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A fitted parameter convergent finite difference scheme for two-parameter singularly perturbed parabolic differential equations

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Abstract. The objective of this paper is to develop a numerical scheme that is uniform in its parameters for a specific type of time-dependent parabolic problem with two perturbation parameters. The existence of these two parameters in the terms with the highest-order derivatives results in the formation of boundary layer(s) in the solution of such problems. Solving these model problems using classical methods does not yield satisfactory results due to the layer behavior. Therefore, nonstandard finite difference schemes have been developed as a means to obtain numerical solutions for these problems. To develop the scheme, we employ the Crank-Nicolson discretization on a uniform time mesh and apply a fitted operator method with a uniform spatial mesh. We have established the stability and convergence of the proposed scheme. The proposed scheme exhibits uniform convergence of second order in the temporal direction and first order in the spatial direction. However, temporal mesh refinements is employed to enhance the order to two in both directions.. Model examples are provided to validate the practicality of the proposed numerical scheme.

Keywords. Singularly perturbation, fitted operator scheme, nonstandard finite difference method, uniform convergence.

1 Introduction

We consider the governing problem with Dirichlet boundary conditions on the domain $\Omega = (0,1) \times (0,T]$ with boundary $\partial\Omega = \bar{\Omega} \backslash \Omega$.

$$L_{\varepsilon,\mu}u(x,t) = \varepsilon u_{xx}(x,t) + \mu a(x,t)u_x(x,t) - b(x,t)u_t(x,t) - c(x,t)u(x,t) = f(x,t), (x,t) \in \Omega \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega_x = (0,1) \tag{1.2}$$

$$u(0,t) = \Phi_0(t), \quad u(1,t) = \Phi_1(t), \quad t \in \Omega_t = (0,T] \tag{1.3}$$

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where $\Omega = \Omega_x \times \Omega_t$ and ε and μ are small positive parameters with $0 < \varepsilon \leq 1$ and $0 \leq \mu \leq 1$. In (1.1)-(1.3), the coefficients $a(x,t)$, $b(x,t)$, $c(x,t)$, for $(x,t) \in \overline{\Omega}$ are assumed to be sufficiently regular and $a(x,\ t) \geq \alpha > 0, \quad b(x,t) \geq \beta > 0, \quad c(x,\ t) \geq \gamma > 0$ and $f(x,\ t)$ is smooth function. We make the assumption of adequate regularity and compatibility conditions at the corners of the domain, as stated in [32]. If all of the aforementioned assumptions are met, then the problem described in (1.1)-(1.3) possesses a unique solution on $\bar{\Omega}$.

The objectives of this study are to develop a numerical scheme that exhibits both accuracy and convergence in a parameter-uniform manner, and to explore its uniform stability.

If the parameters ε and μ approach zero in problem (1.1)-(1.3), boundary layer(s) will arise in the solution at either $x = 0$, $x = 1$ or at both end points. When $\mu = 1$, problem (1.1)-(1.3) represents convection- diffusion problem [17, 40, 44] and in this case a boundary layer of width $O(\varepsilon)$ will occur around the edge $x = 0$. On the other hand, when $\mu = 0$, problem (1.1)-(1.3) represents a parabolic reaction-diffusion problem in [25]. Near $x = 0$ and $x = 1$, thin boundary layers of width $O(\sqrt{\varepsilon})$ exhibited.

O'Malley [33] was the first scholar to introduce and analyze the solution of singularly perturbed two-parameter problems using asymptotic expansion techniques. Again, O'Malley [33, 34] identified that the nature of these problems is strongly affected by the choice of ratio of μ^2 to ε.

The class of time-dependent singularly perturbed problems of convection-diffusion type with two parameters are studied in [30] using non-standard finite difference method. The numerical solution of second-order two-parameter singularly perturbed ordinary differential equations (ODEs) with smooth data [3, 5, 6, 7, 8, 10, 13, 15, 16, 22, 35, 36, 43] and non-smooth data are studied in [24, 41] .

Two-parameter singularly perturbed time-dependent parabolic problems are frequently encountered in the mathematical modeling of various physical phenomena. These problems arise in fields such as fluid dynamics, heat and mass transfer in chemical engineering, quantum mechanics, elasticity, theory of plates and shells, oil and gas reservoir simulation, and magneto-hydrodynamic flow. The study of reaction-diffusion and convection-diffusion problems within the context of two-parameter singularly perturbed boundary value problems has been the subject of research in [10, 12, 14, 21, 23, 2, 38, 37, 39, 40, 18, 42]. Singularly perturbed parabolic problems have also been investigated in [1, 31, 29].

Two-parameter singularly perturbed problems find numerous applications in various applied science fields. However, finding analytical solutions for such problems is either a challenging task or the solutions obtained do not have a closed form. Once again, when seeking numerical solutions, classical finite difference methods frequently produce unstable solutions within the layer region. The convergence and stability of numerical solutions also depend on the small parameters involved. The development of a parameter-uniform numerical method for two-parameter singularly perturbed problems remains an ongoing research area, as indicated in the existing literature. In this study, our focus is on developing a parameter-uniformly convergent numerical scheme to address a specific class of second-order two-parameter singularly perturbed time-dependent problems described in equations (1.1)-(1.3). To derive the scheme, we employ the Crank-Nicolson discretization for the temporal variable, along with a fitted operator finite difference method (FOFDM) on a uniform spatial discretization. The resulting scheme confirms second-order accuracy in time and first-order accuracy in space. Nevertheless, we improve its accuracy to second order in both variables by employing a time mesh refinement technique, which is elaborated upon in Section 5.

Section organization: We have organized the article as follows. We first discuss the qualitative properties such as the bounds of the analytical solution $u(x,t)$ of problem (1.1)- (1.3) and its derivative bounds in Section 2. The development of the numerical scheme of the continuous problem is presented in Section 3. Uniformly stability and convergence analysis of the scheme is discussed in Section 4. In Section 5, we give numerical examples to show the convergent accuracy of the developed scheme. Results and conclusions are presented in Section 6.

Notation: In this work, the maximum norm is denoted as $._{\Omega}$ where Ω represents any bounded and closed subset of $[0,1]\times[0,T]$. The constant C and C_1 used in this paper is a positive generic value that remains independent of the perturbation parameters, ε and μ. Further, we denote $\eta = \min_{(x,t)\in\bar{\Omega}}\left\{\dfrac{c(x,t)}{a(x,t)}\right\} \geq \dfrac{\gamma}{\alpha} > 0$

2 Some qualitative properties of the continuous problem

This section presents a discussion on certain analytical properties of the governing problem (1.1)-(1.3) in a one-dimensional spatial domain, denoted as $\bar{\Omega}$.

Lemma 2.1 (**The minimum principle for the continuous problem**). *Let* $\Psi(x,t)\in C^{2,1}\bar{\Omega}$. *If* $\Psi|_{\partial\Omega}\geq 0$ *and* $\left(L_{\varepsilon,\mu}-\dfrac{\partial}{\partial t}\right)\Psi|_{\Omega}\leq 0$, *then* $\Psi|_{\bar{\Omega}}\geq 0$.

Proof. The proof of this Lemma proceeds by contradiction. Consider an arbitrary point $(x^\star,t^\star)$ in a plane, $\Omega=(0,1)\times(0,T)$ such that $\Psi(x^\star,t^\star)=\min\{\Psi(x,t)\}_{(x^\star,t^\star)\in\bar{\Omega}}$ and again suppose that $\Psi(x^\star,t^\star)<0$. Clearly, $(x^\star,t^\star)\notin\{0,1\}\times\{0,T\}$ by the definition of $(x^\star,t^\star)$. Applying the first and second derivative test for multivariable functions in calculus, we obtain $\Psi_{xx}(x^\star,t^\star)\geq 0$, $\nabla_x\Psi(x^\star,t^\star)=0$, $\nabla_t\Psi(x^\star,t^\star)=0$, and then it is clear that

$$\left(L_{\varepsilon,\mu}-\frac{\partial}{\partial t}\right)\Psi|_{\Omega}\geq 0$$

which is a contradiction. Hence, our initial assumption $\Psi(x^\star,t^\star)<0$ is wrong. So, $\Psi(x^\star,t^\star)|_{\bar{\Omega}}\geq 0$. Since, $(x^\star,t^\star)$ is arbitrary point, we have then, $\Psi(x,t)\geq 0$ for all $(x,t)\in\bar{\Omega}$. See [28] for detailed proof. □

Lemma 2.2 (**Uniform stability of the countinuous problem**). *[11] Let $u(x,t)$ be the solution of (1.1), then $\forall\varepsilon>0,\mu\geq 0$, we have*

$$\|u\|_{\bar{\Omega}}\leq\frac{\xi\|f\|_{\bar{\Omega}}}{\beta}+\|u\|_{\partial\Omega}.$$

Proof. We can prove this Lemma using the concept of barrier functions. Therefore, let us introduce two comparison (barrier) functions as follows:

$$\varphi^{\pm}(x,t)=\frac{\xi\|f\|_{\bar{\Omega}}}{\beta}+\|u\|_{\partial\Omega}\pm u(x,t)\forall(x,t)\in\bar{\Omega}$$

when $x=0$, $\varphi^{\pm}(0,t)=\dfrac{\xi\|f\|_{\bar{\Omega}}}{\beta}+\|u\|_{\partial\Omega}\pm\Phi_0(t)\geq 0$ and

when $x=1$, $\varphi^{\pm}(1,t)=\dfrac{\xi\|f\|_{\bar{\Omega}}}{\beta}+\|u\|_{\partial\Omega}\pm\Phi_1(t)\geq 0$.

Then, we have $\varphi^{\pm}(x,t)\geq 0\ \forall(x,t)\in\partial\Omega$. Again, since $b(x,t)\geq\beta>0$ and $\|f\|_{\bar{\Omega}}\geq f(x,t)\forall(x,t)\in\bar{\Omega}$, we have $-b(x,t)\beta^{-1}\|f\|_{\bar{\Omega}}\pm f(x,t)\leq 0$. This inequality implies

$$L_{\varepsilon,\mu}\varphi^{\pm}(x,t)\leq 0,\quad\forall(x,t)\in\Omega.$$

Applying the minimum principle in Lemma 2.1, we get $\varphi^{\pm}(x,t) \geq 0, \forall (x,t) \in \bar{\Omega}$, which gives the desired estimate. □

Lemma 2.3 (The bounds of the continuous problem and its derivatives). *In the continuous problem described in 1.1)-(1.3), consider the solution u, which can be decomposed as $u = v + w_L + w_R$. Here, v represents the regular component, w_L represents the left singular components, and w_R represents the right singular components. These components are accompanied by a sufficiently large constant C, which remains independent of the perturbation parameters [19]. Then,*

a. $u \leq C$

b. For all non- negative integers i and j $(0 \leq i + 2j \leq 4)$, the derivatives of the solution u of problem (1.1)- (1.3) satisfy

$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\| \leq \begin{cases} C \dfrac{1}{(\sqrt{\varepsilon})^i}, & \text{when } \alpha\mu^2 \leq \eta\varepsilon \\ C \left(\dfrac{\mu}{\varepsilon}\right)^i \left(\dfrac{\mu^2}{\varepsilon}\right)^j, & \text{when } \alpha\mu^2 \geq \eta\varepsilon \end{cases}$$

c. $|w_L(x,t)| \leq Ce^{-\theta_L x}, \quad |w_R(x,t)| \leq Ce^{-\theta_R(1-x)}$

where

$$\theta_L = \begin{cases} \frac{\sqrt{\eta\alpha}}{\sqrt{\varepsilon}}, & \alpha\mu^2 \leq \eta\varepsilon, \\ \frac{\alpha\mu}{\varepsilon}, & \alpha\mu^2 \geq \eta\varepsilon, \end{cases} \qquad \theta_R = \begin{cases} \frac{\sqrt{\eta a}}{2\sqrt{\varepsilon}}, & \alpha\mu^2 \leq \eta\varepsilon \\ \frac{\eta}{2\mu}, & \alpha\mu^2 \geq \eta\varepsilon. \end{cases}$$

Proof. The proof for this can be obtained in [20, 19]. □

Theorem 2.1. ***(Bounds of the derivatives).*** *The regular component $v(x,t)$ satisfies the following bounds for all non-negative integers i and j such that $0 \leq i + 3j \leq 4$.*

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\|_{\bar{\Omega}} \leq \begin{cases} C\left(1 + \dfrac{1}{(\sqrt{\varepsilon})^{i-3}}\right), & \text{for } \alpha\mu^2 \leq \delta\varepsilon \\ C\left(1 + \left(\dfrac{\varepsilon}{\mu}\right)^{3-i}\right), & \text{for } \alpha\mu^2 \geq \delta\varepsilon \end{cases}$$

where, the constant C is independent of both the perturbation parameters ε and μ.

Proof. The detailed proof can be found in [20, 19]. □

3 Derivation of the numerical scheme

3.1 Temporal discretization for the development of semi-discrete scheme

First, we discretize the temporal domain by dividing the given time domain $[0, T]$ using a uniform mesh, such that $\Omega^M = \{t_j = j\triangle t, \ \ j = 0, \ 1, \ 2, \ ..., \ M, \ t_M = T, \ \triangle t = \dfrac{T}{M}\}$ where Ω^M is the set of all mesh points and M is the number of mesh points in time interval $[0, T]$. Next, we employ the implicit Crank-Nicolson discretization in the time direction to formulate the semi-discrete scheme for the problem presented in (1.1)-(1.2). The derivation of the Crank-Nicolson scheme for $U_t(x,t_j)$ at the $(x, j+1/2)$ time step involves using Taylor's series expansion for U^{j+1} and U^j.

$$U^{j+1}(x) = U^{j+1/2}(x) + \frac{\Delta t}{2}\frac{\partial U^{j+1/2}(x)}{\partial t} + \left(\frac{\Delta t}{2}\right)^2 \frac{1}{2!}\frac{\partial^2 U^{j+1/2}(x)}{\partial t^2} + \left(\frac{\Delta t}{2}\right)^3 \frac{1}{3!}\frac{\partial^3 U^{j+1/2}(x)}{\partial t^3} + \dots \tag{3.1}$$

$$U^{j}(x) = U^{j+1/2}(x) - \frac{\Delta t}{2}\frac{\partial U^{j+1/2}(x)}{\partial t} + \left(\frac{\Delta t}{2}\right)^2 \frac{1}{2!}\frac{\partial^2 U^{j+1/2}(x)}{\partial t^2} - \left(\frac{\Delta t}{2}\right)^3 \frac{1}{3!}\frac{\partial^3 U^{j+1/2}(x)}{\partial t^3} + \dots \tag{3.2}$$

Subtracting Eq. (3.2) from Eq. (3.1), and eliminating the term $U^{j+1/2}(x)$, we get

$$\frac{U^{j+1}(x) - U^j(x)}{\Delta t} = \frac{\partial U^{j+1/2}(x)}{\partial t} + O(\Delta t)^3 \tag{3.3}$$

The local truncation error value ($T^{j+1/2}(x)$) obtained from the Taylor's series expansion is

$$T^{j+1/2}(x) = \frac{(\Delta t)^3}{24}\frac{\partial^3 U^{j+1/2}(x)}{\partial t^3} + \text{(higher order terms)}$$

and its order three ($O((\Delta t)^3)$).

After substituting Eq. (3.3) into (1.1)) and rearranging, we obtained the semi-discretized scheme as

$$\begin{aligned}\left[L^{N,M}U(x)\right]^j &\equiv b(x,t_{j+1/2})\left(\frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t}\right) = \varepsilon u_{xx}\left(x,t_{j+1/2}\right) + \\ &\mu a(x,t_{j+1/2})u_x\left(x,t_{j+1/2}\right) - c\left(x,t_{j+1/2}\right)u\left(x,t_{j+1/2}\right) - f\left(x,t_{j+1/2}\right) + O\left((\Delta t)^3\right)\}\end{aligned} \tag{3.4}$$

where

$$u\left(x,t_{j+1/2}\right) = \frac{u(x,t_{j+1}) + u(x,t_j)}{2} + O\left((\Delta t)^3\right), \quad \textit{and}$$
$$f\left(x,t_{j+1/2}\right) = \frac{f(x,t_{j+1}) + f(x,t_j)}{2} + O\left((\Delta t)^3\right)$$

Lemma 3.1. *(**The minimum principle for the semi-discrete scheme**). Assume that $\left[L^M U(x)\right]^{j+1}$ is the discrete operator given in (3.4) and $\Psi^{j+1}(x)$ is any mesh function satisfying $\Psi^{j+1}(x) \geq 0$ on $\partial\Omega$ and $\left[L^M \Psi(x)^{j+1}\right] \leq 0$ on Ω for $0 \leq j \leq M$, then $\Psi^{j+1}(x) \geq 0$ on $\bar{\Omega}$*

Proof. This Lemma can be proven using a proof by contradiction. Therefore, let $r^\star \in \Omega$ be any arbitrary point, such that $\Psi^{j+1}(r^\star) = \min_{x\in\Omega}\Psi^{j+1}(x)$. Again, suppose $\Psi^{j+1}(x) < 0$. It is clear that the set $((r^\star, t_{j+1}) \notin \{(0,t_{j+1}),(1,t_{j+1})\}$. Applying the concept of calculus of several-variable functions, we obtain $(\Psi_{xx})^{j+1}(r^\star) \geq 0$, $(\varphi_x)^{j+1}(s^\star) = 0$. This gives $L^M\Psi(r^\star)^{j+1} > 0$ which contradict to the fact that $L^M\Psi(x)^{j+1} \leq 0$. Therefore, $\Psi^{j+1}(x) \geq 0$ on $\bar{\Omega}$ which is our claim. □

Lemma 3.2. *(The local error estimate). Suppose that* $\|\frac{\partial^k u(x,t)}{\partial t^k}\| \leq C, (x,t) \in \overline{\Omega}, \ k = 0,1,2$. *The local error estimate in temporal direction* $E_{j+1} = U^{j+1}(x) - u(x, t_{j+1})$ *is given by*

$$E_{j+1} \leq C(\Delta t)^3$$

for sufficiently large constant C.

Proof. The time derivative is approximated by the fourth-order Taylor's series expansion of the Crank-Nicholson finite difference discretization in the temporal direction, given by:

$$\frac{U^{j+1}(x) - U^j(x)}{\Delta t} = U_t^{j+1/2}(x) + O((\Delta t)^3) \tag{3.5}$$

Upon substituting Eq. (3.5) into (1.1)-(1.2), we obtain the following equation

$$L_{\varepsilon,\mu} u(x, t_{j+1}) = u_t(x, t_{j+1}) + O((\Delta t)^3)$$

Furthermore, considering E_{k+1} as the semi-discrete minimum operator, we have

$$L^M_{\varepsilon,\mu} E(x, t_{j+1}) = O((\Delta t)^3)$$

Then, by using Lemma 3.1, the bound of the local error is estimated as $E_{j+1} \leq C(\Delta t)^3$ □

Lemma 3.3 (**The global error estimation**). *The global error,* $G_j = U^j(x) - u(x, t_j$ *of the time discretization satisfies*

$$\|G_{j+1}\|_\infty \leq C(\Delta t)^2$$

where C *is a constant independent of* ε, μ *and* Δt.

Proof. By using the estimation of local errors, the global error at $j+1$ nodal points is given as

$$G_{j+1} = \sum_{\xi=1}^{j} E_\xi, j(\Delta t) \leq T$$

$$\begin{aligned}
&= E_1 + E_2 + E_3 + E_4 + ... + E_j \\
&\leq E_1 + E_2 + E_3 + E_4 + ... + E_j \\
&\leq C_1(j)(\Delta t)^3 \\
&\leq C_1 \frac{T}{\Delta t}(\Delta t)^3 = C_1 T(\Delta t)^2, \quad \text{because} \quad j \leq \frac{T}{\Delta t} \\
&\leq C(\Delta t)^2, \quad \text{where,} \quad C = C_1 T
\end{aligned}$$

□

This lemma shows that the convergence of the semi-discrete scheme is order two in time.

Lemma 3.4. *The semi-discrete solution* $U^j(x)$ *and its derivatives satisfy the following bound.*

$$\frac{d^\xi U(x,t_j)}{dx^\xi} \leq C\left(1 + \lambda_1^\xi e^{-p\lambda_1 x} + \lambda_2^\xi e^{-p\lambda_2(1-x)}\right),$$
$$for \quad 0 \leq \xi \leq 4 \ \textit{and} \ p \in (0,1) \ \textit{is any real constant}$$

Proof. This Lemma was proved in [27] □

3.2 Spatial discretization for full-discrete scheme development

The given domain $\bar{\Omega}^N = [0,1]$ is subdivided into N sub-intervals such that

$$x_0 = 0, x_1 = x_0 + i = h, x_2 = x_1 + h, \cdots\ x_N = N(h) = 1$$

Now, the full discretized problem is

$$\begin{aligned}\left[L^{N,M}U\right]_i^j \equiv b(x_i, t_{j+1/2}) \left(\frac{u\left(x_i, t_{j+1}\right) - u\left(x_i, t_j\right)}{\Delta t}\right) = \varepsilon u_{x_i x}\left(x_i, t_{j+1/2}\right) + \\ \mu a(x_i, t_{j+1/2}) u_x\left(x_i, t_{j+1/2}\right) \\ -c\left(x_i, t_{j+1/2}\right) u\left(x_i, t_{j+1/2}\right) - f\left(x_i, t_{j+1/2}\right) + O\left((\Delta t)^2\right)\}\end{aligned} \tag{3.6}$$

Using fitted finite difference methods the fully discrete scheme in (3.6) can be expressed as

$$\begin{aligned}\left[L^{N,M}U\right]_i^j \equiv \frac{1}{2}\left[\varepsilon\delta_x^2 U_i^{j+1} + \mu {a_i}^{j+1} D_x^+ U_i^{j+1} - c_i^{j+1} U_i^{j+1} + \varepsilon\delta_x^2 U_i^j + \mu {a_i}^j D_x^+ U_i^j - c_i^j U_i^j\right] - \\ b_i^{j+1/2} D_t U_i^{j+1/2} = F_i^j \quad i = 0, 1, 2, \cdots, N-1, \quad j = 0, 1, 2, \cdots, M-1\end{aligned} \tag{3.7}$$

where

$$D_x^+ U_i^j = \frac{U_{i+1}^j - U_i^j}{h_x}, \quad D_t U_i^{j+1/2} = \frac{U_i^{j+1} - U_i^j}{\Delta t} \qquad \delta_x^2 U_i^j = \left(\frac{U_{i+1}^j - 2U_i^j + u_{i-1}^j}{\phi_i^2}\right)$$

$$D_x^+ U_i^{j+1} = \frac{U_{i+1}^{j+1} - U_i^{j+1}}{h_x},$$

and

$$\delta_x^2 U_i^{j+1} = \left(\frac{U_{i+1}^{j+1} - 2U_i^{j+1} + u_{i-1}^{j+1}}{\phi_i^2}\right) \qquad F_i^j = \frac{f\left(x_i, t_{j+1}\right) + f\left(x_i, t_j\right)}{2}$$

Again, from [30], the denominator function ϕ_i^2 is given by

$$\phi_i^2(h, \varepsilon, \mu) \equiv \phi_i^2 = \frac{h\varepsilon}{\mu a\left(x_i\right)}\left(\exp\left(\frac{\mu a\left(x_i\right) h}{\varepsilon}\right) - 1\right) \tag{3.8}$$

Using the finite difference schemes and the value of ϕ_i^2 above, Eq. (3.7) can be written in compact form as

$$\left[L^{N,M}U\right]_i^j \equiv \delta^+ U_{i+1}^{j+1} + \delta^c U_i^{j+1} + \delta^- U_{i-1}^{j+1} + \delta_1^+ U_{i+1}^j + \delta_1^c U_i^j + \delta_1^- U_{i-1}^j = F_i^j \tag{3.9}$$

where, the coefficients $\delta^+, \delta_1^+, \delta^c, \delta_1^c, \delta^m$ and δ_1^m are

$$\delta^+ = \left(\frac{\varepsilon}{2\phi(i)^2} + \frac{\mu a_i^{j+1}}{2h}\right), \ \delta_1^+ = \left(\frac{\varepsilon}{2\phi(i)^2} + \frac{\mu a_i^j}{2h}\right), \quad \delta^- = \delta_1^- = \frac{\varepsilon}{2\phi(i)^2}$$

$$\delta^c = \left(\frac{-\varepsilon}{\phi(i)^2} - \frac{\mu a_i^{j+1}}{2h} - \frac{c_i^{j+1}}{2} - \frac{b_i^{j+1}}{\Delta t}\right) \delta_1^c = \left(\frac{-\varepsilon}{\phi(i)^2} - \frac{\mu a_i^j}{2h} - \frac{c_i^j}{2} - \frac{b_i^j}{\Delta t}\right),$$

4 The convergence analysis of the numerical scheme

This section analyzed uniformly stability and uniformly convergence of the developed scheme.

Lemma 4.1. *(**Discrete minimum principle**) Assume that $\left[L^{N,M}U\right]_i^{j+1}$ is the discrete operator given in (3.9) and ϕ_i^{j+1} is any mesh function satisfying $\phi_i^{j+1} \geq 0$ on $\partial\Omega^{N,M}$ and $\left[L^{N,M}\phi\right]_i^{j+1} \leq 0$ on $\Omega^{N,M}$ for $0 \leq i \leq N$, $0 \leq j \leq M$, then $\phi_i^{j+1} \geq 0$ on $\bar{\Omega}^{N,M}$*

Proof. Let s and l be indices such that $\phi_s^{l+1} = \min_{(i,j)} \phi_i^{j+1}$, for $\phi_i^{j+1} \in \bar{\Omega}^{N,M}$. Again, assume that $\phi_s^{l+1} < 0$. It is clear to see that $(s,l) \notin \{1,N\} \times \{1,M\}$, because $\phi_s^{l+1} \geq 0$. It follows that $\phi_{s+1}^{l+1} - \phi_s^{l+1} > 0$ and $\phi_s^{l+1} - \phi_{s-1}^{l+1} < 0$.

$$L^{N,M}\phi_s^{l+1} = \varepsilon\left(\frac{\phi_{s+1}^{l+1} - 2\phi_s^{l+1} + \phi_{s-1}^{l+1}}{\phi_s^2}\right) + \mu a_s^{l+1}\left(\frac{\phi_{s+1}^{l} + \phi_s^{l+1}}{h_s}\right) - c_s^{l+1}\phi_s^{l+1}$$
$$L^{N,M}\phi_s^{l+1} = \varepsilon\left(\frac{\phi_{s+1}^{l+1} - \phi_s^{l+1} + \phi_{s-1}^{l+1} - \phi_{s+1}^{l+1}}{\frac{h\varepsilon}{\mu a(x_s)}\left(\exp\left(\frac{\mu a(x_s)h}{\varepsilon}\right) - 1\right)}\right) + \mu a_s^{l+1}\left(\frac{\phi_{s+1}^{l+1} + \phi_s^{l+1}}{h_s}\right) - c_s^{l+1}\phi_s^{l+1} > 0$$

which is a contradiction to the fact that $L^{N,M}\phi_s^{l+1} \leq 0$. Therefore $\phi_s^{l+1} \geq 0$. The indices s and l being arbitrary, we obtain $\phi_i^{j+1} \geq 0$ in $\bar{\Omega}^{N,M}$. □

Lemma 4.2. *(**Uniform stability estimate**) At any time level t_j, if H_i^{j+1} is any mesh function such that $H_0^{j+1} = H_N^{j+1} = 0$, then*

$$H_i^{j+1} \leq \frac{1}{\Upsilon} \max_{1 \leq i \leq N-1} L^{N,M} H_i^{j+1}, \text{ for } 0 < j < M$$

Theorem 4.1 (Error estimate in the spatial discretization). *Let $U(x_i, t_{j+1})$ and U_i^{j+1} are the exact solution and approximate solution of (1.1)-(1.2) respectively. If N and C are mesh number and sufficiently large constant, then the following error bound holds.*

$$L_{\varepsilon,\mu}^{N,M}\left(U(x_i, t_{j+1}) - U_i^{j+1}\right) \leq CN^{-1} \tag{4.1}$$

Proof. To proof and Lemma 4.2 andTheorem 4.1, see the reference in [28]. □

Combining Lemma 3.3 and Theorem 4.1, we can state the following theorem as main result of this paper.

Theorem 4.2. *(**The main result**) Let $u(x,t)$ be the exact solution of (1.1)- (1.2) and U_i^{j+1} is its numerical approximation obtained using (3.7). Then there exists a constant C independent of ε, μ, h and Δt such that*

$$\max_{0 \leq i \leq N, 0 \leq j \leq M} U_i^{j+1} - u(x_i, t_{j+1}) \leq C(h + \Delta t^2) \tag{4.2}$$

Proof. This Lemma can be proved by combining Lemma 3.3 and Theorem 4.1, and it confirms the developed method is uniformly convergent of second order in time and first order in space directions. □

5 Numerical implementations and discussions

To demonstrate the applicability of the proposed scheme in (3.7), two examples are provided using MATLAB software. The maximum absolute errors and numerical rate of convergence are calculated on the considered meshes (Shishkin mesh type, [26]) using the double mesh principle given in [9] as follow.

$$E_{rr}^{N,M} = \max_{0\le i,j\le N,M} U^{N,M}(x_i,\ t_j) - U^{2N,2M}(x_{2i},\ t_{2j}) \text{ (maximum absolute errors)}$$

$$Roc^{N,M} = \log_2\left(\frac{E_{rr}^{N,M}}{E_{rr}^{2N,2M}}\right) \qquad \text{(rate of convergence)}$$

Example 1. [30] We consider the following time-dependent problem

$$\begin{aligned}
&\varepsilon\frac{\partial^2 u}{\partial x^2} + \mu(1+x)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} - u = 16x^2(1-x)^2, \quad (x,t)\in\Omega=(0,1)\times(0,1]\\
&u(x,0)=0, \quad x\in\bar{\Omega}_x=[0,1]\\
&u(0,t)=0, \quad u(1,t)=0, \quad t\in\bar{\Omega}_t=[0,1]
\end{aligned}$$

Example 2. [4] For our second example, we consider the following time-dependent Initial Value Boundary Problem (IVBP) of

$$\begin{aligned}
&u_t - \varepsilon u_{xx} + \mu(1+\exp(x))u_x + \left(1+x^4x\right)u = 10\exp\left(t^2\right)\left(x^2-x^4\right), (x,t)\in(0,1)\times(0,1]\\
&u(x,0)=x^3(1-x)^3, \quad x\in(0,1)\\
&u(0,t)=u(1,t)=0, \quad t\in(0,1]
\end{aligned}$$

The exact solution of the two given problems are not known. So, we have used the developed scheme in (3.7) to solve such problems and to investigate the applicability of our scheme. Tables 1 and 2 show the result of Example 1 by using the scheme in (3.7). From these tables, results insure that our scheme is first order which confirms the spatial order. But, by using temporal mesh refinement, we have improved this order to two as shown from Tables 3 and 4. Tables 6 and 7 show the numerical implementation of Example 2 using the developed scheme in (3.7), and the results from these tables show that the linear convergence of the scheme. Using similar technique that we did in Tables 3 and 4, we modified the linear convergence of the scheme to quadratic convergence (see Table 8). Comparison of our scheme to existing schemes was done in Tables 5 and 9, and from the comparison our scheme shows better accuracy than the existing schemes. Figure 1 and Figure 2 describe the graphical result of Example 1, while Figure 2 is for Example 2. The plotted figures exhibit that the boundary layer behavior in the solution of the given problem. Also, the log-log plots in Figure 3 is plotted for the considered examples, and these plots support our theoretical error estimates. .

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Table 1: Maximum errors, $Err^{N,M}_{\varepsilon,\mu}$ and rates of convergence, $Roc^{N,M}_{\varepsilon,\mu}$ using scheme (3.7) for Example 1 with $\mu = 2^{-3}$ and different values of ε.

	N $\rightarrow$	16	32	64	128	256
$\varepsilon \downarrow$	M $\rightarrow$	8	16	32	64	128
2^{-6}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	3.6659e-03	1.8261e-03	9.2406e-04	4.5617e-04	2.2918e-04
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.0054	0.9827	1.0184	0.9931	-
2^{-8}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	3.7438e-03	1.8710e-03	9.4711e-04	4.6727e-04	2.3476e-04
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.0007	0.9822	1.0193	0.9931	-
2^{-10}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	3.7424e-03	1.8914e-03	9.7841e-04	4.8873e-04	2.4644e-04
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	0.9845	0.9509	1.0014	0.9878	-
2^{-12}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	3.7399e-03	1.8728e-03	9.5354e-04	4.9208e-04	2.7039e-04
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	0.9978	0.9738	0.9544	0.8639	-
2^{-14}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	3.7399e-03	1.8728e-03	9.5069e-04	4.6984e-04	2.3925e-04
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	0.9978	0.9781	1.0168	0.9912	-
2^{-16}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	3.7399e-03	1.8728e-03	9.5069e-04	4.6983e-04	2.3635e-04
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	0.9978	0.9781	1.0168	0.9912	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	$Err^{N,M}_{\varepsilon,\mu}$	3.7399e-03	1.8728e-03	9.5069e-04	4.6983e-04	2.3635e-04
	$Roc^{N,M}_{\varepsilon,\mu}$	0.9978	0.9781	1.0168	0.9912	-
$Err^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	3.7399e-03	1.8728e-03	9.5069e-04	4.6983e-04	2.3635e-04
$Roc^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	0.9978	0.9781	1.0168	0.9912	-

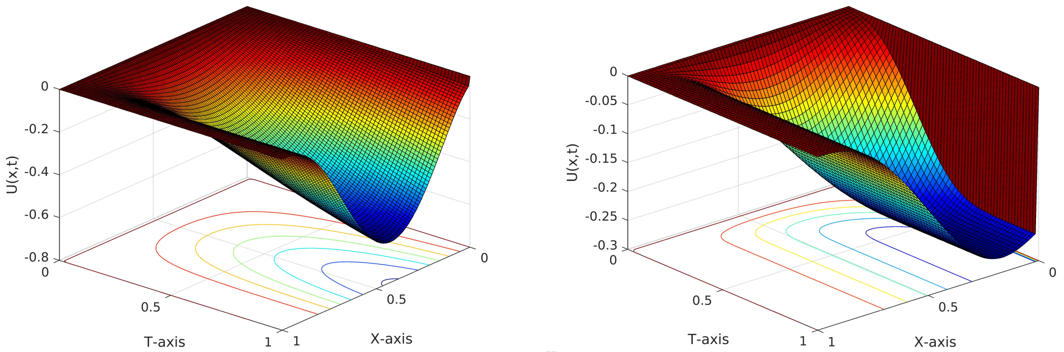

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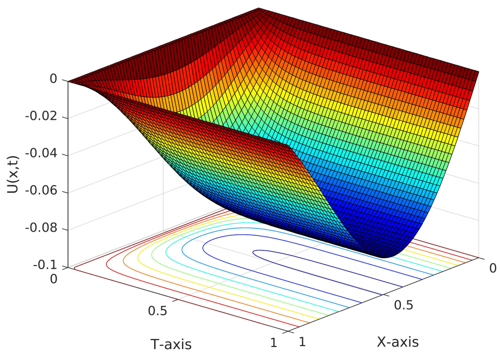
Figure 1: Numerical results of Example 1 using scheme (3.7) (a) for $N = M = 64$, $\varepsilon = 2^{-20}$, $\mu = 2^{-3}$ and (b) for $N = M = 64, \varepsilon = 2^{-20}, \mu = 1$

6 Conclusion of the Study

In this work, we have formulated a fitted operator finite difference method (FOFDM) for solving singularly perturbed partial differential equations with two perturbation parameters. A

Table 2: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (3.7) for Example 1 with $\varepsilon = 10^{-3}$ and different values of μ.

	N $\rightarrow$	16	32	64	128	256
$\mu \downarrow$	M $\rightarrow$	8	16	32	64	128
2^{0}	$Err_{\varepsilon,\mu}^{N,M}$	3.8894e-04	5.4676e-05	1.3382e-05	3.1627e-06	7.0143e-07
	$Roc_{\varepsilon,\mu}^{N,M}$	2.8306	2.0306	2.0811	2.1728	-
2^{-2}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.3064e-03	1.6808e-03	8.2753e-04	4.1419e-04	2.0490e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	0.9761	1.0223	0.9985	1.0154	-
2^{-4}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.2956e-03	1.6376e-03	8.1657e-04	4.1055e-04	2.0407e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	1.0090	1.0039	0.9920	1.0085	-
2^{-6}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.2919e-03	1.6359e-03	8.1575e-04	4.0737e-04	2.0356e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	1.0088	1.0039	1.0018	1.0009	-
2^{-8}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.2909e-03	1.6354e-03	8.1552e-04	4.0725e-04	2.0350e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	1.0088	1.0039	1.0018	1.0009	-
2^{-10}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.2906e-03	1.6353e-03	8.1547e-04	4.0722e-04	2.0349e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	1.0088	1.0039	1.0018	1.0009	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-30}	$Err_{\varepsilon,\mu}^{N,M}$	3.7399e-03	1.8728e-03	9.5069e-04	4.6983e-04	2.3635e-04
	$Roc_{\varepsilon,\mu}^{N,M}$	1.0088	1.0039	1.0018	1.0009	-
$Err_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	3.7399e-03	1.8728e-03	9.5069e-04	4.6983e-04	2.3635e-04
$Roc_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	1.0088	1.0039	1.0018	1.0009	-



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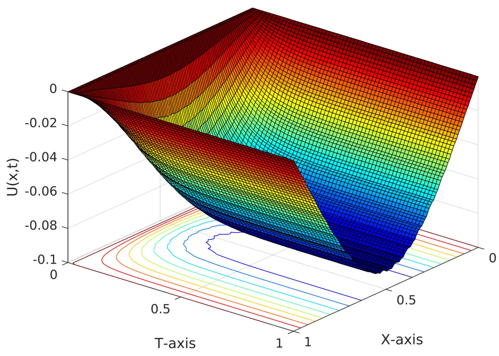

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Figure 2: Numerical results of Example 1 using scheme (3.7) (a) for $N = M = 64$, $\varepsilon = 1 = \mu$ and (b) for $N = 128, M = 64, \varepsilon = 1, \mu = 2^{-40}$

uniform mesh has been considered in both space and time directions while we discretized the given domain. The discretization used the Crank-Nicolson method for time variable and non-standard finite difference method (NSFDM) for space variable. The proposed numerical method is uniformly convergent independent of both the perturbation parameters, ε and μ. The scheme

Table 3: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (3.7) for Example 1 with $\mu = 10^{-3}$ and different values of ε.

	N $\rightarrow$	16	32	64	128
$\varepsilon \downarrow$	M $\rightarrow$	8	32	128	512
10^{0}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.7154e-04	5.1172e-05	1.2743e-05	3.1831e-06
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.8601	2.0056	2.0012	-
10^{-2}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1715e-02	7.4881e-03	1.8460e-03	4.5989e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.0825	2.0202	2.0050	-
10^{-4}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.6848e-02	8.7688e-03	2.1682e-03	5.4171e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.0711	2.0159	2.0009	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.6921e-02	8.8083e-03	2.1882e-03	5.5165e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.0675	2.0091	1.9879	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.6921e-02	8.8083e-03	2.1882e-03	5.5165e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.0675	2.0091	1.9879	-
10^{-10}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.6921e-02	8.8083e-03	2.1882e-03	5.5165e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.0675	2.0091	1.9879	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.6921e-02	8.8083e-03	2.1882e-03	5.5165e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.0675	2.0091	1.9879	-
$Err_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	3.6921e-02	8.8083e-03	2.1882e-03	5.5165e-04
$Roc_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	2.0675	2.0091	1.9879	-

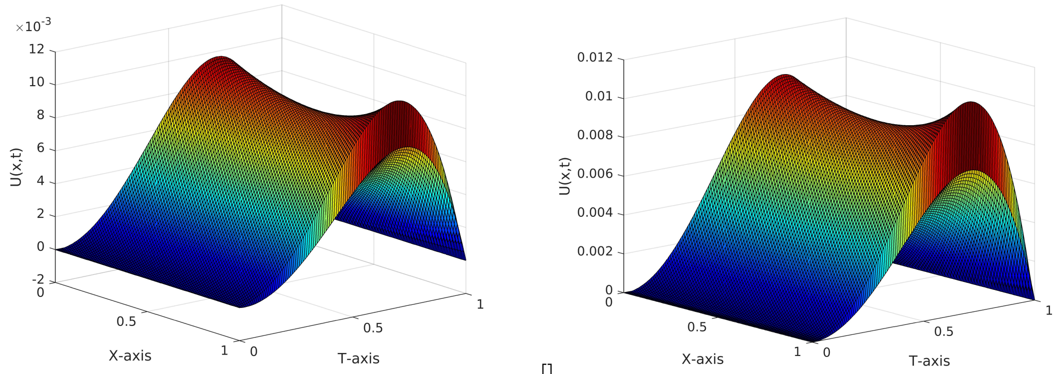

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Figure 3: Numerical results of Example 2 using scheme (3.7) (a) for $N = 128 = M$,$\varepsilon = \mu = 2^{-2}$ and (b) for $N = 128 = M, \varepsilon = \mu = 2^{-12}$

is shown to be first order in space and second order in time theoretically. But, we have improved this order in to second order in both variables as shown in Tables 3, 4 and 8 of section 5. To confirm the theoretical convergence results and to demonstrate the applicability of the proposed

Table 4: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (3.7) for Example 1 with $\mu = 10^{-3}$ and different values of μ.

	N $\rightarrow$	16	32	64	128
$\varepsilon \downarrow$	M $\rightarrow$	8	32	128	512
10^{-2}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.6681e-02	8.7863e-03	2.3760e-03	5.6170e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	2.0617	1.8867	2.0807	-
10^{-4}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.6290e-02	8.6223e-03	2.1288e-03	5.3056e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	2.0734	2.0180	2.0045	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.6288e-02	8.6218e-03	2.1287e-03	5.3053e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	2.0734	2.0180	2.0045	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.6288e-02	8.6218e-03	2.1287e-03	5.3053e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	2.0734	2.0180	2.0045	-
10^{-10}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.6288e-02	8.6218e-03	2.1287e-03	5.3053e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	2.0734	2.0180	2.0045	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	$Err_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	3.6288e-02	8.6218e-03	2.1287e-03	5.3053e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ $\rightarrow$	2.0734	2.0180	2.0045	-
$Err_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	3.6288e-02	8.6218e-03	2.1287e-03	5.3053e-04
$Roc_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	2.0734	2.0180	2.0045	-

Table 5: Comparison of $Err_{\varepsilon,\mu}^{N,M}$ of our scheme in (3.7) with an existing schemes in [30] using Example 1

$\mu = 10^{-2}$, $(N = M) \rightarrow$	16	32	64	128	256
$\varepsilon \downarrow$ **Proposed method**					
2^{-2}	3.5408e-03	1.4406e-03	7.2367e-04	3.6274e-04	1.8160e-04
2^{-6}	7.6891e-03	3.5741e-03	1.6758e-03	8.3861e-04	4.1005e-04
2^{-8}	8.8804e-03	4.8270e-03	2.3954e-03	1.1960e-03	5.9488e-04
scheme in [30]					
2^{-2}	7.10e-3	3.11e-3	1.44e-3	6.89e-4	3.37e-4
2^{-6}	1.95e-2	7.29e-3	2.91e-3	1.27e-3	5.89e-4
2^{-8}	3.21e-2	1.31e-2	4.91e-3	1.92e-3	8.19e-4

scheme, we have implemented two tested examples and results have been provided and presented in tables and graphs. The numerical examples and the graphical results confirm the theoretical analyses and findings. In our study, we focused on a two-parameter time-dependent problem in one spatial dimension. However, future research can explore similar problems in higher spatial dimensions.

Table 6: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (3.7) for Example 2 with $\mu = 10^{-2}$ and different values of ε.

	N $\rightarrow$	32	64	128	256	512
$\varepsilon \downarrow$	M $\rightarrow$	8	16	32	64	128
10^{-2}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.8396e-03	2.4330e-03	1.2184e-03	6.0984e-04	3.0505e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	0.9922	0.9977	0.9985	0.9994	-
10^{-4}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.8440e-03	2.4340e-03	1.2186e-03	6.0991e-04	3.0506e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	0.9929	0.9981	0.9986	0.9995	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.8440e-03	2.4340e-03	1.2186e-03	6.0990e-04	3.0506e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	0.9929	0.9981	0.9986	0.9995	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.8440e-03	2.4340e-03	1.2186e-03	6.0990e-04	3.0506e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	0.9929	0.9981	0.9986	0.9995	-
10^{-10}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.8440e-03	2.4340e-03	1.2186e-03	6.0990e-04	3.0506e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	0.9929	0.9981	0.9986	0.9995	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-40}	$Err_{\varepsilon,\mu}^{N,M}$	4.8440e-03	2.4340e-03	1.2186e-03	6.0990e-04	3.0506e-04
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	0.9929	0.9981	0.9986	0.9995	-
$Err_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	4.8440e-03	2.4340e-03	1.2186e-03	6.0990e-04	3.0506e-04
$Roc_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	0.9929	0.9981	0.9986	0.9995	-

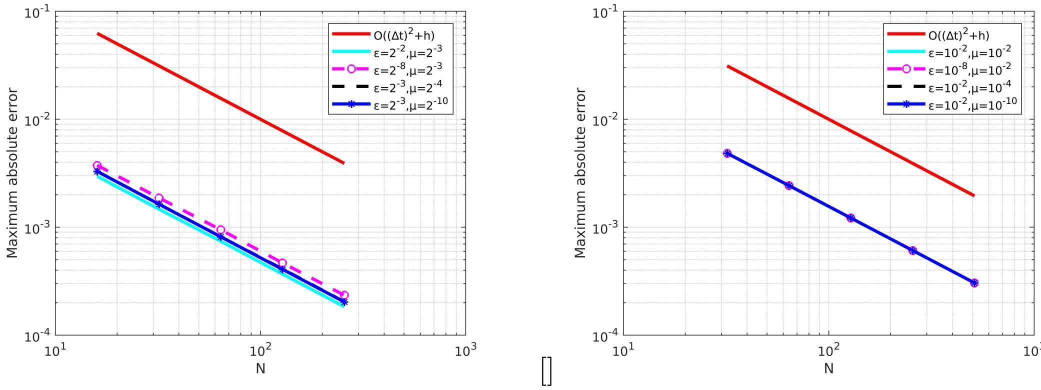

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Figure 4: Log-Log plots of (a) Example 1 and (b) Example 2

Acknowledgments

The authors wish to acknowledge the referees for their constructive suggestions and comments.

Table 7: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (3.7) for Example 2 with $\varepsilon = 10^{-2}$ and different values of μ.

$\mu \downarrow$	N → / M →	32 / 8	64 / 16	128 / 32	256 / 64	512 / 128
10^{-2}	$Err_{\varepsilon,\mu}^{N,M}$ →	4.8396e-03	2.4330e-03	1.2184e-03	6.0984e-04	3.0505e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ →	0.9922	0.9977	0.9985	0.9994	-
10^{-4}	$Err_{\varepsilon,\mu}^{N,M}$ →	4.8396e-03	2.4329e-03	1.2183e-03	6.0983e-04	3.0505e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ →	0.9922	0.9978	0.9984	0.9994	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M}$ →	4.8396e-03	2.4329e-03	1.2183e-03	6.0983e-04	3.0505e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ →	0.9922	0.9978	0.9984	0.9994	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M}$ →	4.8396e-03	2.4329e-03	1.2183e-03	6.0983e-04	3.0505e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ →	0.9922	0.9978	0.9984	0.9994	-
10^{-10}	$Err_{\varepsilon,\mu}^{N,M}$ →	4.8396e-03	2.4329e-03	1.2183e-03	6.0983e-04	3.0505e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ →	0.9922	0.9978	0.9984	0.9994	-
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-40}	$Err_{\varepsilon,\mu}^{N,M}$	4.8396e-03	2.4329e-03	1.2183e-03	6.0983e-04	3.0505e-04
	$Roc_{\varepsilon,\mu}^{N,M}$ →	0.9922	0.9978	0.9984	0.9994	-
$Err_{\varepsilon,\mu}^{N,M}$	→	4.8396e-03	2.4329e-03	1.2183e-03	6.0983e-04	3.0505e-04
$Roc_{\varepsilon,\mu}^{N,M}$	→	0.9922	0.9978	0.9984	0.9994	-

Table 8: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (3.7) for Example 2 with $\mu = 10^{-3}$ and different values of ε.

$\varepsilon \downarrow$	N → / M →	32 / 16	64 / 64	128 / 256	256 / 1024
10^{-4}	$Err_{\varepsilon,\mu}^{N,M}$ →	6.2804e-02	1.6164e-02	4.0679e-03	1.0188e-03
	$Roc_{\varepsilon,\mu}^{N,M}$ →	1.9581	1.9904	1.9974	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M}$ →	6.2807e-02	1.6165e-02	4.0682e-03	1.0189e-03
	$Roc_{\varepsilon,\mu}^{N,M}$ →	1.9581	1.9904	1.9974	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M}$ →	6.2807e-02	1.6165e-02	4.0682e-03	1.0189e-03
	$Roc_{\varepsilon,\mu}^{N,M}$ →	1.9581	1.9904	1.9974	-
⋮	⋮	⋮	⋮	⋮	⋮
10^{-20}	$Err_{\varepsilon,\mu}^{N,M}$ →	6.2807e-02	1.6165e-02	4.0682e-03	1.0189e-03
	$Roc_{\varepsilon,\mu}^{N,M}$ →	1.9581	1.9904	1.9974	-
$Err_{\varepsilon,\mu}^{N,M}$	→	6.2807e-02	1.6165e-02	4.0682e-03	1.0189e-03
$Roc_{\varepsilon,\mu}^{N,M}$	→	1.9581	1.9904	1.9974	-

References

[1] K. Bansal, P. Rai, and K. K. Sharma. Numerical treatment for the class of time dependent singularly perturbed parabolic problems with general shift arguments. *Differ.*, 25(2):327–

Table 9: Comparison of $Err_{\varepsilon,\mu}^{N,M}$ of our scheme in (3.7) with an existing schemes in [4] using Example 2

$\mu = 10^{-4}$, $(N) \rightarrow$	64	128	256	512
$(M) \rightarrow$	16	32	64	128
$\varepsilon \downarrow$ **Proposed method**				
10^{-2}	2.4330e-03	1.2184e-03	6.0984e-04	3.0505e-04
10^{-4}	2.4340e-03	1.2186e-03	6.0991e-04	3.0506e-04
scheme in [4]				
10^{-2}	2.2542e-2	1.1005e-2	5.3920e-3	2.6870e-3
10^{-4}	2.4125e-2	1.1985e-2	5.9160e-3	2.9660e-3

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