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Approximating the fixed points of Suzuki's generalized non-expansive map via an efficient iterative scheme with an application

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Abstract. This paper is aimed at proving the efficiency of a faster iterative scheme called PC^*-iterative scheme to approximate the fixed points for the class of Suzuki's Generalized non-expansive mapping in a uniformly convex Banach space. We will prove some weak and strong convergence results. It is justified numerically that the PC^*-iterative scheme converges faster than many other remarkable iterative schemes. We will also provide numerical illustrations with graphical representations to prove the efficiency of PC^* iterative scheme. As an application of the solution of a fractional differential equation is obtained by using PC^* iterative scheme.

Keywords. Suzuki's generalized non-expansive mapping, iterative scheme, uniformly convex Banach space, fixed point

1 Introduction

Iterative schemes plays an important role in approximating the fixed point in the field of fixed point theory. Various problems in applied sciences uses iterative schemes as an important tool and helps to solve many non-linear problems in different fields like Differential equations, Engineering, Integral equations, Game theory, Approximation theory etc. In 1922 [4], Stefen Banach used Picard iterative scheme [25] to prove the existence of a unique fixed point for a contraction map in the framework of a complete metric space. The generalizations of Banach contraction mapping principle are attained by weakening the contractive conditions and to compensate that the structure of the metric space is enriched by endowing it with some geometrical properties. In 1955 Kranoselekii [21] proved that for non-expansive mapping Picard iteration scheme may fail to converge to a fixed point even if the map T has a unique fixed point. Browder [8], Gohde [14], Kirk [20] studied non expansive maps independently. After this many other iterative schemes were introduced such as Mann[24], Ishikawa [18] so on. An iterative scheme is considered better than the other if it approaches to the fixed point in lesser number of iterations. Over the last decade, the area of approximating fixed points via iterative scheme has become very popular amongst the researchers. The notion of generalized non-expansive mapping was given by Hardy-Rogers [15] in 1973.

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Definition 1. [15] Let X be a nonempty subset of a Banach space Y. A self map $G :\to G$ is said to be a generalized non-expansive map if for all $x, y \in X$,

$$\|Gx - Gy\| \leq a\|x - y\| + b(\|x - Gx\| + \|y - Gy\|) + c(\|x - Gy\| + \|y - Gx\|), \tag{1.1}$$

The class of generalized non-expansive mappings contain the class of non-expansive mappings. The class of generalized non-expansive mappings and non-expansive mappings has been studied by many researchers. Some of the recent and interesting works can be referred to in [1] ,[3], [4],[5],[6], [10], [11], [12], [16],[19], [26], [28], [29], [32],[34], [35] and [36]. In 2008, Suzuki [34] introduced the concept of Suzuki's Generalized non expansive map defined as
A self map $T : C \to C$ where C is a non empty subset of a Banach Space is a Suzuki's Generalized non expansive map if for all $x, y \in C$

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\| \tag{1.2}$$

In 2008, [34] Suzuki proved that the class of Suzuki's generalized non-expansive map contains the class of non-expansive maps.

In this paper, we aim at proving that PC^*-iterative scheme for approximating the fixed points of Suzuki's generalized non-expansive map. The convergence and stability results are proved and numerical examples along with graphical representation are provided. The numerical experiments are performed using Python, a freely available computer programming language.

2 Preliminaries

This section contains some basic definitions and results which are required for our main results. Throughout this paper, we will consider X be a nonempty subset of a Banach space Y.

Definition 2. Let X be a nonempty subset of a Banach space Y. A mapping $F : X \to X$ is said to be a non-expansive map if for all $x, y \in X$,

$$\|Fx - Fy\| \leq \|x - y\|. \tag{2.1}$$

Definition 3. A map $G : X \to X$ is called quasi non-expansive if $Fix(S) \neq \phi$ and $\|Gx - y\| \leq \|x - y\|$ for all $x \in X$ and $y \in Fix(S)$.

Remark 1. A self map $T : [0, 1] \to [0, 1]$ defined as $Tx = 1-x$ doesn't converge to its fixed point $\frac{1}{2}$ via Picard iteration process. The Picard iteration gives an oscillatory sequence x_0, $1 - x_0$, x_0, $1 - x_0$,.... for any initial guess x_0. Clearly it doesn't converge to $\frac{1}{2}$. Hence, the Picard iteration process fails to converge to the fixed point for a non-expansive map.

Definition 4. [19] A normed linear space X is said to be strictly convex if, for $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$ and $\|(1 - \lambda)x + \lambda y\| = 1$ for $\lambda \in (0, 1)$ if and only if $x = y$.

Definition 5. [19] A Banach space $(X, \|.\|)$ is said to be uniformly convex if, for any $\epsilon > 0$ there exists $\delta > 0$ such that for $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| > \epsilon$ we have

$$\frac{1}{2}\|x + y\| < 1 - \delta.$$

Example 1. Every Hilbert space is a uniformly convex Banach space.

Definition 6. [30] A self map S on X, where X is nonempty closed and convex subset of a Banach space is said to satisfy condition I, if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(r) > 0$ for all $r > 0$ such that $d(x, Sx) \geq h(d(x, Fix(S))$ for all $x \in X$, where $d(x, Fix(S)) = inf d(x, t) : t \in Fix(S)$.

In 1953, Mann [24] approximated the fixed points of non-expansive map. For an initial guess $x_o \in X$ he gave the iterative scheme defined as follows.

$$x_{n+1} = (1 - a_n)x_n + a_n Gx_n,$$

where $\{a_n\} \subset (0, 1)$. The scheme defined by Mann [21] fails to converge to a fixed point of pseudo-contractive mappings. In 1974 [18] Ishikawa introduced a two-step iterative scheme to approximate the fixed points of pseudo-contractive mappings as follows

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n Gy_n \\ y_n &= (1 - b_n)x_n + b_n Gx_n, \;\; n \in \mathbb{N} \end{aligned} \tag{2.2}$$

where $\{a_n\}, \{b_n\} \subset (0, 1)$.
X is a nonempty convex subset of a Banach space Y and T is a self map on X, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in $(0, 1)$.

We consider the iterative schemes are M iterative scheme due to Ullah et al.[32], K iteration process due to N. Hussain et al. [17], M^* as defined in [7] and D plus iteration process due to Danish ali et al. [1].

$$\begin{aligned} z_n &= (1 - a_n)x_n + a_n Tx_n \\ y_n &= Tz_n \\ x_{n+1} &= Ty_n, n \in \mathbb{Z}^+. \end{aligned} \tag{1.1}$$

$$\begin{aligned} z_n &= (1 - a_n)x_n + a_n Tx_n \\ y_n &= T((1 - b_n)Tx_n + b_n Tz_n \\ x_{n+1} &= Ty_n, n \in \mathbb{Z}^+. \end{aligned} \tag{1.2}$$

$$\begin{aligned} z_n &= (1 - a_n)x_n + a_n Tx_n \\ y_n &= T((1 - b_n)z_n + b_n Tz_n \\ x_{n+1} &= Ty_n, n \in \mathbb{Z}^+. \end{aligned} \tag{1.3}$$

$$\begin{aligned} z_n &= T((1 - a_n)x_n + a_n Tx_n) \\ y_n &= T((1 - b_n)z_n + b_n Tz_n) \\ x_{n+1} &= T((1 - c_n)Tz_n + c_n Tz_n), n \in \mathbb{Z}^+. \end{aligned} \tag{1.4}$$

We raise a natural question that arises, is it possible to define an iterative scheme that has a faster convergence rate than the schemes defined above when T is Suzuki's Generalized non-expansive map?

As a response to this question, we introduce a new iterative scheme called PC^*-iterative scheme as follows. Let X be a nonempty convex subset of a Banach space Y and T is a self map on X. The sequence $\{x_n\}$ with an initial guess x_0 is defined as

$$\begin{aligned} x_{n+1} &= T^k y_n \\ y_n &= T^k z_n \\ z_n &= T^k((1-a_n)x_n + a_n T^k x_n),\ n \in \mathbb{Z}^+, \end{aligned} \tag{1.5}$$

where $\{a_n\}$ be a sequence in $(0,1)$ and $k = 2,3,4,5$. The PC^*-iterative scheme is collection of four iterative schemes which varies for $k = 2,3,4,5$. We claim that the PC^*-iterative scheme gives a much faster rate of convergence than the iterative schemes (1.1),(1.2),(1.3) and (1.4) for Suzuki's generalized non-expansive map. Also it is to be noted that the PC^*-iterative scheme depends only on one control sequence $\{a_n\}$ whereas the iterative schemes (1.2) and (1.3) are depending on two or more control sequences $\{a_n\}$ $\{b_n\}$ and $\{c_n\}$ in $(0,1)$.

Definition 7. [13] Let X be a nonempty, closed and convex subset of a Banach space Y. A mapping $T : X \to Y$ is said to be demiclosed with respect to $v \in Y$ if for each weakly convergent sequence $\{p_n\}$ at $u \in X$ and Tp_n converges strongly at v implies that $Tu = v$

Definition 8. [25] A Banach space Y is said to satisfy Opial's property if

$$\lim_{n\to\infty} \|p_n - u\| < \liminf_{n\to\infty} \|p_n - v\|$$

holds, for all $v \in Y$ with with $v \neq u$, where $\{p_n\}$ is an arbitrary sequence converges weakly to u in Y.

Definition 9. [13] Let X be a nonempty, closed and convex subset of a Banach space Y. Let $\{p_n\}$ be a bounded sequence in Y and for $u \in X$,

$$r(u,\{p_n\}) = \limsup_{n\to\infty} \|p_n - u\|.$$

The asymptotic radius of $\{p_n\}$ of $\{p_n\}$ relative to X is defined by

$$r(X,\{p_n\}) = inf\{r(u,\{p_n\}) : u \in X\}.$$

The asymptotic center of $\{p_n\}$ relative to X is defined by

$$A(X,\{p_n\}) = \{u \in X : r(u,\{p_n\}) = r(X,\{p_n\})\}.$$

Remark 2. If Y is a uniformly convex Banach space, then $r(X,\{p_n\})$ is singleton.

Proposition 2.1. *[34] Let T be a self mapping on a nonempty subset X of a Banach space Y. Then*

1. *If T is non-expansive then T is Suzuki's generalized non expansive map.*
2. *Every Suzuki's generalized non expansive map with a fixed point is quasi non-expansive.*
3. *If T is Suzuki's generalized non expansive map then $\|x - y\| \le 3\|Tx - y\| + \|x - y\|$ for all $x, y \in X$.*

Lemma 2.1. *[34] Let T be a Suzuki's generalized non-expansive map on a weakly compact convex subset X of a uniformly convex Banach space Y satisfying (1.2). Then T has a fixed point.*

Lemma 2.2. *[34] Let T be a Suzuki's generalized non-expansive self map on a subset X of a Banach space Y with Opial's property. If x_n converges weakly to p and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tz = z$ i.e $I - T$ is demiclosed at zero.*

Lemma 2.3. *[31] Let Y be a uniformly convex Banach space and $0 < a < d_n \leq d < 1$for all $n \in \mathbb{N}$. Assume that $\{p_n\}$ and $\{q_n\}$ are two sequences in Y such that $\limsup_{n\to\infty}\|p_n\| \leq d$, $\limsup_{n\to\infty}\|q_n\| \leq d$ and $\limsup_{n\to\infty}\|d_np_n + (1-d_n)q_n\| = d$ holds, for some $d \geq 0$. Then $\lim_{n\to\infty}\|p_n - q_n\| = 0$.*

3 Main Results

In this section, we will prove weak and strong convergence results.

Theorem 3.1. *Let X be a nonempty closed and convex subset of a uniformly convex Banach space Y. Let $S : X \to X$ be a Suzuki's generalized non-expansive map satisfying (1.2) with $Fix(S)$ non empty. For an arbitrarily chosen $x_0 \in X$, let $\{x_n\}$ be a sequence generated by the iterative scheme PC^* (1.5). Then $\lim_{n\to\infty}\|x_n - t\|$ exists for all $t \in Fix(S)$.*

Proof. Let $t \in Fix(S)$ and $\{x_n\} \subseteq X$. Since S is a Suzuki's generalized non-expansive map, we obtain that

$$\frac{1}{2}\|t - St\| = 0 \leq \|x_n - t\| \Rightarrow \|Sx_n - St\| \leq \|x_n - t\|,$$

for all $x_n \in X$ and for all $p \in Fix(S)$. From the PC^*-iterative scheme (1.5)

$$\begin{aligned}\|z_n - p\| =& \|S^k((1-a_n)x_n + a_nS^kx_n) - t\| \\ \leq& (1-a_n)\|x_n - t\| + a_n\|S^kx_n - t\| \\ \leq& (1-a_n)\|x_n - p\| + a_n\|x_n - p\| \\ =& \|x_n - p\|. \end{aligned} \tag{3.1}$$

Consider

$$\begin{aligned}\|y_n - t\| =& \|S^kz_n - t\| \\ \leq& \|z_n - t\| \\ \leq& \|x_n - t\|. \end{aligned} \tag{3.2}$$

From (3.1),(3.2) and PC^*-iterative scheme (1.5) we obtain

$$\begin{aligned}\|x_{n+1} - t\| =& \|S^ky_n - t\| \\ \leq& \|y_n - t\| \\ \leq& \|z_n - t\| \\ \leq& \|x_n - t\|. \end{aligned} \tag{3.3}$$

Thus $\|x_{n+1} - t\| \leq \|x_n - t\|$. Hence $\{\|x_n - t\|\}$ is a non-increasing sequence and it is bounded below as well. By Monotone convergence theorem $\lim_{n\to\infty}\|x_n - t\|$ exists. □

Theorem 3.2. *Let $S : X \to X$ be a Suzuki's generalised non-expansive map satisfying (1.2), where X is a nonempty closed and convex subset of a uniformly convex Banach space Y. Let $\{x_n\}$ for $n \geq 1$, be a sequence generated by the PC^*-iterative scheme (1.5). Then $Fix(S)$ is nonempty iff $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.*

Proof. Suppose there exists $t \in Fix(S)$. By theorem 3.1, $lim_{n\to\infty}\|x_n - t\|$ exists and $\{x_n\}$ is bounded. Assume that

$$lim_{n\to\infty}\|x_n - t\| = \alpha \tag{3.4}$$

Now from (3.1), (3.2), (3.4) we obtain

$$\limsup_{n\to\infty}\|z_n - t\| \leq \limsup_{n\to\infty}\|x_n - t\| = \alpha. \tag{3.5}$$

$$\limsup_{n\to\infty}\|y_n - t\| \leq \lim_{n\to\infty}\|x_n - t\| = \alpha. \tag{3.6}$$

Since S is a Suzuki's generalized non-expansive map, we have

$$\frac{1}{2}\|t - St\| = 0 \leq \|x_n - t\| \Rightarrow \|Sx_n - St\| \leq \|x_n - t\|,$$

for all $x_n \in X$ and for all $t \in Fix(S)$.

$$\limsup_{n\to\infty}\|Sx_n - t\| = \limsup_{n\to\infty}\|x_n - p\| = \alpha \tag{3.7}$$

Now, by PC^*-iterative scheme (1.5)

$$\begin{aligned}\|x_{n+1} - t\| &= \|S^k y_n - t\| \\ &\leq \|y_n - p\|.\end{aligned}$$

$$\alpha = \lim_{n\to\infty}\|x_{n+1} - p\| \leq inf\|y_n - p\| \tag{3.8}$$

From (3.6) we get

$$\alpha \leq \liminf_{n\to\infty}\|y_n - p\| \leq \limsup_{n\to\infty}\|y_n - p\| \leq \alpha,$$

hence

$$\lim_{n\to\infty}\|y_n - p\| = \alpha. \tag{3.9}$$

Now, from (3.2)

$$\|y_n - t \ = \|S^k z_n - t\| \leq \|z_n - t\|.$$

Thus, we obtain

$$\alpha = \liminf_{n\to\infty}\|y_n - t\| \leq \liminf_{n\to\infty}\|z_n - t\|. \tag{3.10}$$

So (3.5) and (3.10) gives

$$\begin{aligned}&\alpha \leq \liminf_{n\to\infty}\|z_n - t\| \leq \limsup_{n\to\infty}\|z_n - t\| \leq \alpha,\\ &\implies \lim_{n\to\infty}\|z_n - t\| = \alpha.\end{aligned} \tag{3.11}$$

Hence

$$\begin{aligned}\alpha = \lim_{n\to\infty}\|z_n - t\| =& \|S^k((1 - a_n)x_n + a_n S^k x_n) - t\| \\ \leq& \|(1 - a_n)x_n + a_n S^k x_n) - t\| \\ \leq& (1 - a_n)\|x_n - p\| + a_n\|S^k x_n - p\|\end{aligned}$$

$$
\begin{aligned}
&\leq (1-a_n)\|x_n-p\| + a_n\|x_n-p\| \\
&= \|x_n-p\| = \alpha
\end{aligned}
$$

Thus

$$\lim_{n\to\infty}((1-a_n)\|x_n-t\| + a_n\|S^k x_n - t\|) = \alpha. \tag{3.12}$$

Using lemma 2.3, (3.5),(3.7) and (3.12) we obtain,

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0.$$

Conversely, let $\{x_n\}$ be bounded and $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. Let $t \in A(X,\{x_n\})$, using Proposition 2.1, we have

$$
\begin{aligned}
r(St,\{x_n\}) &= \limsup_{n\to\infty}\|x_n - St\| \\
&\leq \limsup_{n\to\infty}(\|x_n-t\| + 3\|Sx_n - x_n\|) \\
&= \limsup_{n\to\infty}\|x_n - t\| \\
&= r(t,\{x_n\}) = r(X,\{x_n\}).
\end{aligned}
$$

Hence, $t \in A(X,\{x_n\})$. Since Y is uniformly convex therefore $A(X,\{x_n\})$ is singleton, implying that $St = t$. □

Theorem 3.3. *Let $S : X \to X$ be Suzuki's generalised non-expansive map satisfying (1.2), where X is a nonempty closed and convex subset of a uniformly convex Banach space Y. Let $\{x_n\}$ for $n \geq 1$, be a sequence generated by the PC-iterative scheme PC^*-iterative scheme (1.5). Assume that Y satisfies Opial's condition then $\{x_n\}$ converges weakly to a point in $Fix(S)$.*

Proof. Suppose, $t \in Fix(S)$ then by theorem 3.1, $\lim_{n\to\infty}\|x_n - t\|$ exists. We will now show that $\{x_n\}$ has a weak sub-sequential limit in $Fix(S)$. Let $\{x_{n_j}\}$ and $\{x_{n_k}\}$ be two sub sequences of $\{x_n\}$ having weak limits t and $\bar{t}$ respectively. By theorem 3.2, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and by lemma 2.3, $I - S$ is demiclosed at zero. Thus $(I-S)t = 0$. Hence $St = t$ and similarly $S\bar{t} = \bar{t}$. Now for the uniqueness part, if $t \neq \bar{t}$, then by using Opial's condition, we have

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n - t\| &= \lim_{n_j\to\infty}\|x_{n_j} - t\| \\
&\leq \lim_{n_j\to\infty}\|x_{n_j} - \bar{t}\| \\
&= \lim_{n\to\infty}\|x_n - \bar{t}\| \\
&= \lim_{n_k\to\infty}\|x_{n_k} - \bar{t}\| \\
&= \lim_{n_k\to\infty}\|x_{n_k} - t\| \\
&= \lim_{n\to\infty}\|x_n - t\|
\end{aligned}
$$

It leads to a contradiction, so $t = \bar{t}$ and $\{x_n\}$ converges weakly to a point in $Fix(S)$. □

Theorem 3.4. *Let $S : X \to X$ be a Suzuki's generalised non-expansive map satisfying (1.2), where X is a nonempty closed and convex subset of a uniformly convex Banach space Y. Let $\{x_n\}$ for $n \geq 1$, be a sequence generated by the PC-iterative scheme PC^*-iterative scheme (1.5). Then $\{x_n\}$ converges to a point $Fix(S)$ if and only if $\liminf_{n\to\infty} d(x_n, Fix(S)){=}0$ or $\limsup_{n\to\infty} d(x_n, Fix(S)){=}0$, where $d(x_n, Fix(S)) = inf\{\|x_n - t\| : t \in Fix(S)\}$.*

Proof. Forward part is obvious. Conversely, assume that $\liminf_{n\to\infty} d(x_n, Fix(S))$=0. By theorem 3.2, $\lim_{n\to\infty} \|x_n - t\|$ exists for all $t \in Fix(S)$. Hence $\lim_{n\to\infty} d(x_n, Fix(S))$=0. We now claim that $\{x_n\}$ is a Cauchy sequence in X. As $\lim_{n\to\infty} d(x_n, Fix(S))$=0, for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\begin{aligned} d(x_n, Fix(S)) &\leq \frac{\epsilon}{2} \\ inf\{\|x_n - t\| : t \in Fix(S)\} &\leq \frac{\epsilon}{2}. \end{aligned}$$

In particular, $inf\{\|x_n - t\| : t \in Fix(S)\} \leq \frac{\epsilon}{2}$ hence, there exists $t \in Fix(S)$ such that

$$\|x_{n_0} - t\| \leq \frac{\epsilon}{2}$$

. Now $m, n \geq x_{n_0}$ and we obtain

$$\begin{aligned} \|x_{m+n} - x_n\| &\leq \|x_{m+n} - t\| + \|x_n - t\| \\ &\leq \|x_{m_0} - t\| + \|x_{m_0} - t\| \\ &= 2\|x_{m_0} - t\| \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is closed and convex subset of a Banach space Y, there exists $t \in X$ such that $\lim_{n\to\infty} x_n = t$. Finally $\lim_{n\to\infty} d(x_n, Fix(S))$=0 implies d$(t, Fix(S))$=0 hence $t \in Fix(S)$. □

Theorem 3.5. *Let X be a nonempty, compact and convex subset of a uniformly convex subset of a Banach space Y and S and $\{x_n\}$ be as defined in theorem 3.1, then the sequence $\{x_n\}$ converges strongly to a fixed point of S.*

Proof. By lemma 2.1, $Fix(S)$ is nonempty. By theorem 3.1, we have $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Here X is compact, hence there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \to t$ strongly for some $t \in X$. Also by proposition 2.1, we get for $j \geq 1$,

$$\|x_{n_j} - St\| \leq 3\|Sx_{n_j} - x_{n_j}\| + \|x_{n_j} - t\|.$$

Letting $j \in \infty$, we get $x_{n_j} \to St$. Hence $St = t$ and by theorem 3.1, $\lim_{n\to\infty} \|x_n - t\|$ exists and thus, t is the strong limit of $\{x_n\}$. □

Theorem 3.6. *Let $S : X \to X$ be a Suzuki's generalised non-expansive map satisfying condition (I) definition 6, where X is a nonempty closed and convex subset of a uniformly convex Banach space Y. Let $\{x_n\}$ for $n \geq 1$, be a sequence generated by the PC^*-iterative scheme (1.5) converges strongly to a fixed point of S.*

Proof. By theorem 3.1, we get

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

By condition (I), definition 6, we have the following

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} h(d(x_n, Fix(S))) \leq \|x_n - Sx_n\| = 0 \\ &\leq \lim_{n\to\infty} h(d(x_n, Fix(S))) = 0. \end{aligned}$$

As $h : [0, \infty) \to [0, \infty)$ is a nondecreasing function. Also $h(0) = 0$ and $h(r) > 0$ for all $r > 0$. We obtain, $\lim_{n\to\infty} d(x_n, Fix(S)) = 0$. Hence by theorem 3.4, $\{x_n\}$ converges strongly to a fixed point of S. □

The following example shows that the generalised non-expansive map need not be Suzuki's generalised non-expansive map.

Example 2. $G : [0, 2] \to [0, 2]$ be a self map defined by

$$S(t) = \begin{cases} 0 & t \neq 2 \\ 0.2 & t = 2 \end{cases}$$

Case(i) $u \in [0, 0.7)$, $v \in [0, 2)$ and $a = \frac{1}{16}$, $b = c = \frac{1}{8}$ we have, $a+2b+2c < 1$. Case(ii) $u \in [0.7, 2]$, $v = 2$ and $u < v$ we have $\|Gu - Gv\| = 0.2$ and we have

$$a\|u - v\| + b(\|u - Gu\| + \|v - Gv\|) + c(\|u - Gv\| + \|v - Gu\|)$$

$$\begin{aligned} &= \frac{1}{16}(u-2) + \frac{1}{8}(u+1.8) + \frac{1}{8}(u+1.8) \\ &= \frac{1}{16}(u-2) + \frac{1}{8}(u+1.8) + \frac{1}{8}(u+1.8) \\ &\geq \frac{1}{16}(0.7-2) + \frac{1}{8}(0.7+1.8) + \frac{1}{8}(0.7+1.8) \\ &= 0.7 \end{aligned}$$

Here, $0.2 \leq 0.7$ and G is satisfying (1.1). But G is not Suzuki's generalized non-expansive map. For $u = 0.1$ and $v = 2$, (1.2) is not satisfied. Therefore a generalised non-expansive map need not be Suzuki's generalized non-expansive map.

In 2020, Ali et.al [2] proved that Suzuki's generalized non-expansive map need not be generalized non-expansive map. Hence these two classes are independent of each other.

4 Numerical Experiments

A program written in Python, a freely available programming language, is used to perform numerical experiments. The following example is from [1], we will verify graphically and numerically that the PC^*-iterative has much faster convergence than (1.1),(1.2),(1.3) and (1.4), in case of a contraction map.

Example 3. [1] Let $X = \mathbb{R}$ be the set of real numbers. Let $S : X \to X$ be defined as $S(x) = \sqrt{x^2 - 8x + 40}$ for all $x \in X$. Choosing the control sequence as $a_n = \frac{3n}{4n+5}$ and $b_n = \frac{2n}{3n+1}$ and $c_n = \frac{4n}{5n+1}$ with an initial guess $x_0 = 40.5$.
We obtain that the PC^*-iterative scheme has an efficient convergence rate than the iterative scheme (1.1)),(1.2),(1.3) and (1.4). It is also observed that in the PC^*-iterative scheme, the higher value of k is improving the convergence rate. The results are presented, graphically by Figure 1 and numerically by Table 1.

Example 4. Let $S : [0, 20] \to [0, 20]$ be defined as $S(x) = \sqrt{x^2 - 7x + 49}$ for all $x \in X$. Choosing the control sequence as $a_n = \frac{n}{4n+5}$ and $b_n = \frac{n}{3n+1}$ and $c_n = \frac{n}{5n+1}$ with an initial guess $x_0 = 5.5$. The results are presented, graphically by Figure 2 and numerically by Table 2.

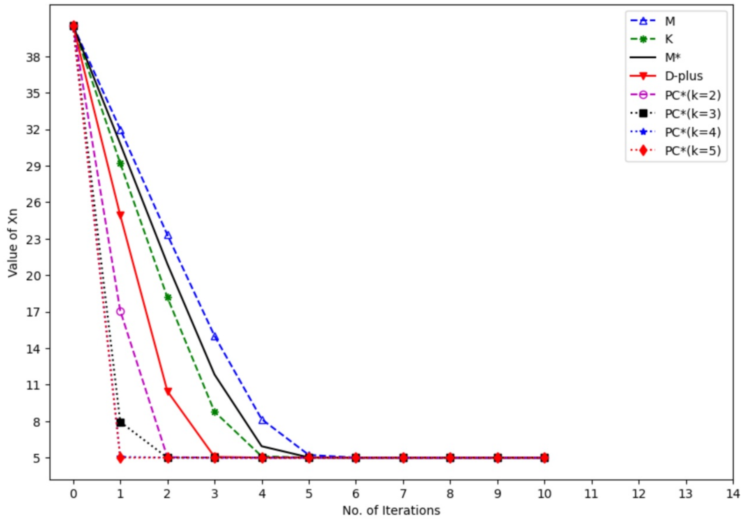

Figure 1: Comparison of iterative schemes (1.1),(1.2),(1.3),(1.4) with (**??**), the PC^*-iterative scheme.

n	M	K	M*	D plus	PC^*(k=2)	PC^*(k=3)	PC^*(k=4)	PC^*(k=5)
	1.1	1.2	1.3	1.4	1.5(k=2)	1.5(k=3)	1.5(k=4)	1.5(k=5)
0	40.5	40.5	40.5	40.5	40.5	40.5	40.5	40.5
1	31.9918	29.2049	30.7999	24.9214	17.0768	7.9362	5.0587	5.0003
2	23.3143	18.1779	20.9261	10.465	5.0408	5	5	5
3	15.0142	8.7779	11.8362	5.0651	5	5	5	5
4	8.1649	5.1145	5.9392	5.0001	5	5	5	5
5	5.2228	5.0007	5.0221	5	5	5	5	5
6	5.0049	5	5.0004	5	5	5	5	5
7	5.0001	5	5	5	5	5	5	5
8	5	5	5	5	5	5	5	5
9	5	5	5	5	5	5	5	5
10	5	5	5	5	5	5	5	5

Table 1: Comparison of iterative schemes (1.1),(1.2),(1.3),(1.4) with (1.5), the PC^*-iterative scheme.

5 Application

Fractional calculus deals with the study of real number powers of a differentiation operator D. A fractional differential equation contains the derivative of a non-integral order. With the help of (1.5), the PC^*-iterative scheme, we will approximate the solution of a nonlinear fractional differential equation of the following form:

$$D^{\gamma}y(u) + D^{\delta}y(u) = g(u, y(u)) \ (0 \leq u \leq 1, 0 < \delta < \gamma < 1), y(0) = y(1) = 0, \tag{5.1}$$

where $g : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Let $C[0, 1]$ be a Banach space of continuous functions from $[0, 1]]$ to $\mathbb{R}$ endowed with supremum norm. The Green's function associated is

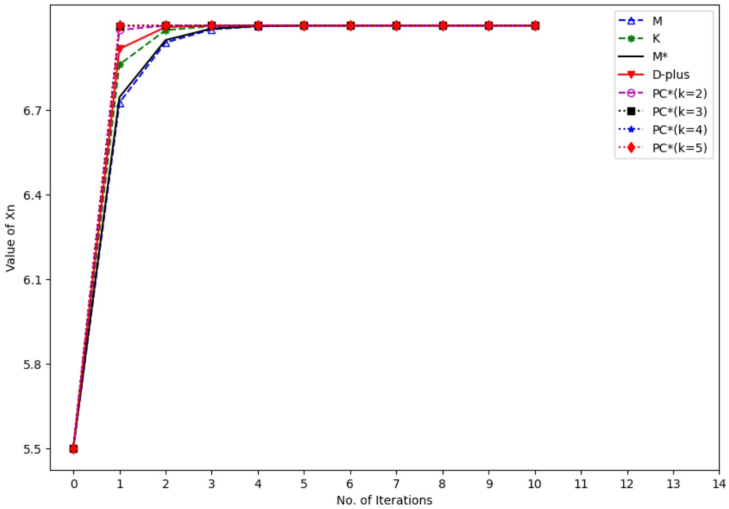

Figure 2: Graphical Comparison of iterative schemes (1.1),(1.2),(1.3),(1.4) with the PC^*-iterative scheme.

n	M	K	M*	D plus	PC^*(k=2)	PC^*(k=3)	PC^*(k=4)	PC^*(k=5)
	1.1	1.2	1.3	1.4	1.5(k=2)	1.5(k=3)	1.5(k=4)	1.5(k=5)
0	5.5	5.5	5.5	5.5	5.5	5.5	5.5	5.5
1	6.7266095	6.8617283	6.7457442	6.9171419	6.9841255	6.9980382	6.9997559	6.9999695
2	6.9396047	6.983358	6.9482864	6.9944972	6.9997813	6.9999967	6.9999999	7
3	6.9863637	6.9979585	6.9892669	6.9996437	6.999997	7	7	7
4	6.9969222	6.9997495	6.9977725	6.9999774	7	7	7	7
5	6.9993078	6.9999693	6.999539	6.9999986	7	7	7	7
6	6.9998449	6.9999962	6.9999048	6.9999999	7	7	7	7
7	6.9999653	6.9999995	6.9999804	7	7	7	7	7
8	6.9999923	6.9999999	6.999996	7	7	7	7	7
9	6.9999983	7	6.9999992	7	7	7	7	7
10	6.9999996	7	6.9999998	7	7	7	7	7

Table 2: Comparison of iterative schemes (1.1),(1.2),(1.3),(1.4) with the PC^*-iterative scheme.

defined as

$$G(u) = u^{\alpha-1}E_{\alpha-\beta,\alpha}(-u^{\alpha-\beta}),$$

where $E_{\alpha-\beta,\alpha}(-u^{\alpha-\beta})$ is the Mittag-Leffler function. We make the following assumption

$$(M_1) : |g(u,a) - g(u,b)| \leq c\|a-b\|$$

, for all $u \in [0,1]$ $a, b \in \mathbb{R}$ and $c \leq \alpha$.

Theorem 5.1. *Let $C[0,1]$ be a Banach space of real continuous functions. Let $C[0,1]$ be endowed with supremum norm. Suppose $\{x_n\}$ be the sequence defined by (1.5), the PC^*-iterative scheme.*

The operator $\Omega : C[0,1] \to C[0,1]$ *defined as follows*

$$\Omega(x(u)) = \int_0^t G(u-t)g(t,x(t)\,dt,$$

for all $u \in [0,1]$ *and* $x \in C[0,1]$*. Let us assume that condition* M_1 *is satisfied. Then* $\{x_n\}$ *converges to a solution of the problem* (5.1) *say* $x^* \in C[0,1]$*.*

Proof. $x^* \in C[0,1]$ is a solution of (5.1) if and only if x^* is a solution of the integral equation

$$x(u) = \int_0^t G(u-t)g(t,x(t))\,dt.$$

Let $x, y \in C[0,1]$ and for all $u \in [0,1]$. Using (M_1), we obtain

$$\begin{aligned}
\|\Omega(x(u)) - \Omega(y(u))\| &= \int_0^t G(u-t)g(t,x(t))\,dt - \int_0^t G(u-t)g(t,y(t))\,dt \\
&\leq \int_0^t G(u-t)(g(t,x(t)) - g(t,y(t)))\,dt \\
&\leq \int_0^t G(u-t)c\|x(t) - y(t)\|\,dt \\
&\leq (sup \int_0^t G(u-t))c\|x-y\|\,dt \\
&\leq \frac{c}{\alpha}\|x-y\|.
\end{aligned}$$

Here $G(u) = u^{\alpha-1}E_{\alpha-\beta,\alpha}(-u^{\alpha-\beta}) \leq u^{\alpha-1}$ for all $u \in [0,1]$. Hence, $sup_{u\in[0,1]} \int_0^t G(u-t)\,dt \leq \frac{1}{\alpha}$. Thus $\|\Omega(x(u)) - \Omega(y(u))\| \leq \|x-y\|$ and Ω is a Suzuki's generalized non-expansive map and main results we obtain that (1.5), the PC^*-iterative scheme converges to the solution of (5.1). □

6 Conclusion

The objective of this paper was to introduce an efficient iterative scheme called the PC^*-iterative scheme for the class of Suzuki's Generalized non-expansive map. We established convergence results for the newly defined PC^*-iterative scheme. The PC^*-iterative scheme is used to solve a fractional differential equation as an application. The numerical experiments validates the fact that the PC^*-iterative scheme is efficient in convergence rate than many other leading iterative schemes.

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