau_\beta(s) = -k_{\bar{\gamma}} \sinh\varphi \cosh\varphi.$$

From (3.14) and (3.20), we obtain

$$\begin{aligned} \kappa(s) &= k_{\bar{\gamma}} \cosh^2\varphi \sin\omega, \\ \tau(s) &= k_{\bar{\gamma}} \cosh^2\varphi \cos\omega, \end{aligned} \tag{3.21}$$

where $\omega(s)$ is a function of s.

Moreover, from (3.14) and (3.18), the relationship between κ and τ can be further expressed as follows:

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' = -\tanh\varphi. \tag{3.22}$$

Substituting relation (3.21) into the above relationship, we find:

$$\omega' = k_{\bar{\gamma}} \sinh\varphi \cosh\varphi.$$

Integrating the above relation gives

$$\omega = \sinh\varphi \cosh\varphi \int k_{\bar{\gamma}} ds, \tag{3.23}$$

where $\varphi = \varphi(s) \geq 0$ is the constant angle between N and U.

Finally, from relation (3.14), we have:

$$T = \frac{1}{\sqrt{\kappa^2+\tau^2}} \left(-\kappa N_\beta + \tau B_\beta\right),$$

or from relation (3.21)

$$T = -\sin\omega \; N_\beta + \cos\omega \; B_\beta, \tag{3.24}$$

Then, we can express the position vector of α as $\alpha(s) = \int T(s) \; ds$. Therefore, we can provide the following theorem and corollaries.

Theorem 3.4. *Let $\bar{\gamma}$ be a timelike curve parameterized by the arc length $s_{\bar{\gamma}}$ lies in a plane with the spacelike normal vector V, and let $\varphi = \varphi(s) \geq 0$ be a constant angle. Then, there is a unit speed spacelike slant helix α with a timelike principal normal in $\mathbb{R}_1^3$ such that $\langle N, V\rangle_L = \sinh\varphi = const.$ for all s, and its position vector of α is given by:*

$$\alpha(s) = \int \left(-\sin\omega \; N_\beta + \cos\omega \; B_\beta\right) ds,$$

and its arc-length parameter s is given by:

$$s = \frac{s_{\bar{\gamma}}}{\cosh\varphi},$$

where $\omega(s)$ is given by (3.23), and N_β and B_β are the principal normal and binormal vectors of the integral curve β of N, respectively.

Corollary 3.5. *The curvatures κ and $\tau \neq 0$ of the slant helix α in $\mathbb{R}_1^3$ are given by:*

$$\kappa(s) = k_{\bar{\gamma}} \cosh^2\varphi \sin\omega,$$

and

$$\tau(s) = k_{\bar{\gamma}} \cosh^2\varphi \cos\omega,$$

where $k_{\bar{\gamma}}$ is the curvature of $\bar{\gamma}$.

Corollary 3.6. *The slant helix α is a general helix if and only if $\omega(s)$ is constant for all s.*

Example 2. Let us consider a timelike curve denoted as

$$\bar{\gamma}(s_{\bar{\gamma}}) = \left(2\sqrt{s_{\bar{\gamma}}}\sinh\left(\sqrt{s_{\bar{\gamma}}}\right) - 2\cosh\left(\sqrt{s_{\bar{\gamma}}}\right), 0, 2\sqrt{s_{\bar{\gamma}}}\cosh\left(\sqrt{s_{\bar{\gamma}}}\right) - 2\sinh\left(\sqrt{s_{\bar{\gamma}}}\right)\right),$$

which is a timelike curve lying in the plane $y = 0$ with a unit normal vector $V = (0, 1, 0)$, and let us choose $\varphi = \ln 2$. Assuming that the curve $\bar{\gamma}$ s chosen such that it is the projection of a timelike curve β onto the plane $y = 0$, from the relation (3.19) the arc-length function $s_{\bar{\gamma}}$ of $\bar{\gamma}$ is

$$s_{\bar{\gamma}} = s\cosh\varphi = \frac{5s}{4}.$$

From relation (3.17), the position vector of β is found as (see Figure 3):

$$\beta(s) = \left(\sqrt{5s}\sinh\left(\frac{\sqrt{5s}}{2}\right) - 2\cosh\left(\frac{\sqrt{5s}}{2}\right), \frac{3s}{4}, \sqrt{5s}\cosh\left(\frac{\sqrt{5s}}{2}\right) - 2\sinh\left(\frac{\sqrt{5s}}{2}\right)\right).$$

and the Frenet apparatus of β is obtained as follows:

$$\begin{aligned}
T_\beta &= \left(\tfrac{5}{4}\cosh\left(\tfrac{\sqrt{5s}}{2}\right), \tfrac{3}{4}, \tfrac{5}{4}\sinh\left(\tfrac{\sqrt{5s}}{2}\right)\right),\\
N_\beta &= \tfrac{\beta''}{\|\beta''\|} = \left(\sinh\left(\tfrac{\sqrt{5s}}{2}\right), 0, \cosh\left(\tfrac{\sqrt{5s}}{2}\right)\right),\\
B_\beta &= T_\beta \times N_\beta = \left(-\tfrac{3}{4}\cosh\left(\tfrac{\sqrt{5s}}{2}\right), -\tfrac{5}{4}, -\tfrac{3}{4}\sinh\left(\tfrac{\sqrt{5s}}{2}\right)\right),\\
\kappa_\beta &= \tfrac{5\sqrt{5}}{16\sqrt{s}},\\
\tau_\beta &= -\tfrac{3\sqrt{5}}{16\sqrt{s}}.
\end{aligned} \tag{3.25}$$

By observing that $\langle T_\beta, V\rangle = \frac{3}{4} = constant$, the curve β is a general timelike helix, although it is not a circular helix. Also, from (3.20) and (3.25) the curvature of $\bar{\gamma}$ is $k_{\bar{\gamma}} = \frac{1}{\sqrt{5s}}$, and so, from (3.23), the angle function $\omega(s)$ is determined to be $\omega = \frac{3\sqrt{5s}}{8}$. Therefore, from Theorem (3.4), the position vector of α is expressed as follows: (see Figure 3)

$$\alpha(s) = \begin{pmatrix}
\int\left(-\sin\left(\frac{3\sqrt{5s}}{8}\right)\sinh\left(\frac{\sqrt{5s}}{2}\right) - \frac{3}{4}\cos\left(\frac{3\sqrt{5s}}{8}\right)\cosh\left(\frac{\sqrt{5s}}{2}\right)\right)ds,\\
\int -\frac{5}{4}\cos\left(\frac{3\sqrt{5s}}{8}\right)ds,\\
\int\left(-\sin\left(\frac{3\sqrt{5s}}{8}\right)\cosh\left(\frac{\sqrt{5s}}{2}\right) - \frac{3}{4}\cos\left(\frac{3\sqrt{5s}}{8}\right)\sinh\left(\frac{\sqrt{5s}}{2}\right)\right)ds
\end{pmatrix}$$

and from Corollary (3.5), its curvatures are given by

$$\kappa(s) = \frac{5\sqrt{5}}{16\sqrt{s}}\sin\left(\frac{3\sqrt{5s}}{8}\right),$$

and

$$\tau(s) = \frac{5\sqrt{5}}{16\sqrt{s}}\cos\left(\frac{3\sqrt{5s}}{8}\right).$$

Thus, the curve α is identified as a slant spacelike helix with a timelike principal normal. However, it does not exhibit the characteristics of a general helix.

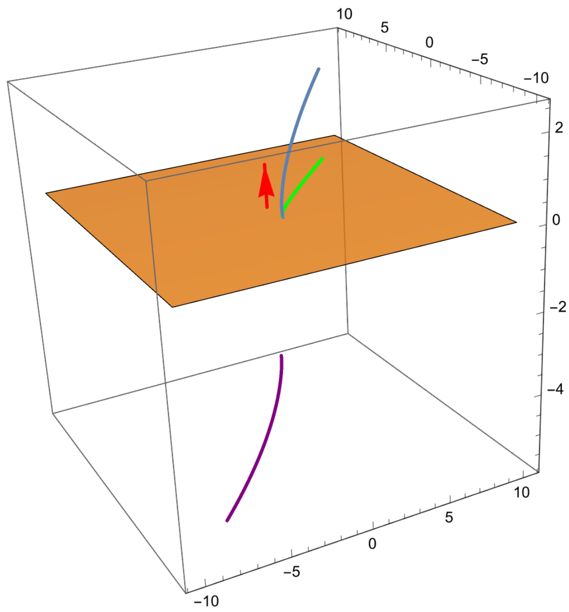

Figure 3: The spacelike slant helix α is purple, the timelike general helix β is blue, the plane curve $\bar{\gamma}$ is green, the plane $y=0$ is brown, and the axis V is red.

3.3 Darboux helices

Consider a unit-speed spacelike curve, denoted as α, with the timelike principal normal N in three-dimensional space $\mathbb{R}_1^3$, and let D be the Darboux vector associated with α:

$$D(s) = \tau(s)\, T(s) + \kappa(s)\, B(s),$$

satisfies the Darboux equations:

$$\begin{aligned} T'(s) &= D(s) \times_L T(s), \\ N'(s) &= D(s) \times_L N(s), \\ B'(s) &= D(s) \times_L B(s). \end{aligned}$$

Assume α to be a Darboux helix with the axis W, such that for all s in the interval I:

$$\langle d(s), W \rangle = c, \tag{3.26}$$

where c is constant. Here, $d(s)$ is the unit Darboux vector of α, defined as:

$$d(s) = \frac{D(s)}{\|D(s)\|} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} B.$$

Now, let us define a curve $\tilde{\beta}$ as the integral curve of the vector d of α:

$$\tilde{\beta} = \int d\, ds. \tag{3.27}$$

Since $\tilde{\beta}' = d$, we can verify that $\left\langle \tilde{\beta}', \tilde{\beta}' \right\rangle_L = 1$, meaning that $\tilde{\beta}$ is a spacelike curve with an arc length equal to s. Furthermore, from relation (3.26), $\tilde{\beta}$ is a unit-speed spacelike general helix with the axis W. Its Frenet apparatus can be given as follows:

$$\begin{array}{l} T_{\tilde{\beta}} = \tilde{\beta}' = d = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} B, \\ N_{\tilde{\beta}} = \frac{\tilde{\beta}''}{\left\| \tilde{\beta}'' \right\|} = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} T - \frac{\tau}{\sqrt{\kappa^2+\tau^2}} B, \\ B_{\tilde{\beta}} = -T_{\tilde{\beta}} \times_L N_{\tilde{\beta}} = N, \\ \kappa_{\tilde{\beta}} = \frac{\kappa\tau' - \tau\kappa'}{\kappa^2+\tau^2}, \\ \tau_{\tilde{\beta}} = \sqrt{\kappa^2+\tau^2}. \end{array} \tag{3.28}$$

Also, the axis W can be expressed as:

$$W = cT_{\tilde{\beta}}(s) + a(s) N_{\tilde{\beta}}(s) + b(s) B_{\tilde{\beta}}(s), \tag{3.29}$$

where $a(s)$ and $b(s)$ are differentiable functions of s, and c is a nonzero constant. Since $\tilde{\beta}$ is a spacelike general helix, we have:

$$\left\langle T_{\tilde{\beta}}(s), W \right\rangle_L = c.$$

Differentiating the above equation with respect to s, and using (2.1), we find that

$$a(s) = 0 \quad \text{and} \quad c\kappa_{\tilde{\beta}} - b\tau_{\tilde{\beta}} = 0, \tag{3.30}$$

where $b(s) = \textit{constant} \neq 0$ for all s.

Let us project the general helix $\tilde{\beta}$ onto a plane $\tilde{P}$ through point $\tilde{\beta}(0)$ and perpendicular to vector W. Then, there exists a curve $\hat{\gamma}$, parameterized by the arc length $s_{\hat{\gamma}}$, lying in the plane $\tilde{P}$ and having W as its normal vector, such that:

$$\tilde{\beta}(s) = \hat{\gamma}(s) + \delta_W W, \tag{3.31}$$

and

$$\frac{d\delta_W}{ds} = \left\langle T_{\tilde{\beta}}(s), W \right\rangle_L,$$

where δ_W is a function of s. The vector W is spacelike since $\hat{\gamma}'$ and $\hat{\gamma}''$ are non-null vectors. Then, from part (iii) of Lemma 2.2 and the relations (3.29) and (3.30), we obtain

$$W = \cosh\phi \, T_{\tilde{\beta}}(s) + \sinh\phi \, B_{\tilde{\beta}}(s),$$

and the curvatures $\kappa_{\tilde{\beta}}$ and $\tau_{\tilde{\beta}}$ of the curve $\tilde{\beta}$ satisfy the relation:

$$\frac{\tau_{\tilde{\beta}}}{\kappa_{\tilde{\beta}}} = \coth\phi. \tag{3.32}$$

where $\phi = \phi(s) \in [0, \pi]$ is the constant angle between $T_{\tilde{\beta}}$ and W. Differentiating relation (3.31) with respect to s, we obtain

$$\frac{d\hat{\gamma}}{ds} = \hat{\gamma}'(s) = T_{\tilde{\beta}} - \cosh\phi W,$$

with

$$\|\hat{\gamma}'\| = \sinh\phi.$$

Therefore, $\hat{\gamma}$ is a timelike curve in the plane $\tilde{P}$ and the arc-length function $s_{\hat{\gamma}}$ of $\hat{\gamma}$ is

$$s_{\hat{\gamma}} = \int_0^s \|\hat{\gamma}'(s)\| \, ds = s \sinh \phi. \tag{3.33}$$

Furthermore, we can express the unit tangent vector $T_{\hat{\gamma}}$ of $\hat{\gamma}$ as

$$T_{\hat{\gamma}} = \frac{1}{\sinh \phi} T_{\tilde{\beta}} - \coth \phi W.$$

Differentiating the last relation with respect to $s_{\hat{\gamma}}$ and using (2.1) and (3.33), we have

$$\frac{dT_{\hat{\gamma}}}{ds_{\hat{\gamma}}} = \frac{dT_{\hat{\gamma}}}{ds}\frac{ds}{ds_{\hat{\gamma}}} = \frac{\kappa_{\tilde{\beta}}}{\sinh^2 \phi} N_{\tilde{\beta}},$$

and since $\dot{T}_{\hat{\gamma}} = k_{\hat{\gamma}} N_{\hat{\gamma}}$,

$$\frac{\kappa_{\tilde{\beta}}}{\sinh^2 \phi} N_{\tilde{\beta}} = k_{\hat{\gamma}} N_{\hat{\gamma}}.$$

It can be easily said that the normal vector $N_{\hat{\gamma}}$ of $\hat{\gamma}$ is parallel to the normal vector $N_{\tilde{\beta}}$ of $\tilde{\beta}$, and thus

$$\kappa_{\tilde{\beta}}(s) = k_{\hat{\gamma}} \sinh^2 \phi. \tag{3.34}$$

Then, from relation (3.32), the torsion of $\tilde{\beta}$ is given by

$$\tau_{\tilde{\beta}}(s) = k_{\hat{\gamma}} \sinh \phi \cosh \phi. \tag{3.35}$$

From (3.28) and (3.35), we obtain

$$\begin{aligned} \kappa(s) &= k_{\hat{\gamma}} \sinh \phi \cosh \phi \sin \bar{\omega}, \\ \tau(s) &= k_{\hat{\gamma}} \sinh \phi \cosh \phi \cos \bar{\omega}, \end{aligned} \tag{3.36}$$

where $\bar{\omega}(s)$ is a function of s.

Moreover, from (3.28) and (3.32) we have

$$\frac{\left(\kappa^2 + \tau^2\right)^{3/2}}{\kappa^2 \left(\frac{\tau}{\kappa}\right)'} = \coth \phi. \tag{3.37}$$

Substituting relation (3.36) into relation (3.37), we obtain

$$\bar{\omega}' = -k_{\hat{\gamma}} \sinh^2 \phi.$$

Integrating the above relation gives

$$\bar{\omega} = -\sinh^2 \phi \int k_{\hat{\gamma}} ds, \tag{3.38}$$

where $\phi \in [0, \pi]$ is the constant angle between d and W.

Furthermore, from relation (3.28) we have

$$T = \frac{1}{\sqrt{\kappa^2 + \tau^2}} \left(\tau T_{\tilde{\beta}} + \kappa N_{\tilde{\beta}} \right),$$

or from relation (3.36)

$$T = \cos \bar{w} \; T_{\tilde{\beta}} + \sin \bar{w} \; N_{\tilde{\beta}}, \tag{3.39}$$

From this, we can get the position vector of α as $\alpha(s) = \int T(s) \; ds$. Therefore, we establish the following theorem and corollaries:

Theorem 3.7. *Let $\hat{\gamma}$ be a timelike curve parameterized by the arc length $s_{\hat{\gamma}}$ lies in a plane with the unit spacelike normal vector W, and let $\phi \in [0,\pi]$ be a constant angle. Then, there exists a unit speed spacelike Darboux helix α with a timelike principal normal N and a unit Darboux vector d in $\mathbb{R}^3_1$ such that $\langle d, W\rangle_L = \sinh\varphi = const.$ for all s, and its position vector of α is given by*

$$\alpha(s) = \int \left(\cos\bar{w}\ T_{\tilde{\beta}} + \sin\bar{w}\ N_{\tilde{\beta}}\right) ds,$$

and its arc-length parameter s is given by:

$$s = \frac{s_{\hat{\gamma}}}{\sinh\phi},$$

where $\bar{\omega}(s)$ is given by (3.38), and $T_{\tilde{\beta}}$ and $N_{\tilde{\beta}}$ are the tangent and principal normal vectors of the integral curve $\tilde{\beta}$ of d, respectively.

Corollary 3.8. *The curvatures κ and $\tau \neq 0$ of the Darboux helix α in $\mathbb{R}^3_1$ are given by:*

$$\kappa(s) = k_{\hat{\gamma}} \sinh\phi\cosh\phi\sin\bar{\omega},$$

and

$$\tau(s) = k_{\hat{\gamma}} \sinh\phi\cosh\phi\cos\bar{\omega},$$

where $k_{\hat{\gamma}}$ is the curvature of $\hat{\gamma}$.

Corollary 3.9. *The Darboux helix α is a general helix if and only if $\bar{\omega}(s)$ is constant for all s.*

Example 3. Let us choose a timelike plane curve

$$\hat{\gamma}(s_{\hat{\gamma}}) = (\sinh s_{\bar{\gamma}}, \cosh s_{\bar{\gamma}}, 0),$$

lying in the plane $z = 0$ with the unit normal $W = (0,0,1)$, and let us choose $\phi = \ln 3$. It is assumed that the curve $\hat{\gamma}$ is the projection of the timelike curve $\tilde{\beta}(s)$ onto the plane $y = 0$. Then, from (3.33), the arc length function $s_{\hat{\gamma}}$ of $\hat{\gamma}$ is

$$s_{\hat{\gamma}} = s\sinh\phi = \frac{4s}{3}.$$

Then, from relation (3.31), the position vector of $\tilde{\beta}$ is found (see Figure 4)

$$\tilde{\beta}(s) = \left(\sinh\left(\frac{4s}{3}\right), \cosh\left(\frac{4s}{3}\right), \frac{5s}{3}\right).$$

and the Frenet apparatus of $\tilde{\beta}$ is obtained as

$$\begin{aligned} T_{\tilde{\beta}} &= \left(\tfrac{4}{3}\cosh\left(\tfrac{4s}{3}\right), \tfrac{4}{3}\sinh\left(\tfrac{4s}{3}\right), \tfrac{5}{3}\right), \\ N_{\tilde{\beta}} &= \frac{\tilde{\beta}''}{\|\tilde{\beta}''\|} = \left(\sinh\left(\tfrac{4s}{3}\right), \cosh\left(\tfrac{4s}{3}\right), 0\right), \\ B_{\tilde{\beta}} &= -T_{\tilde{\beta}} \times N_{\tilde{\beta}} = \left(-\tfrac{5}{3}\cosh\left(\tfrac{4s}{3}\right), -\tfrac{5}{3}\sinh\left(\tfrac{4s}{3}\right), \tfrac{4}{3}\right), \\ \kappa_{\tilde{\beta}} &= \tfrac{16}{9}, \\ \tau_{\tilde{\beta}} &= \tfrac{20}{9}. \end{aligned} \tag{3.40}$$

Since $\left\langle T_{\tilde{\beta}}, W\right\rangle = \frac{5}{3} = constant$, the curve β is a general spacelike helix with a spacelike principal normal. Also, from (3.34) and (3.40), the curvature of $\hat{\gamma}$ is $k_{\hat{\gamma}} = 1$, and so, from (3.38) $\bar{w} = -\frac{16s}{9}$.

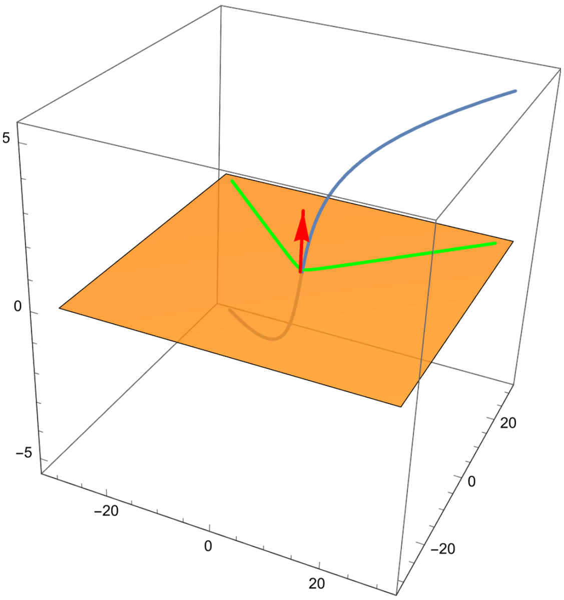

Figure 4: The spacelike general helix $\tilde{\beta}$ is blue, the plane curve $\hat{\gamma}$ is green, the plane $z = 0$ is brown, and the axis W is red.

Therefore, from Theorem (3.7), the position vector of α is given by (see Figure 5)

$$\alpha(s) = \begin{pmatrix} \frac{3}{100}\left(24\cos\left(\frac{16s}{9}\right)\sinh\left(\frac{4s}{3}\right) + 7\sin\left(\frac{16s}{9}\right)\cosh\left(\frac{4s}{3}\right)\right), \\ \frac{3}{100}\left(24\cos\left(\frac{16s}{9}\right)\cosh\left(\frac{4s}{3}\right) + 7\sin\left(\frac{16s}{9}\right)\sinh\left(\frac{4s}{3}\right)\right), \\ \frac{15}{16}\sin\left(\frac{16s}{9}\right) \end{pmatrix}$$

and from Corollary (3.8), its curvatures are given by

$$\kappa(s) = \frac{25}{9}\sin\left(\frac{16s}{9}\right),$$

and

$$\tau(s) = \frac{25}{9}\cos\left(\frac{16s}{9}\right).$$

Then, the curve α is a spacelike Darboux helix with a timelike principal normal, but not a general helix.

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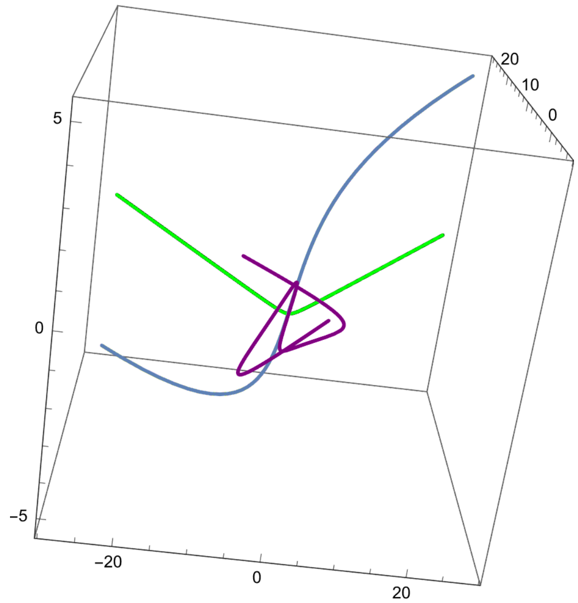

Figure 5: The spacelike Darboux helix α is purple, the timelike general helix $\tilde{\beta}$ is blue, and the plane curve $\hat{\gamma}$ is green.

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