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Existence and decay estimate of global solution for a viscoelastic wave equation with nonlinear boundary source term

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Abstract. In this paper, we consider a viscoelastic wave equation with nonlinear boundary source term. By the combination of Galerkin approximation and potential well methods, we prove the global existence of solutions. Then, we give an decay rate estimate of the energy by making use of the perturbed energy method.

Keywords. Viscoelastic equation, nonlinear boundary source term, global existence, potential well method, exponential stability

1 Introduction

This paper is concerned with the existence and uniform decay rate estimate for the following initial boundary value problem:

$$\begin{gathered} u'' - \Delta u - \Delta u'' + \int_0^t g(t-s)\Delta u(s)ds = 0, \text{ in } \Omega \times (0,\infty), \\ u(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} + \frac{\partial u''}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds = |u|^{p-1}u \quad \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \text{ in } \Omega, \end{gathered} \tag{1.1}$$

where $u = u(x,t)$, $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$ and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with a smooth boundary Γ. Let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and Γ_0, Γ_1 are positive measurable, endowed with the $(n-1)$-dimentional Lebesgue measure. Here, ν is the unit outward normal to Γ. The term $\Delta u''$ represents space-time dispersion and g represents the kernel of memory term, $p > 1$.

Partial differential equations in viscoelastic materials have important physical background and important mathematical significance. The viscous effects are described and characterized by an integral term, and the integral term indicates a dissipative effect. For mathematical analysis on the motions of evolution equations with memory, we refer to [1,2].

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The equation associated with Eq (1.1) is as follows

$$u_{tt} - u_{xx} - u_{xxtt} = 0,$$

which mainly describes a pure dispersion wave process, such as the motion equation of strain-arc wave of linear elastic rod considering transverse inertia and ion-acoustic wave propagation equation in space transformation with weak nonlinear effects (see e.g., [3-6]).

In [7], Han and Wang established the general decay of energy for the equation with integral dissipation and nonlinear damping

$$u'' - \Delta u - \Delta u'' + \int_0^t g(t-s)\Delta u(s)ds + |u'|^{m-2}u' = 0, \text{ in } \Omega \times (0, \infty).$$

Xu, Yang and Liu [8] studied an initial value problem for nonlinear viscoelastic wave equation with strong damping and dispersive terms,

$$u'' - \Delta u - \Delta u'' - \Delta u' + \int_0^t g(t-s)\Delta u(s)ds + u' = |u|^{p-1}u, \text{ in } \Omega \times (0, \infty),$$

By introducing a family of potential wells, they obtained the invariant sets and proved existence and nonexistence of global weak solutions with low initial energy. In the high energy case, they also established a blow-up result with arbitrary positive initial energy.

For the viscoelastic equation with nonlinear boundary condition, Yu et al [9] considered with the following viscoelastic wave equation with nonlinear boundary damping-source interactions.

$$\begin{gathered}
u'' - \Delta u - \Delta u'' - \Delta u' + \int_0^t g(t-s)\Delta u(s)ds = 0, \text{ in } \Omega \times (0, \infty),\\
u(x,t) = 0 \quad \text{on } \Gamma_0 \times (0, \infty),\\
\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} + \frac{\partial u''}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + |u'|^{m-2}u' = |u|^{p-2}u \quad \text{on } \Gamma_1 \times (0, \infty),\\
u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \text{ in } \Omega,
\end{gathered}$$

Under some appropriate assumptions on the boundary damping-source terms and the relaxation function g, they established general decay and blow-up results associated with solution energy. Estimates of the lifespan of solutions are also given.

Recently, Yu et al.[10] considered with the following viscoelastic wave equation with acoustic boundary conditions

$$\begin{gathered}
u'' - \Delta u - \Delta u'' - \Delta u' + \int_0^t g(t-s)\Delta u(s)ds + |u'|^{m-2}u' = |u|^{p-2}u, \text{ in } \Omega \times (0, \infty),\\
u(x,t) = 0 \quad \text{on } \Gamma_0 \times (0, \infty),\\
\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} + \frac{\partial u''}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + y(u') = h(x)y'(t) \quad \text{on } \Gamma_1 \times (0, \infty),\\
u' + f(x)y'(u) + q(x)y(t) = 0 \quad \text{on } \Gamma_1 \times (0, \infty),\\
u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \text{ in } \Omega\\
y(x,0) = y_0(x) \quad \text{on } \Gamma_1.
\end{gathered}$$

Under some small assumption on the relaxation function g and h, they proved that the solutions blow up in finite time if the positive initial energy satisfies a suitable condition.

More recently, Peyravi in Ref. [11] investigated a system of viscoelastic wave equations with nonlinear boundary source term of the form

$$
\begin{gathered}
(u_i)'' - \Delta(u_i) - \Delta(u_i)'' - \Delta(u_i)' + \int_0^t g_i(t-s)\Delta u_i(s)ds = 0, \text{ in } \Omega \times (0,T), \\
u_i(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,T), \\
\frac{\partial u_i}{\partial \nu} + \frac{\partial (u_i)'}{\partial \nu} + \frac{\partial (u_i)''}{\partial \nu} - \int_0^t g_i(t-s)\frac{\partial u_i}{\partial \nu}(s)ds + f_i(u_i) = 0 \quad \text{on } \Gamma_1 \times (0,T), \\
u_i(x,0) = \varphi_i(x), \quad (u_i)'(x,0) = \psi_i(x), \text{ in } \Omega,
\end{gathered}
$$

where $i = 1, ..., l (l \geq 2)$. They established decay and blow-up results associated with solution energy under quite restrictive assumptions on the relaxation function g.

Motivated by the above researches, in the present work we consider the viscoelastic wave equation with nonlinear boundary source term, we prove the existence of global weak solutions by the combination of Galerkin approximation and potential well methods. Then, we give an decay rate estimate of the energy by making use of the perturbed energy method.

Then, we give an decay rate estimate of the energy by using only one simple auxiliary functional

2 Preliminaries

Throughout this paper, the following inner products and norms are used for precise statement:

$$
(u,v) = \int_\Omega u(x)v(x)dx, \quad (u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)d\Gamma,
$$

$$
\|u\|_q = \|u\|_{L^q(\Omega)}, \quad \|u\|_{\Gamma_1,q} = \|u\|_{L^q(\Gamma_1)}.
$$

and the Hilbert space

$$
V = \left\{ u \in H^1(\Omega) | \ u = 0 \text{ on } \Gamma_0 \right\}.
$$

Since Γ_0 has positive $(n-1)$ dimensional Lebesgue measure, by Poincar'e inequality, we can endow V with the equivalent norm $\|u\|_V = \|\nabla u\|_2$ (see [12]).

Now we present the hypotheses for the main results.

(H1) For the nonlinear boundary term, assume

$$
\begin{cases} 1 < p < \infty, & \text{if} \quad n = 1,2, \\ 1 < p \leq \frac{n}{n-2}, & \text{if} \quad n \geq 3. \end{cases}
$$

(H2) We assume that $g : \mathbb{R}^+ \longrightarrow (0,\infty)$ is a bounded C^1 function satisfying

$$
l(t) = 1 - \int_0^t g(s)ds \geq 1 - \int_0^\infty g(s)ds = l > 0,
$$

and there exists a positive constant ξ such that

$$
g'(t) \leq -\xi g(t) \quad \forall t \geq 0.
$$

For simplicity of notations, we let

$$g * w = \int_0^t g(t-s)w(s)ds, \quad g \circ w = \int_0^t g(t-s)\|w(t)-w(s)\|_2^2 ds.$$

By direct calculations, we find

$$\begin{aligned}(g * w, w_t) &= -\frac{1}{2}g(t)\|w(t)\|_2^2 - \frac{d}{dt}\left\{\frac{1}{2}(g \circ w) - \frac{1}{2}\left(\int_0^t g(s)ds\right)\|w(t)\|_2^2\right\} \\ &\quad + \frac{1}{2}(g' \circ w), \end{aligned} \tag{2.1}$$

Next, we will introduce some functionals which are all associated with the potential wells. The energy associated with problem (1.1) is given by

$$\begin{aligned}\mathcal{E}(t) = \mathcal{E}(u, u') &= \frac{1}{2}\left\{\|u'\|_2^2 + \|\nabla u'\|_2^2 + l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)\right\} \\ &\quad - \frac{1}{p+1}\|u\|_{\Gamma_1, p+1}^{p+1} \\ &\geq \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|\nabla u'\|_2^2 + \mathcal{J}(u), \end{aligned} \tag{2.2}$$

$$\begin{aligned}\mathcal{J}(u) &= \frac{l}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u) - \frac{1}{p+1}\|u\|_{\Gamma_1, p+1}^{p+1} \\ &= \frac{p-1}{2(p+1)}\left(l\|\nabla u\|_2^2 + (g \circ \nabla u)\right) + \frac{1}{p+1}\mathcal{I}(u), \end{aligned}$$

$$\mathcal{I}(u) = l\|\nabla u\|_2^2 + g \circ \nabla u - \|u\|_{\Gamma_1, p+1}^{p+1}.$$

The corresponding potential well set is defined in the form

$$\mathcal{W} = \{u \in V |\ \mathcal{I}(u) > 0,\ \mathcal{J}(u) < d_1\} \cup \{0\}.$$

For $t \geq 0$, we define

$$d(t) = \inf_{u \in V \setminus \{0\}} \sup_{\lambda \geq 0} \mathcal{J}(\lambda u).$$

The following lemma is similar to the lemmas of Ref. [9] with slight modification.

Lemma 2.1. *Let the assumptions* $(H1)$ *and* $(H2)$ *hold, then for* $t \geq 0$*, we have*

$$0 < d_1 \leq d(t) \leq d_2(u) = \sup_{\lambda \geq 0} \mathcal{J}(\lambda u),$$

where

$$d_1 = \frac{p-1}{2(p+1)}\left(\frac{l}{C_{p+1}^2}\right)^{\frac{p+1}{p-1}},$$

$$d_2(u) = \frac{p-1}{2(p+1)}\left(\frac{l\|\nabla u\|_2^2 + g \circ \nabla u}{\|u\|_{\Gamma_1, p+1}^2}\right)^{\frac{p+1}{p-1}},$$

where C_{p+1} *is the optimal constant of the Sobolev embedding* $V \hookrightarrow L^{p+1}(\Gamma_1)$*.*

We recall the trace Sobolev embedding inequality

$$V \hookrightarrow L^q(\Gamma_1) \quad \text{for} \quad 2 \le q < \frac{2(n-1)}{n-2}$$

and the embedding inequality

$$\|u\|_{\Gamma_1,q} \le C_q \|\nabla u\|_2,$$

where C_q is the optimal constant.

Lemma 2.2. *Assume that* $(H1)$, $(H2)$ *hold. Let* u *be a solution of problem* (1.1). *Then, the energy functional* $\mathcal{E}(t)$ *of problem* (1.1) *is non-increasing. Moreover, the following energy inequality holds:*

$$\frac{d}{dt}\mathcal{E}(t) = \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 \le 0. \tag{2.3}$$

For simplicity, we define the weak solutions of (1.1) over the interval $\Omega \times [0,T)$, with $0 \le t < T$ being the maximal existence time.

Definition 1. A function $u = u(x,t)$ is called a weak solution of (1.1) on the interval $\Omega \times [0,T)$, if

$$u \in L^\infty(0,T;V), \quad u' \in L^\infty(0,T;V) \;\text{ and }\; u'' \in L^\infty(0,T;V)$$

satisfy the following conditions:

(1) for any $\phi \in V$ and a.e. $t \in [0,T)$, we have

$$(u'',\phi) + (\nabla u, \nabla\phi) + (\nabla u'', \nabla \phi) - \left(\int_0^t g(t-s)\nabla u(s)ds, \nabla\phi\right) = (|u|^{p-1}u, \phi)_{\Gamma_1},$$

(2) $(u(x,0), u'(x,0)) = (u_0(x), u_1(x))$ in $V \times V$.

(3) the energy inequality

$$\mathcal{E}(t) \le \mathcal{E}(0)$$

holds for any $0 \le t < T$.

The following Lemma is similar to the Lemma 3(1) of [13] with slight modification.

Lemma 2.3. *Let the assumptions* $(H1)$, $(H2)$ *hold and* u *be a solution of problem* (1.1). *Further assume that* $u_0, u_1 \in V$, *we have*
If $\mathcal{E}(0) < d_1$, $\mathcal{I}(u_0) > 0$ *or* $\|u_0\|_V = 0$, *then the solution* $u \in \mathcal{W}$ *for all* $t \in [0,T)$.

3 Existence of global weak solutions

In this section, we are going to obtain the existence of global weak solutions for the problem (1.1) with the initial conditions $\mathcal{E}(0) < d_1$ and $\mathcal{I}(u_0) > 0$ or $\|u_0\|_V = 0$ by the combination of Galerkin approximation and potential well methods benefited from the ideas of [13]-[14].

Theorem 3.1. *Suppose that the assumptions* $(H1)$, $(H2)$ *hold and* $u_0(x)$, $u_1(x) \in V$. *If in addition, the initial data satisfy* $\mathcal{E}(0) < d_1$, $\mathcal{I}(u_0) > 0$ *or* $\|u_0\|_V = 0$, *then the problem* (1.1) *admits a global weak solution*

$$u \in L^\infty(0,\infty;V), \quad u' \in L^\infty(0,\infty;V), \quad u'' \in L^\infty(0,\infty;V) \ \ \textit{and} \ \ u \in \mathcal{W}.$$

Proof. Let $\{\omega_j(x)\}$ be a complete orthogonal system in V. We suppose that the approximate weak solution u_k of the problem (1.1) can be written

$$u_k(t) = \sum_{j=1}^{k} \eta_{kj}(t)\omega_j(x), \quad k = 1, 2, ...,$$

According to Galerkin's method, these coefficients η_{kj} need to satisfy the following initial value problem of nonlinear ordinary integro-differential equations

$$\begin{aligned}&(u_k'', \omega_j) + (\nabla u_k, \nabla\omega_j) + (\nabla u_k'', \nabla\omega_j) - \left(\int_0^t g(s)\nabla u_k(s)ds, \nabla\omega_j\right)\\ &= (|u_k|^{p-1}u_k, \omega_j)_{\Gamma_1}, \quad j = 1, 2, ..., k,\end{aligned} \tag{3.1}$$

$$u_k(x,0) = \sum_{j=1}^{k} \eta_{kj}(0)\omega_j(x) \to u_0(x) \ \text{ in } \ V, \quad k \to \infty, \tag{3.2}$$

$$u_k'(x,0) = \sum_{j=1}^{k} \eta_{kj}'(0)\omega_j(x) \to u_1(x) \ \text{ in } \ V, \quad k \to \infty. \tag{3.3}$$

We will prove that the initial value problem (3.1) − (3.3) of the nonlinear integro-differential equations have global weak solutions in the interval $[0,\infty)$. Furthermore, we show that the solutions of the problem (1.1) can be approximated by the functions u_k.

Now, multiplying (3.1) by $\eta_{jk}'(t)$ and summing for $j = 1, ..., k$ and using identity (2.1), we have

$$\frac{d}{dt}\mathcal{E}_k(t) = -\frac{1}{2}g(t)\|\nabla u_k\|_2^2 + \frac{1}{2}(g' \circ \nabla u_k) \le 0, \tag{3.4}$$

where $\mathcal{E}_k(t)$ is the energy functional (2.2) for Galerkin solutions u_k.

In view of (2.4) and (3.4), we obtain

$$\frac{1}{2}\|u_k'\|_2^2 + \frac{1}{2}\|\nabla u_k'\|_2^2 + \mathcal{J}(u_k) \le \mathcal{E}_k(t) \le \mathcal{E}_k(0), \ 0 \le t < T. \tag{3.5}$$

By the hypotheses $\mathcal{E}(0) < d_1$ and $\mathcal{I}(u_0) > 0$ or $\|u_0\|_V = 0$, it follows that $u_0 \in \mathcal{W}$. So applying (3.2) and (3.3), we further get that $\mathcal{E}_k(0) < d_1$, $\mathcal{I}(u_k(0)) > 0$ and then $u_k(0) \in \mathcal{W}$ for sufficiently large k. In what follows, from the (3.5) and Lemma 2.3, we can obtain $u_k(t) \in \mathcal{W}$ for sufficiently large k and $0 \le t < \infty$ such that

$$\begin{aligned}\mathcal{E}_k(t) \ge \mathcal{J}(u_k) \ &= \ \frac{p-1}{2(p+1)}\left(l\|\nabla u_k\|_2^2 + g \circ \nabla u_k\right) + \frac{1}{p}\mathcal{I}(u_k)\\ &\ge \ \frac{p-1}{2(p+1)}\left(l\|\nabla u_k\|_2^2 + g \circ \nabla u_k\right)\\ &\ge \ 0.\end{aligned} \tag{3.6}$$

Hence, we have from (3.6) and (3.5) that

$$\frac{1}{2}\|u_k'\|_2^2+\frac{1}{2}\|\nabla u_k'\|_2^2+\frac{p-1}{2(p+1)}\left(l(t)\|\nabla u_k\|_2^2+g\circ\nabla u_k\right)$$
$$\leq \mathcal{E}_k(t)<d_1, \tag{3.7}$$

for sufficiently large k and $t\in[0,\infty)$. By (3.7) , we have that

$$\|u_k'\|_2^2<2d_1,\quad 0\leq t<\infty, \tag{3.8}$$

$$\|\nabla u_k'\|_2^2<2d_1,\quad 0\leq t<\infty, \tag{3.9}$$

$$\|\nabla u_k\|_2^2<\frac{2(p+1)}{l(p-1)}d_1,\quad 0\leq t<\infty, \tag{3.10}$$

Using the Sobolev inequality and (3.10), it follows that

$$\begin{aligned}\int_{\Gamma_1}\left||u_k|^{p-1}u_k\right|^{\frac{p+1}{p}}d\Gamma &= \int_{\Gamma_1}|u_k|^{p+1}\,d\Gamma\\ &= \|u_k\|_{\Gamma_1,p+1}^{p+1}\\ &\leq C_{p+1}^{p+1}\|\nabla u_k\|_2^{p+1}\\ &\leq C_{p+1}^{p+1}\left(\frac{2(p+1)}{l(p-1)}d_1\right)^{\frac{p+1}{2}},\quad 0\leq t<\infty,\end{aligned} \tag{3.11}$$

Integrating (3.4) from 0 to t, we get

$$\frac{1}{2}\int_0^t g(t-s)\|\nabla u_k(s)\|_2^2ds-\frac{1}{2}\int_0^t(g'\circ\nabla u_m)(s)ds=E_k(0)-E_k(t),$$

which together with (3.6), (3.7) and condition $(H2)$ yields that

$$\int_0^t g(t-s)\|\nabla u_k(s)\|_2^2ds\leq 2E_k(0)\leq 2d_1, \tag{3.12}$$

Now,multiplying (3.1) by $\delta_{kj}''(t)$ and summing for $j=1,...,k$, we obtain

$$(u_k''(t),u'')+(\nabla u_k,\nabla u_k'')+(\nabla u_k'',\nabla u_k'')-\left(\int_0^t g(t-s)\nabla u_k(s)ds,\nabla u_k''\right)$$
$$=(|u_k|^{p-1}u_k,u_k'')_{\Gamma_1}, \tag{3.13}$$

Applying the Hölder's, Young's, Sobolev inequalities and trace theorem, we have form (3.13) and $(H1)$ that

$$\begin{aligned}&\|u_k''\|_2^2+\|\nabla u_k''\|_2^2\\ =\;&-(\nabla u_k,\nabla u_k'')+\left(\int_0^t g(t-s)\nabla u_k(s)ds,\nabla u_k''\right)+(|u_k|^{p-1}u_k,u_k'')_{\Gamma_1}\\ \leq\;&2\gamma\|\nabla u_k''\|_2^2+\frac{\gamma C_2^2}{2}\|\nabla u_k''\|_2^2+\frac{C_{2p}^{2p}}{8\gamma}\|\nabla u_k\|_2^{2p}+\frac{1}{4\gamma}\|\nabla u_k\|_2^2\\ &+\frac{1}{4\gamma}\int_0^t g(s)ds\int_0^t g(t-s)\|\nabla u_k(s)\|_2^2ds,\end{aligned} \tag{3.14}$$

Let us take γ small enough such that $1-2\gamma-\frac{\gamma C_2^2}{2}>0$. So, by a simple calculation, (3.14) becomes

$$\begin{aligned}
&\|u_k''\|_2^2+\left(1-2\delta-\frac{\delta C_2^2}{2}\right)\|\nabla u_k''\|_2^2\\
\leq\;&\frac{C_{2p}^{2p}}{8\delta}\|\nabla u_k\|_2^{2p}+\frac{1}{4\delta}\|\nabla u_k\|_2^2\\
&+\frac{1}{4\delta}\int_0^t g(s)ds\int_0^t g(t-s)\|\nabla u_k(s)\|_2^2ds, \qquad (3.15)
\end{aligned}$$

Using (3.10) and (3.12), we can obtain from (3.15) the following inequality

$$\|\nabla u_k''\|_2^2\leq K,\quad 0\leq t<\infty, \qquad (3.16)$$

where K is a positive constant independent of time t.

Furthermore, by (3.11), we get

$$\begin{aligned}
\left|(|u_k|^{p-1}u_k,u_k)_{\Gamma_1}\right| &= \|u_k\|_{\Gamma_1,p+1}^{p+1}\\
&\leq C_{p+1}^{p+1}\left(\frac{2(p+1)}{l(p-1)}d_1\right)^{\frac{p+1}{2}},\quad 0\leq t<\infty, \qquad (3.17)
\end{aligned}$$

The estimates (3.8)-(3.10), (3.16) and (3.17) permit us to obtain a subsequences of $\{u_k\}$ which from now on will be also denoted by $\{u_k\}$ and functions u, χ such that

$$u_k\to u\ \text{ weak star in }\ L^\infty(0,\infty;V),\quad k\to\infty, \qquad (3.18)$$

$$u_k'\to u'\ \text{ weak star in }\ L^\infty(0,\infty;V),\quad k\to\infty, \qquad (3.19)$$

$$u_k''\to u''\ \text{ weakly }\ L^\infty(0,\infty;V),\quad k\to\infty, \qquad (3.20)$$

$$|u_k|^{p-1}u_k\to\chi\ \text{ weak star in }\ L^\infty(0,\infty;L^{\frac{p+1}{p}}(\Gamma_1)),\quad k\to\infty, \qquad (3.21)$$

Since $V\hookrightarrow L^2(\Omega)$ is compactness, we have, thanks to Aubin-Lions theorem, that

$$u_k\to u\ \text{ strongly in }\ L^\infty(0,\infty;L^2(\Omega)),\quad k\to\infty, \qquad (3.22)$$

$$u_k'\to u'\ \text{ strongly in }\ L^\infty(0,\infty;L^2(\Omega)),\quad k\to\infty, \qquad (3.23)$$

$$u_k''\to u''\ \text{ strongly in }\ L^\infty(0,\infty;L^2(\Omega)),\quad k\to\infty, \qquad (3.24)$$

and further using Lemma 1.3 in [15], we can deduce

$$|u_k|^{p-1}u_k\to|u|^{p-1}u=\chi\ \text{ weak star in }\ L^\infty(0,\infty;L^{\frac{p+1}{p}}(\Gamma_1)),\quad k\to\infty, \qquad (3.25)$$

Taking $k\to\infty$ in the (3.1) and then making use of (3.18)-(3.20) and (3.25), we obtain

$$\begin{aligned}
&(u'',\omega_j)+(\nabla u,\nabla\omega_j)+(\nabla u'',\nabla\omega_j)\\
&-\left(\int_0^t g(t-s)\nabla u(s)ds,\nabla\omega_j\right)=(|u|^{p-1}u,\omega_j)_{\Gamma_1} \qquad (3.26)
\end{aligned}$$

Considering that the basis $\{\omega_j(x)\}_{j=1}^\infty$ are dense in V, we choose a function $\phi\in V$ having the form $\phi=\sum_{j=1}^\infty\eta_j\omega_j(x)$, where $\{\eta_j\}_{j=1}^\infty$ are given functions.

Multiplying (3.26) by η_j and then summing for $j=1,.....$, it follows that

$$\begin{aligned}
&(u'',\phi)+(\nabla u,\nabla\phi)+(\nabla u'',\nabla\phi)\\
&-\left(\int_0^t g(t-s)\nabla u(s)ds,\nabla\phi\right)=(|u|^{p-1}u,\phi)_{\Gamma_1},\quad\forall\phi\in V.
\end{aligned}$$

Hence, the global existence of weak solutions is completed. □

4 Decay estimate

In this section, we shall prove the energy decay estimate of the global solutions obtained in the previous section by making use of the perturbed energy method coupled with some technical Lemmas inspired by the contribution of [16].

Theorem 4.1. *Let the assumptions* $(H1)$, $(H2)$ *hold and* $u_0(x)$, $u_1(x) \in V$. *Further assume that* $\mathcal{E}(0) < d_1$ *and* $\mathcal{I}(u_0) > 0$. *Then there exist two positive constants* M *and* κ *such that the solution of the problem* (1.1) *satisfies*

$$\mathcal{E}(t) \leq Me^{-\kappa t}, \quad t \geq 0. \tag{4.1}$$

For this purpose, we introduce the functional

$$\mathcal{E}_\epsilon(t) = \mathcal{E}(t) + \epsilon\psi(t), \tag{4.2}$$

where

$$\psi(t) = -\int_\Omega uu'dx - \int_\Omega \nabla u \cdot \nabla u'dx - 2\int_0^t \|\nabla u(s)\|_2^2 ds, \tag{4.3}$$

Lemma 4.1. *Let* $u \in L^\infty(0,\infty;V)$ *be the solution of* (1.1) *and* $\mathcal{E}(0) < d_1$, $\mathcal{I}(u_0) > 0$, *then we have*

$$\mathcal{E}(t) \geq \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|\nabla u'\|_2^2 + \frac{(p-1)l}{2(p+1)}\|\nabla u\|_2^2. \tag{4.4}$$

Proof. From $\mathcal{E}(0) < d_1$, $\mathcal{I}(u_0) > 0$ and Lemma 2.3, we can obtain $u(t) \in \mathcal{W}$ for $0 \leq t < \infty$. Thus we have

$$\begin{aligned}
\mathcal{E}(t) &= \frac{1}{2}\left\{\|u'\|_2^2 + \|\nabla u'\|_2^2 + l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)\right\} \\
&\quad -\frac{1}{p+1}\|u\|_{\Gamma_1,p+1}^{p+1} \\
&\geq \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|\nabla u'\|_2^2 + \mathcal{J}(u) \\
&\geq \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|\nabla u'\|_2^2 + \frac{p-1}{2(p+1)}\left(l\|\nabla u\|_2^2 + (g \circ \nabla u)\right) + \frac{1}{p+1}\mathcal{I}(u) \\
&\geq \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|\nabla u'\|_2^2 + \frac{(p-1)l}{2(p+1)}\|\nabla u\|_2^2.
\end{aligned}$$

□

In order to derive the energy decay estimate (4.1), we divide the proof into two cases.

Case 1

Suppose that

$$\int_0^t \|\nabla u(s)\|_2^2 ds \leq \mathcal{E}(t) \tag{4.5}$$

for all $t \geq 0$. "In this situation, we first prove two lemmas, "and" then based on them, we complete the proof".

Lemma 4.2. *We have* $\left|\mathcal{E}_\epsilon(t) - \mathcal{E}(t)\right| \leq \epsilon C_1 \mathcal{E}(t)$, $\forall t \geq 0$ *and* $\forall \epsilon > 0$.

Proof. From (4.3) and (4.4), we have

$$\begin{aligned}
\left|\psi(t)\right| &\leq \left|\int_\Omega uu'dx\right| + \left|\int_\Omega \nabla u \cdot \nabla u' dx\right| + 2\int_0^t \|\nabla u(s)\|_2^2 ds \\
&\leq \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u'\|_2^2 + 2\mathcal{E}(t) \\
&\leq \frac{1}{2}\left\{\|u'\|_2^2 + \|\nabla u'\|_2^2\right\} + \frac{1}{2}(1+B^2)\|\nabla u\|_2^2 + 2\mathcal{E}(t) \\
&\leq 3\mathcal{E}(t) + \frac{1}{2}(1+B^2)\frac{2(p+1)}{l(p-1)}\mathcal{E}(t) \\
&= \left[3 + (1+B^2)\frac{p+1}{l(p-1)}\right]\mathcal{E}(t),
\end{aligned}$$

where B is the optimal constant satisfying the Poincaré inequality $\|u\|_2 \leq B\|\nabla u\|_2$.

Hence we have

$$\left|\mathcal{E}_\epsilon(t) - \mathcal{E}(t)\right| = \epsilon\left|\psi(t)\right| \leq \epsilon C_1 \mathcal{E}(t)$$

where $C_1 = 3 + (1+B^2)\frac{p+1}{l(p-1)} > 0$. □

Lemma 4.3. *There exists* $\epsilon_1 > 0$ *such that*

$$\frac{d}{dt}\mathcal{E}_\epsilon(t) \leq -2\epsilon\mathcal{E}(t) \quad \forall t \geq 0 \quad \textit{and} \quad \forall \epsilon \in (0, \epsilon_1]. \tag{4.6}$$

Proof. Differentiating $\psi(t)$ and using (1.1), we have

$$\begin{aligned}
\frac{d}{dt}\psi(t) &= -\|u'\|_2^2 - \int_\Omega u''udx - \|\nabla u'\|_2^2 - \int_\Omega \nabla u \cdot \nabla u'' dx - 2\|\nabla u\|_2^2 \\
&= -\|u'\|_2^2 - \|\nabla u'\|_2^2 - \|\nabla u\|_2^2 - \|u\|_{\Gamma_1,p+1}^{p+1} \\
&\quad - \int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx \\
&\leq -\|u'\|_2^2 - \|\nabla u'\|_2^2 - \|\nabla u\|_2^2 - \int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx. \qquad (4.7)
\end{aligned}$$

From Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
-\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx &= \int_0^t g(t-s)\int_\Omega \nabla u(t)\left[\nabla u(t) - \nabla u(s)\right]dxds\Big| \\
&\quad - \int_0^t g(s)ds\|\nabla u(t)\|_2^2 \\
&\leq \frac{1}{2}\int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 dx + \frac{1}{2}\int_0^t g(s)ds\|\nabla u(t)\|_2^2 \\
&\quad - \int_0^t g(s)ds\|\nabla u(t)\|_2^2 \\
&\leq \frac{1}{2}\int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 dx = \frac{1}{2}(g\circ\nabla u).
\end{aligned}$$

Therefore

$$\frac{d}{dt}\psi(t) \le -\|u'\|_2^2 - \|\nabla u'\|_2^2 - \|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u).$$

Subtracting and adding $2\mathcal{E}(t)$ yields

$$\frac{d}{dt}\psi(t) \quad \le \quad -2\mathcal{E}(t) + \frac{3}{2}(g \circ \nabla u).$$

Moreover, by $(H2)$, we get from (2.3) that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 \\ &\le \frac{1}{2}(g' \circ \nabla u) \\ &\le -\frac{\xi}{2}(g \circ \nabla u). \end{aligned}$$

Hence from (4.2) and the above inequality, we get

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_\epsilon(t) &= \frac{d}{dt}\mathcal{E}(t) + \epsilon\frac{d}{dt}\psi(t) \\ &\le -2\epsilon\mathcal{E}(t) - \frac{1}{2}(g \circ \nabla u)\left[\xi - 3\epsilon\right]. \end{aligned} \tag{4.8}$$

Defining

$$\epsilon_1 = \frac{\xi}{3},$$

we conclude by taking $\epsilon \in (0, \epsilon_1]$ in (4.8) that

$$\frac{d}{dt}\mathcal{E}_\epsilon(t) \le -2\epsilon\mathcal{E}(t). \tag{4.9}$$

This concludes the proof of Lemma 4.3.

Let us define $\epsilon_2 = \min\left\{\frac{1}{2C_1}, \epsilon_1\right\}$ and consider $\epsilon \in (0, \epsilon_2]$. From Lemma 4.2 we have

$$(1 - \epsilon C_1)\mathcal{E}(t) \le \mathcal{E}_\epsilon(t) \le (1 + \epsilon C_1)\mathcal{E}(t), \tag{4.10}$$

and so

$$\frac{1}{2}\mathcal{E}(t) \le \mathcal{E}_\epsilon(t) \le \frac{3}{2}\mathcal{E}(t). \tag{4.11}$$

From (4.11) we get

$$-2\epsilon\mathcal{E}(t) \le -\frac{4}{3}\epsilon\mathcal{E}_\epsilon(t). \tag{4.12}$$

Hence from (4.12) and (4.9), we obtain

$$\frac{d}{dt}\mathcal{E}_\epsilon(t) \le -\frac{4}{3}\epsilon\mathcal{E}_\epsilon(t). \tag{4.13}$$

Integrating the last inequality over $[0, t]$, we get

$$\mathcal{E}_\epsilon(t) \le \mathcal{E}_\epsilon(0)e^{-\frac{4}{3}\epsilon t}. \tag{4.14}$$

From (4.14) and (4.11), we have

$$\mathcal{E}(t) \le 3\mathcal{E}(0)e^{-\frac{4}{3}\epsilon t}, \quad t \ge 0, \quad \forall \epsilon \in (0, \epsilon_2], \tag{4.15}$$

that is,

$$\mathcal{E}(t) \le Me^{-\kappa t} \quad \forall t \ge 0, \tag{4.16}$$

where $M = 3\mathcal{E}(0)$ and $\kappa = \frac{4}{3}\epsilon$. The proof of Case 1 is completed.

Case 2

Suppose that there exists a $t_0 \geq 0$ such that

$$\int_0^{t_0} \|\nabla u(s)\|_2^2 ds > \mathcal{E}(t_0). \tag{4.17}$$

Without loss of generality, we suppose that $t_0 > 0$ and is the first one such that the above inequality holds. This falls out that

$$\int_0^{t_0} \|\nabla u(s)\|_2^2 ds \leq \mathcal{E}(t) \quad \forall t \in [0, t_0). \tag{4.18}$$

Then along the line of proofs for Case 1, we deduce that (4.16) holds for each $t \in [0, t_0)$. On the other hand, the non-increasing property of $\mathcal{E}(t)$, we easily get that for all $t \geq t_0$,

$$\begin{aligned} \|\nabla u(t)\|_2^2 &\leq C\mathcal{E}(t) \\ &\leq C\mathcal{E}(t_0) \\ &\leq C\int_0^{t_0} \|\nabla u(s)\|_2^2 ds \\ &\leq C\int_0^{t} \|\nabla u(s)\|_2^2 ds, \end{aligned} \tag{4.19}$$

where C is a positive constant. Then by the Gronwall inequality, we infer from (4.19) that $u = 0$ as $t \geq t_0$. Therefore, the decay property of u is trivial for $t \geq t_0$. Case 2 is proved. Combining Case 1 and Case 2, we complete the proof Theorem 4.1. □

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