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# Dynamics for a viscoelastic wave equation with nonlocal nonlinear dissipation and logarithmic nonlinearity: blow-up solutions, lifespan estimates and asymptotic stability

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**Abstract**. This paper investigates the instability of a class of wave equations with a non-local nonlinear damping term

$$v_{tt} - \Delta v + (g * \Delta v)(t) + \sigma(\|\nabla v\|_2^2)\phi(v_t) = |v|^{p-2}v \ln |v|^k,$$

where $(x,t) \in \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$, $\sigma$ represents the nonlocal coefficient and $\phi$ is the nonlinear damping term. By considering suitable assumptions on the functions $\sigma$ and $\phi$, the exponents $p$ and $k$, the relaxation function $g$ and the initial data, and by making use of differential inequality technique, we establish the occurrence of finite time blow up of solutions at low and arbitrary high positive initial energy levels. Moreover, lower bounds for the lifespan of solutions are derived in both cases. Asymptotic stability for the solution energy is also investigated by employing the energy perturbation method. This work extends and complements some previous results in the literature.

***Keywords.*** Viscoelastic wave equations, blow-up, asymptotic stability, nonlocal damping, logarithmic nonlinearity

## 1 Introduction

In this article, we investigate the wave equation

$$v_{tt} - \Delta v + (g * \Delta v)(t) + \sigma(\|\nabla v\|_2^2)\phi(v_t) = f(v), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

subject to the boundary and initial conditions

$$v(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T), \tag{1.2}$$

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\Gamma$, $T > 0$, $f(v) = |v|^{p-2}v \ln |v|^k$, $p > 2$, $\phi(s) = k_0 s|s|^\alpha + k_1 s$ with $\alpha > 0$, $k_0 \geq 0$, $k_1 \geq 0$, $k > 0$, $\sigma(s) = s^\theta$, $(\theta > 0)$ is a nonlocal coefficient, and $(g * \Delta v)(t) = \int_0^t g(t-s)\Delta v(s)ds$.

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Consider the problem (1.1) - (1.3) with $g \equiv 0$, $\sigma \equiv 1$ and the source nonlinearity characterized by $f(v) = |v|^{p-2}v$. In this case, the equation (1.1) simplifies to a damped nonlinear wave equation that has been the subject of extensive study. For $k_0 = 0$ and $k_1 = 1$, Levine [25] demonstrated a blow-up phenomenon associated with negative initial energy in the abstract version of (1.1)-(1.3). The author employed a methodology that utilized convexity arguments, later termed the concavity method (see also [26]). Following Levine's work, Pucci and Serrin [44] established the global nonexistence of solutions with positive initial energy for abstract evolution equations with linear damping effects. Georgiev and Todorova [11] extended Levine's results to the case of nonlinear damping (i.e. when $k_0 = 1$ and $k_1 = 0$). They investigated the interactions between the nonlinear damping term and the source term to obtain their influence on global behavior and finite-time blow-up. Todorova [48] further generalized these findings, achieving results on global existence and finite-time blow-up in the entire space $\mathbb{R}^n$ when the data are of compact support and the initial energy is sufficiently negative. Levine and Todorova [27] also studied the existence of solutions and finite-time blow-up behavior for cases with positive initial energy in the whole space $\mathbb{R}^n$. It is noteworthy that the absence of the source term ensures global existence in time for arbitrary initial data, as demonstrated in the works by Harraux and Zuazua [17] and Kopackova [19]. In the case where $\sigma \equiv 0$, meaning the damping term is absent, numerous studies have explored how the source term can lead to the existence of blow-up solutions, including the contributions of Payne and Sattinger [41], Shatah and Strauss [45], Antonini and Merle [3], along with the studies by Merle and Zagg [32, 33, 34, 35] and the references therein.

In recent years, a considerable amount of researches have been carried out on nonlinear wave equations that incorporate logarithmic nonlinearities. Notably, among the pioneering studies in this field are the works of Bialynicki-Birula and Myslinski [5, 6], where the authors investigated the model represented by

$$\begin{cases} v_{tt} - \Delta v + v = \varepsilon v \ln|v|, & (x,t) \in \Omega \times (0,T), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \\ v(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \end{cases} \tag{1.4}$$

where $\Omega = [a,b]$ and $\varepsilon \in [0,1]$. It is pertinent to note that models similar to (1.4) are often applied in quantum mechanics (for further information, see [49]). In the work of Cazenave and Haraux [8], the authors conducted an investigation of problem (1.4) in $\mathbb{R}^3$ with a focus on global solutions. In a separate study, Gorka [12] utilized the Galerkin method to prove the existence of weak solutions to (1.4). Subsequently, Bartkowski and Gorka [4] studied existence of classical and weak solutions to (1.4) within the one-dimensional case. A generalized form of (1.4) is presented by Hiramatsu et al. [18], wherein the authors considered the equation

$$v_{tt} - \Delta v + v + v_t + |v|^2 v = v \ln|v|^2. \tag{1.5}$$

With the given initial and boundary conditions, numerical analysis was conducted on problem (1.5). Furthermore, Han [16] studied the existence of global weak solutions to (1.5) in $\mathbb{R}^3$. Di et al. [22] investigated the existence and uniqueness of weak solutions for the equation

$$v_{tt} - \Delta v_t - \Delta v = |v|^{p-2} v \ln|v|, \quad p > 2. \tag{1.6}$$

In addition, an investigation was conducted to determine the polynomial and exponential energy decay rates of the solutions energy. This was done under appropriate conditions on the initial data. Moreover, the authors proved blow-up properties of the solutions through the use of integral-differential inequalities, the concavity method, and energy perturbation technique. In relation to this, some references can be made to other researches. For instance, in [52], the authors obtained estimates for the blow-up time bounds for the solutions of (1.6) with arbitrary initial energy. In another study [15], a generalized form of (1.6)

$$v_{tt} - \Delta v + (g * \Delta v)(t) - \Delta v_t = |v|^{p-2} v \ln |v|, \quad p > 2,$$

investigated. The authors obtained blow-up solutions for weak solutions with positive initial energy. Later, Liao [28] reviewed these results under arbitrary initial energy and proved that how solutions blow up with arbitrary and positive initial energy in finite time. Several investigations have been conducted pertaining to nonlinear wave equations that involve logarithmic-type nonlinearities. Although it is impractical to discuss each one in this particular context, certain recent analyses in this field are noteworthy. Specifically, concerning wave equations that involve logarithmic nonlinearity, the works [14, 2, 9, 21, 42, 43, 40, 13, 29, 31] warrant attention.

Wave equations with nonlocal effects have attracted significant attention from researchers in recent years. As one of the pioneering studies in this area, Lange-Menzala [23] investigated the decay rates of global solutions of the equation

$$v_{tt} + \Delta^2 v + M(\|\nabla v\|_2) v_t = 0, \tag{1.7}$$

where $M(s) > 0$ and the term $M(\|\nabla v\|2)v_t$ is recognized as a nonlocal Kuramoto-Shivashinsky dissipative term. In fact, the authors proved the existence of global classical solutions by using Fourier transform and showed that how the solutions decay with a rate of $t^{-\gamma}$. After that, Cavalcanti et al. [7] extended (1.7) to

$$v_{tt} + \Delta^2 v - (g * \Delta^2 v)(t) + M(\|\nabla v\|_2) v_t = 0.$$

The authors established existence of weak and strong solutions and obtained a decay rate for weak solutions. Regarding the nonlocal terms with weakly nonlinear dissipative mechanisms, Narciso in [37] considered the equation

$$v_{tt} + \Delta^2 v + l_1(v)|v_t|^\gamma v_t + l_2(v)|v|^\rho v = h,$$

where $\gamma > 0$, $\rho > 0$, $h$ is a function in $L^2$, $l_1(v) = M(\|\nabla v\|_2^2)$ and $l_2(v) = N(\|v\|_{\rho+2}^{\rho+2})$, in which $M$ and $N$ are functions of class $C^1$, with $M(s) > 0$ and $N(s) \geq 0$. The author, while proving the well-posedness of the associated problem, under suitable conditions on the functions and initial data, investigated the stability and asymptotic behavior of the solutions. In another recent study conducted by the same author [38], the well-posedness and stability of a wave equation of Kirchhoff type with nonlocal terms in the form

$$v_{tt} - \phi(\|\nabla v\|_2^2)\Delta v + \sigma(\|\nabla v\|_2^2) g(v_t) + f(v) = h,$$

are investigated where $f$ and $g$ are nonlinear functions and $\phi$ and $\sigma$ defined on $\mathbb{R}_0^+$. Recently, Narciso et al. [39] considered the eqaution

$$v_{tt} + \Delta^2 v - M(\|\nabla v\|_2^2)\Delta v + \|\Delta v\|_2^{2\alpha} |v_t|^\alpha v_t = 0,$$

and showed that for regular initial data the energy associated with the corresponding problem goes to zero when time goes to infinity. In this connection, we can also remember the work by Zhang and Li [51] in which the authors considered the equation

$$v_{tt} - \Delta v + \|\nabla v\|_2^{2l} |v_t|^{m-2} v_t = |v|^{p-2} v, \quad p, m > 2,$$

and obtained stability of solutions under appropriate conditions. Then, by providing sufficient conditions on the initial data, they showed that the solutions blow up in the case $p > l + mp$. Li et al. [20], considered a viscoelastic wave equation in the form

$$v_{tt} - \Delta v + (g * \Delta v)(t) + N(\|\nabla v\|_2^2)v_t = f(v), \tag{1.8}$$

in which $f$ is a function with polynomial structure and $N$ is the non-local weak damping coefficient. After proving the existence of strong and local weak solutions by constructing a second-order differential inequality under appropriate conditions on the initial data and the function $g$, the authors proved that the solutions blow up with positive initial energy in a finite time and obtained an upper bound for the lifespan. In a recent research, the authors in [50] studied the equation (1.8) with $g = 0$ and $f(v) = |v|^{p-2}v \ln |v|$. After proving the well-posedness of the local and global solutions, the asymptotic algebraical and exponential stability of the solutions were investigated. Also, the non-existence of arbitrary solutions proved in the case where the initial energy of the solutions considered to be negative. There exists a multitude of comprehensive explorations pertaining to wave equations that are accompanied by nonlocal damping terms. For instance, in the case of nonlocal weak damping terms, one can reference the work of [24], where $N(\|v_t\|_2^2)v_t$ is exemplified. In the instance of non-local strong damping, reference may be made to the study conducted by [30], where $N(\|\nabla v\|_2^2)\Delta v_t$ is examined. Furthermore, for non-local strong damping of the form $\left(\|\Delta v\|_2^2 + \|v_t\|_2^2\right)^q \Delta v_t$, one may consider [46], while for nonlocal damping terms in the form $N(\|\nabla v\|_2^2)(-\Delta v_t)^\theta$, references may be made to both [10] and [47].

Taking into consideration the prior researches conducted on the impact of nonlocal damping terms in hyperbolic equations, this article aims to study the problem (1.1) - (1.3) in terms of solution instability. Specifically, our investigation focuses on the instability of solutions with low and high positive initial energy levels. By imposing certain conditions on the exponents $p$ and $\alpha$, the function $g$, and the initial data, we prove finite-time blow-up of solutions with positive initial energy. Additionally, we will provide estimates for the lower bound of the blow-up time in both cases for the initial energy. It is noteworthy to mention that when $k_0 = 0$ and $k_1 > 0$, indicating a nonlocal linear damping effect, the non-existence of global solutions can be obtained by following the procedure presented in [20], alongside obtaining an upper bound for the blow-up time. The results of our work extend and complement some recent studies. For instance, the case of $g = 0$ and $\phi(v_t) = v_t$ was investigated in [50] where the non-existence of global solutions obtained for negative initial energy. Similarly, [20] studied (1.8) with a nonlinear source with polynomial structure, where the existence of blow-up solutions proved in the case that $\phi(v_t) = v_t$. In addition, in [51], the stability and instability of the solutions, when $g = 0$ are considered.

In the present article, the definitions, lemmas, and necessary material required to prove the theorems and lemmas throughout the article are presented in section 2. Subsequently, in section 3, an analysis of the blowing up solutions at low initial energy is conducted. In section 4, non-existence of global solutions is derived under arbitrary high positive initial energy. Finally, section 5 provides a comprehensive analysis of asymptotic stability, offering insights into the general decay rates for solution energy.

## 2 Material and assumptions

In this section, we will present the notations, definitions, lemmas and generally what we need in this article. We use $(.,.)$ to show the inner production in $L^2$:

$$(u, v) = \int_\Omega u(x)v(x)dx, \quad \|u\|_2^2 = \int_\Omega |u(x)|^2 dx, \quad u, v \in L^2(\Omega).$$

$L^p(\Omega)$ is the $p-$Lebesgue integrable functions with the norm $\|.\|_p$. $H^1(\Omega) = W^{1,2}(\Omega)$, $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ and

$$(u, v)_{H_0^1} = (\nabla u, \nabla v), \quad \|u\|_{H_0^1} = \|\nabla u\|_2, \quad u, v \in H_0^1(\Omega).$$

$C_q$ shows the best constant of the following embedding([1]):

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega), \qquad 2 \le q \le \bar{q}, \qquad \bar{q} = \begin{cases} \frac{2n}{n-2}, & n \ge 3, \\ +\infty & n = 1, 2. \end{cases} \tag{2.1}$$

For the functions $g$ and the exponent $p$, we assume:

($A_1$) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a function in $C^1(0, +\infty)$ is so that

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = \ell > 0, \qquad g'(t) \le 0, \quad \forall t \ge 0.$$

($A_2$) $2 < p < +\infty$ if $n = 1, 2$, and $2 < p < \frac{2(n-1)}{n-2}$ for $n \ge 3$.

**Definition 1** (**Weak Solution**)**.** Suppose $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. $v(x,t)$ is the weak solution of problem (1.1)-(1.3) on $\Omega \times (0,T)$, whenever $v \in C\big([0,T]; H_0^1(\Omega) \times L^2(\Omega)\big)$, $v_t \in C\big([0,T]; H_0^1(\Omega) \times L^2(\Omega)\big) \cap L^{\alpha+2}\big(\Omega \times (0,T)\big)$ and

$$\begin{aligned}(v_t, \varphi) &+ \int_0^t \big(\nabla v(s), \nabla\varphi\big)ds - \int_0^t \left(\int_0^s g(s-\tau)\nabla v(\tau)d\tau, \nabla\varphi\right)ds \\ &+ \int_0^t \sigma(\|\nabla v(s)\|_2^2)\big(\phi(v_t(s)), \varphi\big)ds = (v_1, \varphi) + \int_0^t \big(|v(s)|^{p-2}v(s)\ln|v(s)|^k, \varphi\big)ds.\end{aligned}$$

satisfies for every $\varphi \in H_0^1(\Omega)$.

Existence of weak solutions can be obtained by adopting the arguments in [38], [29] and [20]. Hear, we omit the proof.

**Definition 2** (**Energy Functional**)**.** The energy functional corresponding to the problem (1.1)-(1.3) defined as

$$E[v](t) = E(t) = \frac{1}{2}\int_\Omega |v_t|^2 dx + J[v](t), \qquad \forall v \in H_0^1(\Omega),$$

where

$$J[v](t) = J(t) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla v\|_2^2 + \frac{1}{2}(g \diamond \nabla v)(t) - \frac{1}{p}\int_\Omega |v|^p \ln|v|^k dx + \frac{k}{p^2}\|v\|_p^p,$$

and

$$(g \diamond \nabla v)(t) = \int_0^t g(t-s)\int_\Omega |\nabla v(t,x) - \nabla v(s,x)|^2 dx ds.$$

**Lemma 2.1** ([36])**.** *For differentiable functions $g = g(t)$ and $v = v(.,t)$ we have*

$$\begin{aligned}\int_\Omega v_t(t)(g * v)(t)dx = &-\frac{1}{2}g(t)\int_\Omega |v(x,t)|^2dx + \frac{1}{2}(g' \diamond v)(t) - \frac{1}{2}\frac{d}{dt}(g \diamond v)(t) \\ &+ \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\int_\Omega |v(x,t)|^2dx\right).\end{aligned}$$

**Lemma 2.2.** *For each $v \in H_0^1(\Omega)$, the energy of the problem (1.1)-(1.3) is decreasing in time and*

$$\begin{aligned} E(t) = E(0) &+ \frac{1}{2}\int_0^t (g' \diamond \nabla v)(s) ds - \frac{1}{2}\int_0^t g(s)\|\nabla v(s)\|_2^2 ds \\ &- k_0 \int_0^t \|\nabla v(s)\|_2^{2\theta} \|v_t(s)\|_{\alpha+2}^{\alpha+2} ds - k_1 \int_0^t \|\nabla v(s)\|_2^{2\theta} \|v_t(s)\|_2^2 ds. \end{aligned} \tag{2.2}$$

*Proof.* The result is obtained by multiplying (1.1) by $v_t$ and then integrating over $\Omega \times (0,t)$. □

**Lemma 2.3** ([15] - Lemma 2.1)**.** *For every $\varsigma > 0$,*

$$0 \le \ln s \le \frac{s^{\varsigma}}{e\varsigma}, \qquad s > 1.$$

**Lemma 2.4.** *Suppose that*

$$E_\jmath = Q(\gamma_\jmath), \qquad \gamma_\jmath = \left(\frac{p\jmath\ell}{k(p+\jmath)C_{p+\jmath}^{p+\jmath}}\right)^{\frac{1}{p+\jmath-2}}, \qquad 0 < \jmath < \jmath^*, \tag{2.3}$$

*where $Q(\gamma) = \frac{\ell}{2}\gamma^2 - \frac{k}{p\jmath}C_{p+\jmath}^{p+\jmath}\gamma^{p+\jmath}$, and*

$$\jmath^* = \begin{cases} \frac{2n}{n-2} - p, & n \ge 3, \\ +\infty & n = 1, 2. \end{cases} \tag{2.4}$$

*Then*

- (**a**) *If $E(0) < E_\jmath$ and $\|\nabla v_0\|_2 > \gamma_\jmath$, then for every $t \ge 0$: $\|\nabla v(t)\|_2 > \gamma_\jmath$.*
- (**b**) *If $E(0) < E_\jmath$ and $\|\nabla v_0\|_2 < \gamma_\jmath$, then for every $t \ge 0$: $\|\nabla v(t)\|_2 < \gamma_\jmath$.*

*Proof.* (**a**) Using the definition of the energy functional, ($A_1$), Lemma 2.3 and (2.1), for each $0 < \jmath < \jmath^*$, we have

$$\begin{aligned} E(t) \ge J(t) \ge& \frac{\ell}{2}\|\nabla v\|_2^2 + \frac{1}{2}(g \diamond \nabla v)(t) + \frac{k}{p^2}\|v\|_p^p - \frac{1}{p}\int_\Omega |v|^p \ln |v|^k dx \\ \ge& \frac{\ell}{2}\|\nabla v\|_2^2 - \frac{k}{p}\int_{\{x\in\Omega:|v|\le 1\}} |v|^p \ln |v| \; dx - \frac{k}{p}\int_{\{x\in\Omega:|v|>1\}} |v|^p \ln |v| \; dx \\ \ge& \frac{\ell}{2}\|\nabla v\|_2^2 - \frac{k}{p}\int_{\{x\in\Omega:|v|>1\}} |v|^p \ln |v| \; dx \\ \ge& \frac{\ell}{2}\|\nabla v\|_2^2 - \frac{k}{pe\jmath}\|v\|_{p+\jmath}^{p+\jmath} \\ \ge& \frac{\ell}{2}\|\nabla v\|_2^2 - \frac{k}{p\jmath}C_{p+\jmath}^{p+\jmath}\|\nabla v\|_2^{p+\jmath} = Q(\gamma(t)), \end{aligned} \tag{2.5}$$

where $\gamma(t) = \|\nabla v(t)\|_2$. By differentiating $Q$ with respect to $\gamma$, we get

$$Q'(\gamma) = \ell\gamma - \frac{k(p+\jmath)}{p\jmath}C_{p+\jmath}^{p+\jmath}\gamma^{p+\jmath-1}.$$

Now, it is clear that $Q'(\gamma_\jmath) = 0$ and $Q''(\gamma_\jmath) = \ell(2-p-\jmath) < 0$. Therefore, $Q$ obtains its maximum value at $\gamma = \gamma_\jmath$. Also, $Q$ is increasing on $(0, \gamma_\jmath)$ and decreasing on $(\gamma_\jmath, +\infty)$, $Q(\gamma) \xrightarrow{\gamma \to +\infty} -\infty$ and

$$\max_{0<\gamma<+\infty} Q(\gamma) = Q(\gamma_\jmath) = E_\jmath.$$

Now, suppose that $t'_\jmath = \inf\{t > 0 : \|\nabla v(t)\|_2 = \gamma_\jmath\}$. Using the continuity of the mapping $t \mapsto \|\nabla v(t)\|_2$ we will have $\|\nabla v(t'_\jmath)\|_2 = \gamma_\jmath$. Therefore, using (2.5) and Lemma 2.2, we obtain

$$E_\jmath > E(0) \geq E(t'_\jmath) \geq Q(\|\nabla v(t'_\jmath)\|_2) = Q(\gamma_\jmath) = E_\jmath,$$

which is clearly a contradiction.

(**b**) The proof is similar to part (**a**). □

Finally, we present the following lemma which will be used throughout the proofs. By the assumption $(\mathrm{A}_2)$ we have:

**Lemma 2.5** ([36]-Lemma 2.3)**.** *Suppose that (2.1) holds. There exists a constant $C > 1$ such that for every $2 \leq s \leq q$:*

$$\|v\|_q^s \leq C\big(\|v\|_q^q + \|\nabla v\|_2^2\big), \qquad v \in H_0^1(\Omega).$$

# 3 Blow up for low initial energy

In this section, we study the blow-up solutions of the problem (1.1)-(1.3) for low positive initial energy. Our main results reads as follows.

**Theorem 3.1.** *Assume that $v_0 \in H_0^1(\Omega)$, $v_1 \in L^2(\Omega)$ and the assumptions $(\mathrm{A}_1) - (\mathrm{A}_2)$ hold. If $\|\nabla v_0\|_2 > \gamma_\jmath, 0 < \jmath < \jmath^*$, and the memory kernel, the initial energy and the exponents $p, \alpha$ and $\theta$ satisfy*

- $(\mathrm{L}_1)$ $\ell > \dfrac{1}{\big((1-\kappa)p-1\big)^2}$ *such that* $0 < \kappa < 1 - \frac{2}{p}$.
- $(\mathrm{L}_2)$ $E(0) < q\widetilde{E}_\jmath$ *where* $0 < q < 1$ *and* $\widetilde{E}_\jmath = \frac{\gamma_\jmath^2}{2p^2(1-\kappa)^2}\Big(\ell\big((1-\kappa)p-1\big)^2 - 1\Big)$.
- $(\mathrm{L}_3)$ $p - 2 > \alpha > 0$ *and* $0 < \theta < \dfrac{p+\jmath-\alpha-2}{p+\jmath}$.

*then the solution of problem (1.1)-(1.3) blows up in a finite time. Furthermore, the blow-up time $T^\star$ satisfies the estimate*

$$T^\star > \int_{W(0)}^{+\infty} \frac{dy}{\beta_0 y + \beta_1 y^\xi + \beta_2 y^\zeta}, \tag{3.1}$$

*where $W(0) = \int_\Omega |u_0|^p \ln |u_0| dx$ and $\beta_0$, $\beta_1$, $\beta_2$, $\xi$ and $\zeta$ are positive constants which will be determined in the proof.*

*Proof.* On contrary, suppose that $v$ is a global solution of problem (1.1)-(1.3). Define

$$Z(t) = \mathcal{E}_\jmath - E(t), \quad \forall t \geq 0, \qquad \mathcal{E}_\jmath \in \big(0, q\widetilde{E}_\jmath\big).$$

First, we show $\widetilde{E}_\jmath < E_\jmath$. Using the definition of $Q$, we have

$$E_\jmath = \frac{\ell(p+\jmath-2)}{2(p+\jmath)}\gamma_\jmath^2.$$

On the other hand

$$\begin{aligned}\widetilde{E}_\jmath &= \frac{\gamma_\jmath^2}{2p^2(1-\kappa)^2}\left(\ell\big((1-\kappa)p-1\big)^2-1\right)\\ &< \frac{\ell\gamma_\jmath^2}{p(1-\kappa)}\left(\frac{(1-\kappa)p}{2}-1\right) = \ell\gamma_\jmath^2\left(\frac{1}{2}-\frac{1}{p(1-\kappa)}\right)\\ &< \ell\gamma_\jmath^2\left(\frac{p-2}{2p}\right) < \ell\gamma_\jmath^2\left(\frac{p+\jmath-2}{2(p+\jmath)}\right)\\ &= E_\jmath.\end{aligned}$$

Considering that $E(0) < \mathcal{E}_\jmath < E_\jmath$, $Z(0) > 0$ and using the Lemma 2.2, we have $Z'(t) \geq 0$. So, using $(\mathrm{A}_1)$, from the Lemma 2.4 - (**a**), we get

$$\begin{aligned}&\mathcal{E}_\jmath - \left[\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla v\|_2^2 + \frac{1}{2}(g\diamond\nabla v)(t) + \frac{k}{p^2}\|v\|_p^p\right]\\ &\leq E_\jmath - \left[\frac{1}{2}\|v_t\|_2^2 + \frac{\ell}{2}\|\nabla v\|_2^2 + \frac{1}{2}(g\diamond\nabla v)(t) + \frac{k}{p^2}\|v\|_p^p\right]\\ &\leq E_\jmath - \frac{\ell}{2}\|\nabla v\|_2^2 \leq E_\jmath - \frac{\ell}{2}\gamma_\jmath^2 = -\frac{k}{p\jmath}C_{p+\jmath}^{p+\jmath}\gamma_\jmath^{p+\jmath} < 0.\end{aligned}$$

Hence,

$$0 < Z(0) \leq Z(t) \leq \frac{1}{p}\int_\Omega |v|^p \ln|v|^k dx \leq \frac{k}{pe\jmath}\int_\Omega |v|^{p+\jmath}dx. \tag{3.2}$$

Define

$$\Psi(t) = Z^{1-\varsigma}(t) + \varepsilon(v, v_t),$$

where

$$0 < \varsigma < \min\left\{\frac{(p+\jmath)(1-\theta)-\alpha-2}{(p+\jmath)(\alpha+1)}, \frac{p-2}{2p}\right\}. \tag{3.3}$$

Using (1.1) and (1.2) we obtain

$$\begin{aligned}\Psi'(t) =&(1-\varsigma)Z^{-\varsigma}(t)Z'(t) + \varepsilon\|v_t\|_2^2 - \varepsilon\|\nabla v\|_2^2\\ &-\varepsilon k_0\|\nabla v\|_2^{2\theta}\int_\Omega vv_t|v_t|^\alpha dx - \varepsilon k_1\|\nabla v\|_2^{2\theta}\int_\Omega vv_t dx\\ &+\varepsilon\int_\Omega |v|^p\ln|v|^k dx + \varepsilon\int_\Omega\int_0^t g(t-s)\nabla v(t).\nabla v(s)dsdx.\end{aligned} \tag{3.4}$$

For the last term on the right side of (3.4), for every $\delta > 0$, we have

$$\begin{aligned}&\int_\Omega\int_0^t g(t-s)\nabla v(t).\nabla v(s)dsdx\\ &= \int_\Omega\int_0^t g(t-s)(\nabla v(s)-\nabla v(t)).\nabla v(t)dsdx + \int_0^t g(s)ds\|\nabla v(t)\|_2^2\\ &\geq \left(\int_0^t g(s)ds - \delta\right)\|\nabla v\|_2^2 - \frac{1}{4\delta}\left(\int_0^t g(s)ds\right)(g\diamond\nabla v)(t)\\ &\geq \left(\int_0^t g(s)ds - \delta\right)\|\nabla v\|_2^2 - \frac{1-\ell}{4\delta}(g\diamond\nabla v)(t).\end{aligned} \tag{3.5}$$

Then, by the energy functional, (3.4) and (3.5), for every $0<\kappa<1-\frac{2}{p}$, we have

$$\begin{aligned}\Psi'(t) \geq & (1-\varsigma)Z^{-\varsigma}(t)Z'(t)+\varepsilon\left(1+\frac{(1-\kappa)p}{2}\right)\|v_t\|_2^2\\&+\varepsilon\left[\left(\frac{(1-\kappa)p}{2}-1\right)\left(1-\int_0^t g(s)ds\right)-\delta\right]\|\nabla v\|_2^2\\&+\varepsilon\left(\frac{(1-\kappa)p}{2}-\frac{1-\ell}{4\delta}\right)(g\diamond\nabla v)(t)+\frac{k\varepsilon}{p}(1-\kappa)\|v\|_p^p\\&+\varepsilon\kappa\int_\Omega |v|^p\ln|v|^k dx-\varepsilon p(1-\kappa)\mathcal{E}_\jmath+\varepsilon p(1-\kappa)Z(t)\\&-\varepsilon k_0\|\nabla v\|_2^{2\theta}\int_\Omega v v_t|v_t|^\alpha dx-\varepsilon k_1\|\nabla v\|_2^{2\theta}\int_\Omega v v_t dx.\end{aligned} \tag{3.6}$$

Using Young's inequality, for every $\delta_1>0$ and $\delta_2>0$, we have

$$\begin{aligned}&\int_\Omega v v_t|v_t|^\alpha dx\leq\frac{\delta_1^{\alpha+2}}{\alpha+2}\|v\|_{\alpha+2}^{\alpha+2}+\frac{\alpha+1}{\alpha+2}\delta_1^{-\frac{\alpha+2}{\alpha+1}}\|v_t\|_{\alpha+2}^{\alpha+2},\\&\int_\Omega v v_t dx\leq\frac{\delta_2}{2}\|v\|_2^2+\frac{1}{2\delta_2}\|v_t\|_2^2.\end{aligned} \tag{3.7}$$

Also, according to the Lemma 2.2 we have $k_0\|\nabla v\|_2^{2\theta}\|v_t\|_{\alpha+2}^{\alpha+2}\leq Z'(t)$ and $k_1\|\nabla v\|_2^{2\theta}\|v_t\|_2^2\leq Z'(t)$. Therefore, considering (3.7), the inequality (3.6) will be as follows

$$\begin{aligned}\Psi'(t) \geq & (1-\varsigma)Z^{-\varsigma}(t)Z'(t)+\varepsilon\left(1+\frac{(1-\kappa)p}{2}\right)\|v_t\|_2^2+\varepsilon\kappa\int_\Omega |v|^p\ln|v|^k dx\\&+\varepsilon\left[\left(\frac{(1-\kappa)p}{2}-1\right)\left(1-\int_0^t g(s)ds\right)-\delta\right]\|\nabla v\|_2^2\\&+\varepsilon\left(\frac{(1-\kappa)p}{2}-\frac{1-\ell}{4\delta}\right)(g\diamond\nabla v)(t)+\frac{k\varepsilon}{p}(1-\kappa)\|v\|_p^p\\&-\varepsilon p(1-\kappa)\mathcal{E}_\jmath-\varepsilon\|\nabla v\|_2^{2\theta}\left(\frac{k_0\delta_1^{\alpha+2}}{\alpha+2}\|v\|_{\alpha+2}^{\alpha+2}+\frac{k_1\delta_2}{2}\|v\|_2^2\right)\\&+\varepsilon p(1-\kappa)Z(t)-\varepsilon\left[\left(\frac{\alpha+1}{\alpha+2}\right)\delta_1^{-\frac{\alpha+2}{\alpha+1}}+\frac{1}{2\delta_2}\right]Z'(t).\end{aligned} \tag{3.8}$$

Suppose that $\delta_1^{-\frac{\alpha+2}{\alpha+1}}=\mathcal{K}_1 Z^{-\varsigma}(t)$ and $\delta_2^{-1}=\mathcal{K}_2 Z^{-\varsigma}(t)$. Then, $\delta_1^{\alpha+2}=\mathcal{K}^{-(\alpha+1)}Z^{\varsigma(\alpha+1)}(t)$ and $\delta_2=\mathcal{K}_2^{-1}Z^{\varsigma}(t)$. Moreover, by choosing $\delta=\frac{1-\ell}{2(1-\kappa)p}$ and using the conditions ($\mathrm{A}_1$) and ($\mathrm{L}_1$), for the coefficient of the fourth term on the right side of (3.8), we get

$$\begin{aligned}&\left(\frac{(1-\kappa)p}{2}-1\right)\left(1-\int_0^t g(s)ds\right)-\frac{1-\ell}{2(1-\kappa)p}\\&\qquad>\left(\frac{(1-\kappa)p}{2}-1\right)\ell-\frac{1-\ell}{2(1-\kappa)p}:=a_\ell>0.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\Psi'(t) \geq & \left[1-\varsigma-\varepsilon\left(\frac{\alpha+1}{\alpha+2}\right)\mathcal{K}_1-\frac{\varepsilon}{2}\mathcal{K}_2\right]Z^{-\varsigma}(t)Z'(t)+\varepsilon\left(1+\frac{(1-\kappa)p}{2}\right)\|v_t\|_2^2 \\ & +a_\ell\varepsilon\|\nabla v\|_2^2+\frac{k\varepsilon}{p}(1-\kappa)\|v\|_p^p+\varepsilon\kappa\int_\Omega |v|^p\ln|v|^k dx+\varepsilon p(1-\kappa)Z(t) \\ & -\varepsilon p(1-\kappa)\mathcal{E}_\jmath-\left(\frac{\varepsilon k_0}{\alpha+2}\right)\mathcal{K}_1^{-(\alpha+1)}Z^{\varsigma(\alpha+1)}(t)\|\nabla v\|_2^{2\theta}\|v\|_{\alpha+2}^{\alpha+2} \\ & -\left(\frac{\varepsilon k_1}{2}\right)\mathcal{K}_2^{-1}Z^{\varsigma}(t)\|\nabla v\|_2^{2\theta}\|v\|_2^2.\end{aligned} \tag{3.9}$$

Now, we estimate the eighth term on the right side of (3.9). Using the condition ($L_3$) and (3.2), we have

$$\begin{aligned}& Z^{\varsigma(\alpha+1)}(t)\|\nabla v\|_2^{2\theta}\|v\|_{\alpha+2}^{\alpha+2} \\ & \quad\leq |\Omega|^{\frac{p-\alpha-2}{p}}Z^{\varsigma(\alpha+1)}(t)\|\nabla v\|_2^{2\theta}\|v\|_p^{\alpha+2} \\ & \quad\leq |\Omega|^{\frac{p-\alpha-2}{p}}\left(\frac{k}{p}\right)^{\varsigma(\alpha+1)}\|\nabla v\|_2^{2\theta}\|v\|_p^{\alpha+2}\left(\int_\Omega |v|^p\ln|v|dx\right)^{\varsigma(\alpha+1)} \\ & \quad\leq C_1\|\nabla v\|_2^{2\theta}\|v\|_{p+\jmath}^{\alpha+2+\varsigma(p+\jmath)(\alpha+1)},\end{aligned} \tag{3.10}$$

where $C_1=|\Omega|^{\left(\frac{p-\alpha-2}{p}+\frac{\jmath(\alpha+2)}{p(p+\jmath)}\right)}\left(\frac{k}{pe\jmath}\right)^{\varsigma(\alpha+1)}$. Similarly, for the ninth term, we get

$$Z^{\varsigma}(t)\|\nabla v\|_2^{2\theta}\|v\|_2^2 \leq C_2\|\nabla v\|_2^{2\theta}\|v\|_{p+\jmath}^{2+\varsigma(p+\jmath)}, \tag{3.11}$$

where $C_2=|\Omega|^{\frac{p+\jmath-2}{p+\jmath}}\left(\frac{k}{pe\jmath}\right)^{\varsigma}$. By using Young's inequality, taking (3.3) and Lemma 2.5 into account, from (3.10) we obtain

$$\begin{aligned}Z^{\varsigma(\alpha+1)}(t)\|\nabla v\|_2^{2\theta}\|v\|_{\alpha+2}^{\alpha+2} & \leq C_1\|\nabla v\|_2^{2\theta}\|v\|_{p+\jmath}^{\alpha+2+\varsigma(p+\jmath)(\alpha+1)} \\ & \leq C_1\theta\|\nabla v\|_2^2+C_1(1-\theta)\|v\|_{p+\jmath}^{\frac{\alpha+2+\varsigma(p+\jmath)(\alpha+1)}{1-\theta}} \\ & \leq C_1\big(\theta+C(1-\theta)\big)\|\nabla v\|_2^2+C_1C(1-\theta)\|v\|_{p+\jmath}^{p+\jmath}.\end{aligned} \tag{3.12}$$

By (3.3) it is not difficult to see that $\frac{2+\varsigma(p+\jmath)}{1-\theta}<p+\jmath$. Thus, by the similar way in (3.12), from (3.11) we get

$$\begin{aligned}Z^{\varsigma}(t)\|\nabla v\|_2^{2\theta}\|v\|_2^2 & \leq C_2\|\nabla v\|_2^{2\theta}\|v\|_{p+\jmath}^{2+\varsigma(p+\jmath)} \\ & \leq C_2\theta\|\nabla v\|_2^2+C_2(1-\theta)\|v\|_{p+\jmath}^{\frac{2+\varsigma(p+\jmath)}{1-\theta}} \\ & \leq C_2\big(\theta+C(1-\theta)\big)\|\nabla v\|_2^2+C_2C(1-\theta)\|v\|_{p+\jmath}^{p+\jmath}.\end{aligned} \tag{3.13}$$

Therefore, from (3.12),(3.13) and using the fact that $\int_\Omega |u|^p\ln|u|^k dx > pZ(0)$, the estimate (3.9) becomes as

$$
\begin{aligned}
\Psi'(t) \geq & \left[1-\varsigma-\varepsilon\left(\frac{\alpha+1}{\alpha+2}\right)\mathcal{K}_1-\frac{\varepsilon}{2}\mathcal{K}_2\right] Z^{-\varsigma}(t)Z'(t)+\varepsilon\left(1+\frac{(1-\kappa)p}{2}\right)\|v_t\|_2^2 \\
& +\mu a_\ell \varepsilon\|\nabla v\|_2^2+\frac{k\varepsilon}{p}(1-\kappa)\|v\|_p^p+\varepsilon\kappa p Z(0)+\varepsilon p(1-\kappa)Z(t)-\varepsilon p(1-\kappa)\mathcal{E}_\jmath \\
& +\varepsilon\left[(1-\mu)a_\ell-\big(\theta+C(1-\theta)\big)\left(\frac{k_0C_1\mathcal{K}_1^{-(\alpha+1)}}{\alpha+2}+\frac{k_1C_2\mathcal{K}_2^{-1}}{2}\right)\right]\|\nabla v\|_2^2 \\
& -\varepsilon C(1-\theta)\left(\frac{k_0C_1\mathcal{K}_1^{-(\alpha+1)}}{\alpha+2}+\frac{k_1C_2\mathcal{K}_2^{-1}}{2}\right)\|v\|_{p+\jmath}^{p+\jmath}.
\end{aligned} \tag{3.14}
$$

where $0<\mu<1$ is an arbitrary constant. Using the Lemma 2.4 - (**a**) and ($\mathrm{L}_2$), for each $\mu\in[q,1)$, we have

$$
\begin{aligned}
\mu a_\ell & \|\nabla v\|_2^2-p(1-\kappa)\mathcal{E}_\jmath \\
& > \mu a_\ell\gamma_\jmath^2-p(1-\kappa)\mathcal{E}_\jmath \\
& > \mu\gamma_\jmath^2\left[\left(\frac{(1-\kappa)p}{2}-1\right)\ell-\frac{1-\ell}{2(1-\kappa)p}\right]-pq(1-\kappa)\widetilde{E}_\jmath \\
& = (\mu-q)\left(\frac{\ell\big((1-\kappa)p-1\big)^2-1}{2(1-\kappa)p}\right)\gamma_\jmath^2 \\
& \geq 0.
\end{aligned} \tag{3.15}
$$

Since $v$ is global, there exists a constant $\mathcal{C}>0$ such that $\|v\|_{p+\jmath}^{p+\jmath}\leq\mathcal{C}$ for all $t\geq 0$. Then, we choose $\mathcal{K}_1$ and $\mathcal{K}_2$ so large that

$$
\begin{cases}
b:=(1-\mu)a_\ell-\big(\theta+C(1-\theta)\big)\left(\frac{k_0C_1\mathcal{K}_1^{-(\alpha+1)}}{\alpha+2}+\frac{k_1C_2\mathcal{K}_2^{-1}}{2}\right)>0, \\
\kappa pZ(0)-\mathcal{C}C(1-\theta)\left(\frac{k_0C_1\mathcal{K}_1^{-(\alpha+1)}}{\alpha+2}+\frac{k_1C_2\mathcal{K}_2^{-1}}{2}\right)>0.
\end{cases} \tag{3.16}
$$

Now, we choose $\varepsilon$ small enough so that

$$
0<\varepsilon<\frac{2(\alpha+2)(1-\varsigma)}{2(\alpha+1)\mathcal{K}_1+(\alpha+2)\mathcal{K}_2}. \tag{3.17}
$$

Therefore, using (3.15)-(3.17), the inequality (3.14) can be summarized as follows

$$
\Psi'(t)\geq K_0\big(\|v_t\|_2^2+\|\nabla v\|_2^2+\|v\|_p^p+Z(t)\big), \quad \forall t\geq 0, \tag{3.18}
$$

in which

$$
K_0=\varepsilon\min\left\{1+\frac{(1-\kappa)p}{2},\frac{k}{p}(1-\kappa),p(1-\kappa),b\right\}.
$$

Choosing $\varepsilon$ small enough so that $\Psi(0)\geq 0$, according to (3.18), we can say that for every $t\geq 0$, $\Psi(t)\geq 0$. On the other hand, using Hölder and Young inequalities, we have

$$
\begin{aligned}
|(v,v_t)|^{\frac{1}{1-\varsigma}} & \leq \|v\|_2^{\frac{1}{1-\varsigma}}\|v_t\|_2^{\frac{1}{1-\varsigma}} \\
& \leq |\Omega|^{\frac{p-2}{2p(1-\varsigma)}}\|v\|_p^{\frac{1}{1-\varsigma}}\|v_t\|_2^{\frac{1}{1-\varsigma}} \\
& \leq |\Omega|^{\frac{p-2}{2p(1-\varsigma)}}\left(\frac{1}{\eta_1}\|v\|_p^{\frac{\eta_1}{1-\varsigma}}+\frac{1}{\eta_2}\|v_t\|_2^{\frac{\eta_2}{1-\varsigma}}\right),
\end{aligned} \tag{3.19}
$$

where $\eta_1, \eta_2 > 0$ and $\eta_1^{-1} + \eta_2^{-1} = 1$. By choosing $\eta_2 = 2(1-\varsigma)$, we have $\eta_1 = 2(1-\varsigma)/(1-2\varsigma)$. It is also clear from (3.3) that $\frac{2}{1-2\varsigma} \leq p$. Therefore, using the Lemma 2.5, the estimate (3.19) will be in the form

$$\begin{aligned} |(v,v_t)|^{\frac{1}{1-\varsigma}} &\leq \frac{|\Omega|^{\frac{p-2}{2p(1-\varsigma)}}}{2(1-\varsigma)} \left((1-2\varsigma)\|v\|_p^{\frac{2}{1-2\varsigma}} + \|v_t\|_2^2\right) \\ &\leq \frac{|\Omega|^{\frac{p-2}{2p(1-\varsigma)}}}{2(1-\varsigma)} \left(C(1-2\varsigma)\big(\|\nabla v\|_2^2 + \|v\|_p^p\big) + \|v_t\|_2^2\right). \end{aligned} \tag{3.20}$$

which yields

$$\begin{aligned} \Psi^{\frac{1}{1-\varsigma}}(t) &= \big(Z^{1-\varsigma}(t) + \varepsilon(v,v_t)\big)^{\frac{1}{1-\varsigma}} \\ &\leq 2^{\frac{\varsigma}{1-\varsigma}}\left(Z(t) + \varepsilon^{\frac{1}{1-\varsigma}}|(v,v_t)|^{\frac{1}{1-\varsigma}}\right) \\ &\leq K_1\big(Z(t) + \|\nabla v\|_2^2 + \|v\|_p^p + \|v_t\|_2^2\big), \qquad \forall t \geq 0, \end{aligned} \tag{3.21}$$

where

$$K_1 = 2^{\frac{\varsigma}{1-\varsigma}}\left(1 + \frac{\varepsilon^{\frac{1}{1-\varsigma}}|\Omega|^{\frac{p-2}{2p(1-\varsigma)}}}{2(1-\varsigma)}\Big(1 + C(1-2\varsigma)\Big)\right).$$

Therefore, by combining the inequalities (3.18) and (3.21) we arrive at the following differential inequality

$$\Psi^{\frac{1}{1-\varsigma}}(t) \leq \frac{K_1}{K_0}\Psi'(t),$$

which shows that $\psi$ blows up at a finite time $T^\star$ and

$$0 < T^\star < \frac{K_1(1-\varsigma)}{K_0\varsigma\Psi^{\frac{\varsigma}{1-\varsigma}}(0)}.$$

Now, by (3.21) we have

$$\lim_{t\to T^{\star -}} \big(Z(t) + \|\nabla v\|_2^2 + \|v\|_p^p + \|v_t\|_2^2\big) = +\infty.$$

This contradicts the assumption that $v$ is global in time.

To prove (3.1), by the Lemma 2.4-(**a**) and definition of the solution energy, we have

$$\frac{1}{p}\int_\Omega |v|^p \ln|v|^k dx + E_\jmath \geq \frac{\ell}{2}\|\nabla v\|_2^2 > \frac{\ell}{2}\gamma_\jmath^2.$$

Thus, we get

$$\int_\Omega |v(t)|^p \ln|v(t)| dx > \frac{p}{k}\left(\frac{\ell}{2}\gamma_\jmath^2 - E_\jmath\right) = \frac{p\ell\gamma_\jmath^2}{k(p+\jmath)}. \tag{3.22}$$

Next, we define the function $W$ as follows:

$$W(t) = \int_\Omega |v(t)|^p \ln|v(t)| dx.$$

By differentiating $W$ with respect to $t$, we have

$$W'(t) = p\int_\Omega |v|^{p-2}vv_t \ln|v| dx + \int_\Omega |v|^{p-2}vv_t dx. \tag{3.23}$$

Now, we consider the following two cases:

**Case I:** $n = 1, 2$. For the logarithmic integral in (3.23), considering that for every $x > 1$, $\ln(x) < x$, and also using (3.22), we get

$$\begin{aligned}
\int_\Omega |v|^{p-2} v v_t \ln |v| dx
&\le \int_\Omega |v|^{p-1} |v_t| \big| \ln |v| \big| dx \\
&= \int_{\{x\in\Omega\,:\,|v|\le 1\}} |v|^{p-1} |v_t| \big| \ln |v| \big| dx + \int_{\{x\in\Omega\,:\,|v|\ge 1\}} |v|^{p-1} |v_t| \big| \ln |v| \big| dx \\
&\le \frac{1}{e(p-1)} \int_\Omega |v_t| dx + \int_{\{x\in\Omega\,:\,|v|\ge 1\}} |v|^p |v_t| dx \\
&\le \frac{|\Omega|}{e(p-1)} + \frac{1}{2}\left(1 + \frac{1}{e(p-1)}\right) \int_\Omega |v_t|^2 dx + \frac{1}{2}\int_\Omega |v|^{2p} dx \\
&\le d_1 \int_\Omega |v|^p \ln |v| dx + d_2 \int_\Omega |v_t|^2 dx + d_3 \left(\int_\Omega |\nabla v|^2 dx\right)^p,
\end{aligned} \tag{3.24}$$

in which

$$d_1 = \frac{k|\Omega|(p+\jmath)}{p\ell\gamma_\jmath^2 e(p-1)}, \quad d_2 = \frac{1}{2}\left(1 + \frac{1}{e(p-1)}\right), \quad d_3 = \frac{C_{2p}^{2p}}{2}.$$

For the second term on the right side of (3.23), we have

$$\begin{aligned}
\int_\Omega |v|^{p-2} v v_t dx \le \int_\Omega |v|^{p-1} |v_t| dx &\le \frac{1}{2}\int_\Omega |v|^{2(p-1)} dx + \frac{1}{2}\int_\Omega |v_t|^2 dx \\
&\le d_4 \left(\int_\Omega |\nabla v|^2 dx\right)^{p-1} + d_5 \int_\Omega |v_t|^2 dx,
\end{aligned} \tag{3.25}$$

where $d_4 = \frac{C_{2(p-1)}^{2(p-1)}}{2}$ and $d_5 = \frac{1}{2}$.

**Case II:** $n \ge 3$. Using Young's inequality, we obtain

$$W'(t) \le \frac{p}{2} \int_\Omega |v|^{2(p-1)} \big| \ln |v| \big|^2 dx + \frac{1}{2}\int_\Omega |v|^{2(p-1)} dx + \left(\frac{p+1}{2}\right) \int_\Omega |v_t|^2 dx. \tag{3.26}$$

To estimate the first integral on the right side of (3.26), by Lemma 2.3, for each $\tau > 0$, we have

$$\begin{aligned}
\int_\Omega |v|^{2(p-1)} \big| \ln |v| \big|^2 dx
&= \int_{\{x\in\Omega\,:\,|v|\le 1\}} |v|^{2(p-1)} \big| \ln |v| \big|^2 dx + \int_{\{x\in\Omega\,:\,|v|>1\}} |v|^{2(p-1)} \big| \ln |v| \big|^2 dx \\
&\le \frac{|\Omega|}{e^2(p-1)^2} + \frac{1}{e\tau} \int_{\{x\in\Omega\,:\,|v|>1\}} |v|^{2(p-1+\tau)} dx \\
&\le \frac{|\Omega|}{e^2(p-1)^2} + \frac{1}{e\tau} \int_\Omega |v|^{2(p-1+\tau)} dx.
\end{aligned} \tag{3.27}$$

According to (A$_2$), $\tau$ can be chosen as $\tau = \frac{n}{n-2} + 1 - p$. With this choice of $\tau$, using (2.1) and also applying (3.22), the inequality (3.27) will be as follows:

$$\begin{aligned}\int_\Omega |v|^{2(p-1)} \big| \ln |v| \big|^2 dx &\leq \frac{k|\Omega|(p+\jmath)}{p\ell\gamma_\jmath^2 e^2 (p-1)^2} \int_\Omega |v|^p \ln |v| dx + \frac{1}{e\tau} \int_\Omega |v|^{\frac{2n}{n-2}} dx \\ &\leq d_6 \int_\Omega |v|^p \ln |v| dx + d_7 \left( \int_\Omega |\nabla v|^2 dx \right)^{\frac{n}{n-2}},\end{aligned} \tag{3.28}$$

where

$$d_6 = \frac{k|\Omega|(p+\jmath)}{p\ell\gamma_\jmath^2 e^2 (p-1)^2}, \quad d_7 = \frac{(n-2) C_{\frac{2n}{n-2}}^{\frac{2n}{n-2}}}{e\big(2(n-1) - p(n-2)\big)}.$$

For the second integral in (3.26), according to (A$_2$), the estimate obtained in (3.25) can be used. On the other hand, using definition of the solution energy and (3.22), we arrive at the following two estimates for $\|v_t\|_2$ and $\|\nabla v\|_2$:

$$\begin{aligned}&\int_\Omega |v_t|^2 dx < d_8 \int_\Omega |v|^p \ln |v| dx, \qquad d_8 = \frac{2k}{p}\left(1 + \frac{(p+\jmath)E_\jmath}{\ell\gamma_\jmath^2}\right), \\ &\int_\Omega |\nabla v|^2 dx < d_9 \int_\Omega |v|^p \ln |v| dx, \qquad d_9 = \frac{2k}{p\ell}\left(1 + \frac{(p+\jmath)E_\jmath}{\ell\gamma_\jmath^2}\right).\end{aligned} \tag{3.29}$$

Therefore, using the estimates (3.24) - (3.29), we have

$$W'(t) \leq \beta_0 W(t) + \beta_1 \big(W(t)\big)^\xi + \beta_2 \big(W(t)\big)^\zeta, \tag{3.30}$$

where in each of the two cases, $\xi = p-1$ and for the first case

$$\zeta = p, \qquad \beta_0 = p(d_1 + d_2 d_8) + d_5 d_8, \qquad \beta_1 = d_4 d_9^{p-1}, \qquad \beta_2 = p d_3 d_9^p,$$

and for the second case

$$\zeta = \frac{n}{n-2}, \qquad \beta_0 = \frac{p}{2} d_6 + \left(\frac{p+1}{2}\right) d_8, \qquad \beta_1 = \frac{1}{2} d_4 d_9^{p-1}, \qquad \beta_2 = \frac{p}{2} d_7 d_9^{\frac{n}{n-2}}.$$

It is clear from the Theorem 3.1 that

$$\lim_{t \to T^{\star -}} W(t) = +\infty.$$

By integrating (3.30) over $(0,t)$, $(0 < t < T^\star)$ we have

$$\int_{W(0)}^{W(t)} \frac{dy}{\beta_0 y + \beta_1 y^\xi + \beta_2 y^\zeta} \leq t$$

The proof is completed when we let $t \to T^{\star -}$. □

# 4 Blow up for high initial energy

In this section we investigate the blow up property of solutions at arbitrary high positive initial energy. The main result reads from the following theorem.

**Theorem 4.1** (Blow up when $E(0)>0$)**.** *Assume that the conditions* ($\mathrm{A}_1$)-($\mathrm{A}_2$) *hold. If*
($\mathrm{H}_1$) $p-2>\alpha>0$ *and* $0<\theta<\dfrac{p-\alpha-2}{p}$,
($\mathrm{H}_2$) $\ell>\frac{1}{\left(p(1-\varsigma)-1\right)^2}$ *where* $0<\varsigma<1-\frac{2}{p}$,
($\mathrm{H}_3$) $0<k<\dfrac{2ep\big(v(x,0),v_t(x,0)\big)}{C_*\epsilon^\star|\Omega|}\sqrt{\left(\frac{p(1-\epsilon^\star)}{2}-1\right)(\ell-\epsilon^\star)-\frac{1-\ell}{2p(1-\epsilon^\star)}}$,
*where* $C_*$ *is the best constant of the embedding inequality* $\|v\|_2\le C_*\|v\|_{H_0^1}$, $\epsilon^\star\in(0,1)$ *is the first positive root of the equation*

$$\frac{2(\alpha+1)\big(\rho(\epsilon^\star)\big)^{\frac{\alpha(\alpha+2)}{\alpha+1}}+\alpha+2}{C_*(1-\epsilon^\star)\big(\rho(\epsilon^\star)\big)^{\alpha+2}}=\frac{p(\alpha+2)}{\sqrt{\left(\frac{p(1-\epsilon^\star)}{2}-1\right)(\ell-\epsilon^\star)-\frac{1-\ell}{2p(1-\epsilon^\star)}}},\tag{4.1}$$

*where* $\rho(\epsilon)$ *is given in* (4.10),
($\mathrm{H}_4$) *the initial energy and the initial values* $v_0\in H_0^1(\Omega)$ *and* $v_1\in L^2(\Omega)$ *satisfy*

$$\frac{\big(v(x,0),v_t(x,0)\big)-\xi}{pC_*(1-\epsilon^\star)}>\frac{E(0)}{2\sqrt{\left(\frac{p(1-\epsilon^\star)}{2}-1\right)(\ell-\epsilon^\star)-\frac{1-\ell}{2p(1-\epsilon^\star)}}}>0,$$

*where* $\xi$ *is a positive constant determined in* (4.14)*. Then solution of* (1.1)-(1.3) *blows up at a finite time* $T^\star$*. Furthermore, the lifespan* $T^\star$ *satisfies the following estimate*

$$T^\star>\int_{\mathcal{W}(0)}^{+\infty}\frac{dy}{\alpha_0+\alpha_1 y+\alpha_2 y^\zeta},\tag{4.2}$$

*where* $\mathcal{W}(0)=\frac{1}{2}\|v_1\|_2^2+\frac{1}{2}\|\nabla v_0\|_2^2$ *and the positive constants* $\alpha_0$, $\alpha_1$, $\alpha_2$ *and* $\zeta$ *will be determined in the proof.*

*Proof.* We assume that $v$ is a global solution of the problem (1.1)-(1.3). Let

$$\omega(t)=\int_\Omega v(x,t)v_t(x,t)dx-\xi,\qquad t\ge 0,$$

where $\xi>0$ is a constant which will be determined later. Then, by the boundary condition (1.2), we easily obtain

$$\begin{aligned}\frac{d}{dt}\omega(t)=&\|v_t\|_2^2-\|\nabla v\|_2^2+\int_\Omega\int_0^t g(t-s)\nabla v(t).\nabla v(s)dsdx\\&-k_0\|\nabla v\|_2^{2\theta}\int_\Omega|v_t|^\alpha v_t v dx-k_1\|\nabla v\|_2^{2\theta}\int_\Omega v v_t dx+\int_\Omega|v|^p\ln|v|^k dx.\end{aligned}\tag{4.3}$$

For any $0<\epsilon<1$, we get

$$\begin{aligned}&\int_0^t\int_\Omega g(t-s)\nabla v(t).\nabla v(s)dxds\\&\qquad=\|\nabla v\|_2^2\int_0^t g(s)ds+\int_0^t\int_\Omega g(t-s)\nabla v(t).\big(\nabla v(s)-\nabla v(t)\big)dxds\\&\qquad\ge\left(1-\frac{1}{2p(1-\epsilon)}\right)\|\nabla v\|_2^2\int_0^t g(s)ds-\frac{p(1-\epsilon)}{2}(g\diamond\nabla v)(t),\end{aligned}\tag{4.4}$$

By Young's inequality, we have

$$
\begin{aligned}
k_0\|\nabla v\|_2^{2\theta}\int_\Omega |v_t|^\alpha v_t v dx &\le \frac{k_0\varepsilon_1^{\alpha+2}}{\alpha+2}\|\nabla v\|_2^{2\theta}\|v\|_{\alpha+2}^{\alpha+2} \\
&\quad + k_0\varepsilon_1^{-\frac{\alpha+2}{\alpha+1}}\left(\frac{\alpha+1}{\alpha+2}\right)\|\nabla v\|_2^{2\theta}\|v_t\|_{\alpha+2}^{\alpha+2}, \\
k_1\|\nabla v\|_2^{2\theta}\int_\Omega v_t v dx &\le \frac{k_1\varepsilon_2}{2}\|\nabla v\|_2^{2\theta}\|v\|_2^2 + \frac{k_1}{2\varepsilon_2}\|\nabla v\|_2^{2\theta}\|v_t\|_2^2,
\end{aligned} \tag{4.5}
$$

for any $\varepsilon_1, \varepsilon_2 > 0$. Hence, from (4.3)-(4.5) and using the energy identity, we obtain

$$
\begin{aligned}
\frac{d}{dt}\omega(t) \ge &\left(1+\frac{p}{2}(1-\epsilon)\right)\|v_t\|_2^2 + \epsilon\int_\Omega |v|^p \ln|v|^k dx + \frac{k(1-\epsilon)}{p}\|v\|_p^p \\
&+ \left[\left(\frac{p(1-\epsilon)}{2}-1\right)\ell - \frac{1-\ell}{2p(1-\epsilon)}\right]\|\nabla v\|_2^2 - p(1-\epsilon)E(t) \\
&- \frac{k_0\varepsilon_1^{\alpha+2}}{\alpha+2}\|\nabla v\|_2^{2\theta}\|v\|_{\alpha+2}^{\alpha+2} - k_0\varepsilon_1^{-\frac{\alpha+2}{\alpha+1}}\left(\frac{\alpha+1}{\alpha+2}\right)\|\nabla v\|_2^{2\theta}\|v_t\|_{\alpha+2}^{\alpha+2} \\
&- \frac{k_1\varepsilon_2}{2}\|\nabla v\|_2^{2\theta}\|v\|_2^2 - \frac{k_1}{2\varepsilon_2}\|\nabla v\|_2^{2\theta}\|v_t\|_2^2.
\end{aligned} \tag{4.6}
$$

Using Young's inequality, Lemma 2.5 and the assumption ($H_1$), we have

$$
\begin{aligned}
\|\nabla v\|_2^{2\theta}\|v\|_{\alpha+2}^{\alpha+2} &\le \theta\|\nabla v\|_2^2 + (1-\theta)\|v\|_{\alpha+2}^{\frac{\alpha+2}{1-\theta}} \\
&\le \theta\|\nabla v\|_2^2 + \mathcal{C}_1(1-\theta)\|v\|_p^{\frac{\alpha+2}{1-\theta}} \\
&\le \theta\|\nabla v\|_2^2 + \mathcal{C}_1 C(1-\theta)\left(\|\nabla v\|_2^2 + \|v\|_p^p\right) \\
&= \left(\theta + \mathcal{C}_1 C(1-\theta)\right)\|\nabla v\|_2^2 + \mathcal{C}_1 C(1-\theta)\|v\|_p^p,
\end{aligned} \tag{4.7}
$$

where $\mathcal{C}_1 = |\Omega|^{\frac{p-\alpha-2}{p(1-\theta)}}$. Similarly,

$$
\begin{aligned}
\|\nabla v\|_2^{2\theta}\|v\|_2^2 &\le \theta\|\nabla v\|_2^2 + (1-\theta)\|v\|_2^{\frac{2}{1-\theta}} \\
&\le \theta\|\nabla v\|_2^2 + \mathcal{C}_2(1-\theta)\|v\|_p^{\frac{2}{1-\theta}} \\
&\le \theta\|\nabla v\|_2^2 + \mathcal{C}_2 C(1-\theta)\left(\|\nabla v\|_2^2 + \|v\|_p^p\right) \\
&= \left(\theta + \mathcal{C}_2 C(1-\theta)\right)\|\nabla v\|_2^2 + \mathcal{C}_2 C(1-\theta)\|v\|_p^p,
\end{aligned} \tag{4.8}
$$

where $\mathcal{C}_2 = |\Omega|^{\frac{p-2}{p(1-\theta)}}$. Thus, by Lemma 2.2, letting $\varepsilon_2 = \varepsilon_1^{\alpha+2}$ and using the fact that $x^p \ln|x| >$

$-\frac{1}{ep}$ on $0<x<1$, for any $0<\epsilon<1-\frac{2}{p}$, we get

$$\begin{aligned}
&\frac{d}{dt}\left[\omega(t)-\left(\frac{\alpha+1}{\alpha+2}\varepsilon_1^{-\frac{\alpha+2}{\alpha+1}}+\frac{1}{2}\varepsilon_1^{-(\alpha+2)}\right)E(t)\right]\\
&\qquad\geq\left(1+\frac{p}{2}(1-\epsilon)\right)\|v_t\|_2^2+\epsilon\int_\Omega |v|^p\ln|v|^k dx-p(1-\epsilon)E(t)\\
&\qquad\quad+\left[\left(\frac{p(1-\epsilon)}{2}-1\right)\ell-\frac{1-\ell}{2p(1-\epsilon)}\right]\|\nabla v\|_2^2\\
&\qquad\quad+\left[\frac{k(1-\epsilon)}{p}-C(1-\theta)\varepsilon_1^{\alpha+2}\left(\frac{\mathcal{C}_1k_0}{\alpha+2}+\frac{\mathcal{C}_2k_1}{2}\right)\right]\|v\|_p^p\\
&\qquad\quad-\varepsilon_1^{\alpha+2}\left(\frac{k_0\big(\theta+\mathcal{C}_1C(1-\theta)\big)}{\alpha+2}+\frac{k_1\big(\theta+\mathcal{C}_2C(1-\theta)\big)}{2}\right)\|\nabla v\|_2^2\\
&\qquad\geq\left(1+\frac{p}{2}(1-\epsilon)\right)\|v_t\|_2^2-\frac{\epsilon k|\Omega|}{ep}-p(1-\epsilon)E(t)\\
&\qquad\quad+\gamma(\epsilon)\|\nabla v\|_2^2+\left(\frac{k(1-\epsilon)}{p}-\varepsilon_1^{\alpha+2}Q_1(C,\mathcal{C}_1,\mathcal{C}_2)\right)\|v\|_p^p\\
&\qquad\quad+\left(\left(\frac{p(1-\epsilon)}{2}-1\right)\epsilon-\varepsilon_1^{\alpha+2}Q_2(C,\mathcal{C}_1,\mathcal{C}_2)\right)\|\nabla v\|_2^2,
\end{aligned} \tag{4.9}$$

where

$$Q_1(C,\mathcal{C}_1,\mathcal{C}_2)=C(1-\theta)\left(\frac{\mathcal{C}_1k_0}{\alpha+2}+\frac{\mathcal{C}_2k_1}{2}\right),$$
$$Q_2(C,\mathcal{C}_1,\mathcal{C}_2)=\frac{k_0\big(\theta+\mathcal{C}_1C(1-\theta)\big)}{\alpha+2}+\frac{k_1\big(\theta+\mathcal{C}_2C(1-\theta)\big)}{2},$$

and

$$\gamma(\epsilon)=\left(\frac{p(1-\epsilon)}{2}-1\right)(\ell-\epsilon)-\frac{1-\ell}{2p(1-\epsilon)}.$$

We see that $\gamma(\epsilon)\to\gamma(0)=\left(\frac{p}{2}-1\right)\ell-\frac{1-\ell}{2p}$ as $\epsilon\to 0^+$. By the assumption ($H_2$) we have

$$\ell>\frac{1}{\big(p(1-\varsigma)-1\big)^2}>\frac{1}{(p-1)^2},$$

which means that $\gamma(0)>0$. Now, to use the intermediate value theorem, we consider two possibilities for $\ell$ and $1-\frac{2}{p}$. (i): $1-\frac{2}{p}<\ell$. In this case, $\gamma(\epsilon)\to-\frac{1-\ell}{p^2}<0$ as $\epsilon\to\left(1-\frac{2}{p}\right)^-$. (ii): $\ell\leq 1-\frac{2}{p}$. In this case we also have $\gamma(\epsilon)\to-\frac{1}{2p}<0$ when $\epsilon\to\ell^-$. We set $\epsilon'=\min\left\{\ell,1-\frac{2}{p}\right\}$. Hence, there exists a proper $\epsilon_0\in(0,\epsilon')\subset(0,1)$ such that $\gamma(\epsilon)>0$ on $(0,\epsilon_0)$ and $\gamma(\epsilon_0)=0$. Next, we let

$$\varepsilon_1=\rho(\epsilon):=\min\left\{\left(\frac{k(1-\epsilon)}{Q_1(C,\mathcal{C}_1,\mathcal{C}_2)}\right)^{\frac{1}{\alpha+2}},\left(\frac{\big(p(1-\epsilon)-2\big)\epsilon}{2Q_2(C,\mathcal{C}_1,\mathcal{C}_2)}\right)^{\frac{1}{\alpha+2}}\right\},\quad\forall\epsilon\in(0,\epsilon_0). \tag{4.10}$$

By (4.10) it is clear that

$$\rho(\epsilon)\longrightarrow\rho(\epsilon_0)>0\quad\text{as}\quad\epsilon\to\epsilon_0^-,\qquad\text{and}\qquad\rho(\epsilon)\longrightarrow 0\quad\text{as}\quad\epsilon\to 0^+. \tag{4.11}$$

Therefore, by (4.10), from (4.9) we arrive at

$$\begin{aligned}\frac{d}{dt}\Big[\omega(t)-\omega_1(\epsilon)E(t)\Big] &\geq \Big(1+\frac{p}{2}(1-\epsilon)\Big)\,\|v_t\|_2^2+\gamma(\epsilon)\|\nabla v\|_2^2-\frac{\epsilon k|\Omega|}{ep}-p(1-\epsilon)E(t)\\ &\geq \Big(1+\frac{p}{2}(1-\epsilon)\Big)\,\|v_t\|_2^2+\frac{\gamma(\epsilon)}{C_*^2}\|v\|_2^2-\frac{\epsilon k|\Omega|}{ep}-p(1-\epsilon)E(t)\\ &\geq \frac{2}{C_*}\sqrt{\gamma(\epsilon)\Big(1+\frac{p}{2}(1-\epsilon)\Big)}(v,v_t)-\frac{\epsilon k|\Omega|}{ep}-p(1-\epsilon)E(t)\\ &\geq \varphi(\epsilon)\left((v,v_t)-\frac{\epsilon k|\Omega|}{ep\varphi(\epsilon)}-\omega_2(\epsilon)E(t)\right),\quad \forall\epsilon\in(0,\epsilon_0),\end{aligned} \tag{4.12}$$

where

$$\varphi(\epsilon)=\frac{2\sqrt{\gamma(\epsilon)}}{C_*},\quad \omega_1(\epsilon)=\frac{\alpha+1}{\alpha+2}\big(\rho(\epsilon)\big)^{-\frac{\alpha+2}{\alpha+1}}+\frac{1}{2}\big(\rho(\epsilon)\big)^{-(\alpha+2)},\quad \omega_2(\epsilon)=\frac{p(1-\epsilon)}{\varphi(\epsilon)}. \tag{4.13}$$

Hence, from (4.11) and (4.13), when $\epsilon\to 0^+$ we have

$$\varphi(\epsilon)\longrightarrow\frac{2\sqrt{\gamma(0)}}{C_*},\quad \omega_1(\epsilon)\longrightarrow+\infty,\quad \omega_2(\epsilon)\longrightarrow\frac{p\,C_*}{2\sqrt{\gamma(0)}},$$

and if $\epsilon\to\epsilon_0^-$, we deduce that

$$\varphi(\epsilon)\longrightarrow 0,\quad \omega_1(\epsilon)\longrightarrow\frac{\alpha+1}{\alpha+2}\big(\rho(\epsilon_0)\big)^{-\frac{\alpha+2}{\alpha+1}}+\frac{1}{2}\big(\rho(\epsilon_0)\big)^{-(\alpha+2)},\quad \omega_2(\epsilon)\longrightarrow+\infty.$$

Thus, there exists a constant $\epsilon^\star\in(0,\epsilon_0)$ such that $\omega_1(\epsilon^\star)=\omega_2(\epsilon^\star)$. Then, taking $\xi$ to be

$$\xi=\frac{\epsilon^\star k|\Omega|}{ep\varphi(\epsilon^\star)}, \tag{4.14}$$

from (4.12) we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\omega(t)-\omega_2(\epsilon^\star)E(t)\Big) &\geq \varphi(\epsilon^\star)\Big((v,v_t)-\xi-\omega_2(\epsilon^\star)E(t)\Big)\\ &=\varphi(\epsilon^\star)\Big(\omega(t)-\omega_2(\epsilon^\star)E(t)\Big).\end{aligned} \tag{4.15}$$

Let $H(t)=\omega(t)-\omega_2(\epsilon^\star)E(t)$. Then, by ($H_3$) it holds that $\omega(0)>0$ and hence from ($H_4$) we deduce that $H(0)=\omega(0)-\omega_2(\epsilon^\star)E(0)>0$. Therefore, by (4.15) we get

$$H(t)\geq e^{\varphi(\epsilon^\star)t}H(0),\qquad \forall t\geq 0. \tag{4.16}$$

At this point we consider the following two cases:

**Case I**: $E(t)\geq 0$ for all $t\geq 0$. Then, (4.16) results

$$(v,v_t)\geq\xi+\omega_2(\epsilon^\star)E(t)+e^{\varphi(\epsilon^\star)t}H(0)>e^{\varphi(\epsilon^\star)t}H(0),\qquad \forall t\geq 0.$$

Hence we get

$$\|v(t)\|_2^2\geq\|v_0\|_2^2+\frac{2H(0)}{\varphi(\epsilon^\star)}\Big(e^{\varphi(\epsilon^\star)t}-1\Big),\qquad \forall t\geq 0, \tag{4.17}$$

which shows that the solutions grows exponentially when $t \longrightarrow +\infty$. On the other hand, we estimate the norm $\|v\|_2$ from above. Since, $v$ is global we have

$$\sigma_0 \leq \|\nabla v(t)\|_2^{2\theta} \leq \sigma_1, \qquad \forall t \geq 0,$$

where $\sigma_0$ and $\sigma_1$ are some positive constants. Then,

$$\begin{aligned} \|v(t)\|_2 &\leq \|v_0\|_2 + \int_0^t \|v_t(s)\|_2 ds \\ &\leq \|v_0\|_2 + |\Omega|^{\frac{\alpha}{2(\alpha+2)}} \int_0^t \|v_t(s)\|_{\alpha+2} ds \\ &\leq \|v_0\|_2 + |\Omega|^{\frac{\alpha}{2(\alpha+2)}} t^{\frac{\alpha+1}{\alpha+2}} \left( \int_0^t \|v_t(s)\|_{\alpha+2}^{\alpha+2} ds \right)^{\frac{1}{\alpha+2}} \\ &\leq \|v_0\|_2 + |\Omega|^{\frac{\alpha}{2(\alpha+2)}} t^{\frac{\alpha+1}{\alpha+2}} \left( \int_0^t \frac{\|\nabla v(s)\|_2^{2\theta} \|v_t(s)\|_{\alpha+2}^{\alpha+2}}{\sigma_0} ds \right)^{\frac{1}{\alpha+2}} \\ &\leq \|v_0\|_2 + |\Omega|^{\frac{\alpha}{2(\alpha+2)}} t^{\frac{\alpha+1}{\alpha+2}} \left( \frac{E(0) - E(t)}{k_0 \sigma_0} \right)^{\frac{1}{\alpha+2}} \\ &\leq \|v_0\|_2 + |\Omega|^{\frac{\alpha}{2(\alpha+2)}} t^{\frac{\alpha+1}{\alpha+2}} \left( \frac{E(0)}{k_0 \sigma_0} \right)^{\frac{1}{\alpha+2}}. \end{aligned} \tag{4.18}$$

Obviously, the inequality (4.18) contradicts with (4.17).

**Case II:** There exists $t' > 0$ such that $E(t') < 0$. Due to Lemma 2.2, $E(t) < 0$ for all $t > t'$. Then, by recalling the inequality (2.5) with $t = t'$ and $0 < \jmath < \jmath^*$, we have

$$\|\nabla v(t')\|_2 > \left( \frac{p\ell_\jmath}{2kC_{p+\jmath}^{p+\jmath}} \right)^{\frac{1}{p+\jmath-2}} > \left( \frac{p\ell_\jmath}{k(p+\jmath)C_{p+\jmath}^{p+\jmath}} \right)^{\frac{1}{p+\jmath-2}} = \gamma_\jmath.$$

Moreover,

$$0 < \theta < \frac{p - \alpha - 2}{p} < \frac{p + \jmath - \alpha - 2}{p + \jmath}.$$

Hence, by the fact that $E(t') < 0 < q\widetilde{E}_0$ together with ($H_1$) and ($H_2$), all assumptions stated in Theorem 3.1 have been fulfilled. Therefore, the solutions blow up at a finite time.

To prove (4.2), we define

$$\begin{aligned} \mathcal{W}(t) = \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla v\|_2^2 \\ + \frac{1}{2}(g \diamond \nabla v)(t) + \int_0^t \|\nabla v(s)\|_2^{2\theta} \int_\Omega \phi\big(v_t(s)\big) dx ds. \end{aligned}$$

By definition of the energy functional we have

$$\mathcal{W}(t) = \frac{1}{p}\int_\Omega |v|^p \ln|v|^k dx - \frac{k}{p^2}\|v\|_p^p + E(t) + \int_0^t \|\nabla v(s)\|_2^{2\theta} \int_\Omega \phi\big(v_t(s)\big) dx ds.$$

Then, from the Lemma 2.2, we get

$$\begin{aligned}\mathcal{W}'(t) &= \int_\Omega |v|^{p-2}vv_t \ln|v|^k dx + E'(t) + \|\nabla v(t)\|_2^{2\theta}\int_\Omega \phi(v_t(t))dx \\ &\le \int_\Omega |v|^{p-2}vv_t\ln|v|^k dx.\end{aligned} \tag{4.19}$$

Next, we consider the following two cases:

**Case I:** $n = 1, 2$. Similar to the proof lines of (3.24), we obtain

$$\begin{aligned}\int_\Omega vv_t|v|^{p-2}\ln|v|^k dx &\le \frac{k|\Omega|}{e(p-1)} + kd_2\int_\Omega |v_t|^2dx + d_3\left(\int_\Omega |\nabla v|^2 dx\right)^p \\ &\le \frac{k|\Omega|}{e(p-1)} + 2kd_2\mathcal{W}(t) + k\left(\frac{2}{\ell}\right)^p d_3\Big(\mathcal{W}(t)\Big)^p.\end{aligned} \tag{4.20}$$

**Case II:** $n \ge 3$. Similar to proof lines in (3.27) and (3.28), we get

$$\begin{aligned}&\int_\Omega |v|^{p-2}vv_t\ln|v|^k dx \\ &\quad\le \frac{k|\Omega|}{2e^2(p-1)^2} + \frac{k}{2e}\left(\frac{n-2}{2(n-1)-p(n-2)}\right)\int_\Omega |v|^{\frac{2n}{n-2}}dx + \frac{k}{2}\int_\Omega |v_t|^2 dx \\ &\quad\le \frac{k|\Omega|}{2e^2(p-1)^2} + \frac{k}{2e}\left(\frac{(n-2)C_{\frac{2n}{n-2}}^{\frac{2n}{n-2}}}{2(n-1)-p(n-2)}\right)\left(\int_\Omega |\nabla v|^2dx\right)^{\frac{n}{n-2}} + \frac{k}{2}\int_\Omega |v_t|^2dx \\ &\quad\le \frac{k|\Omega|}{2e^2(p-1)^2} + \frac{k}{2}\left(\frac{2}{\ell}\right)^{\frac{n}{n-2}} d_7\Big(\mathcal{W}(t)\Big)^{\frac{n}{n-2}} + k\mathcal{W}(t).\end{aligned} \tag{4.21}$$

Therefore, by the estimates (4.20)-(4.21), we arrive at

$$\mathcal{W}'(t) \le \alpha_0 + \alpha_1\mathcal{W}(t) + \alpha_2\Big(\mathcal{W}(t)\Big)^\zeta, \tag{4.22}$$

where for the first case

$$\zeta = p, \qquad \alpha_0 = \frac{k|\Omega|}{e(p-1)}, \qquad \alpha_1 = 2kd_2, \qquad \alpha_2 = k\left(\frac{2}{\ell}\right)^p d_3,$$

and for the second case

$$\zeta = \frac{n}{n-2}, \qquad \alpha_0 = \frac{k|\Omega|}{2e^2(p-1)^2}, \qquad \alpha_1 = k, \qquad \alpha_2 = \frac{k}{2}\left(\frac{2}{\ell}\right)^{\frac{n}{n-2}} d_7.$$

From the first part of Theorem, we know that $\|\nabla v(t)\|_2 \longrightarrow +\infty$ as $t \to T^{\star -}$. Moreover, by definition of $\mathcal{W}$ we have $\mathcal{W}(t) \ge \frac{\ell}{2}\|\nabla v(t)\|_2^2$. Hence,

$$\lim_{t\to T^{\star-}} \mathcal{W}(t) = +\infty.$$

Then, an integration of (4.22) over $(0, t)(0 < t < T^\star)$ yields

$$\int_{\mathcal{W}(0)}^{\mathcal{W}(t)} \frac{dy}{\alpha_0 + \alpha_1 y + \alpha_2 y^\zeta} \le t.$$

Letting $t \to T^{\star-}$ completes the proof. □

# 5 Asymptotic stability

In this section, we obtain the solution energy decay rates for the problem (1.1) - (1.3). First, we make the following assumptions on the function $g$, initial energy and the exponent $\alpha$:

(S$_1$) There exists a non-increasing and differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \le -\zeta(t)g(t), \quad \forall t \ge 0, \qquad \int_0^{+\infty} \zeta(s)ds = +\infty.$$

(S$_2$) $0 < E(0) < \min\left\{ E_\jmath, \frac{(p-2)\ell}{2p}\left(\frac{e\jmath p\ell}{8k(p-1)C_{p+\jmath}^{p+\jmath}}\right)^{\frac{2}{p+\jmath-2}}, \frac{(p-2)\ell}{2p}\left(\frac{p^2}{8kC_p^p}\right)^{\frac{2}{p-2}}\right\}$, with $0 < \jmath < \jmath^*$.

(S$_3$) $0 < \alpha \le \alpha^* = \begin{cases} \frac{4}{n-2}, & n = 3, 4, ... \\ +\infty, & n = 1, 2. \end{cases}$

**Lemma 5.1.** *Under the conditions of Lemma 2.4 -* (**b**) *solutions of problem* (1.1) - (1.3) *are bounded in time.*

*Proof.* Define the functional

$$I[v](t) = I(t) = \left(1 - \int_0^t g(s)ds\right) \|\nabla v\|_2^2 + (g \diamond \nabla v)(t) - \int_\Omega |u|^p \ln |u|^k dx.$$

Then

$$J[v](t) = \left(\frac{p-2}{2p}\right)\left[\left(1 - \int_0^t g(s)ds\right) \|\nabla v\|_2^2 + (g \diamond \nabla v)(t)\right] + \frac{k}{p^2}\|v\|_p^p + \frac{1}{p}I[v](t). \tag{5.1}$$

By using (A$_1$) and Lemma 2.3, we have

$$\begin{aligned} I[v](t) &\ge \ell\|\nabla v\|_2^2 - \int_{\{x\in\Omega:|v|\le 1\}} |v|^p \ln|v|^k dx - \int_{\{x\in\Omega:|v|>1\}} |v|^p \ln|v|^k dx \\ &\ge \ell\|\nabla v\|_2^2 - \int_{\{x\in\Omega:|v|>1\}} |v|^p \ln|v|^k dx \\ &\ge \ell\|\nabla v\|_2^2 - \frac{k}{e\jmath}\int_{\{x\in\Omega:|v|>1\}} |v|^{p+\jmath} dx \\ &\ge \ell\|\nabla v\|_2^2 - \frac{k}{e\jmath}\|v\|_{p+\jmath}^{p+\jmath}. \end{aligned}$$

Then, by Lemma 2.4 - (**b**), we get

$$
\begin{aligned}
I[v](t) &\geq \ell\|\nabla v\|_2^2 - \frac{k}{e\sigma}C_{p+\jmath}^{p+\jmath}\|\nabla v\|_2^{p+\jmath} \\
&\geq \frac{k}{\jmath e}C_{p+\jmath}^{p+\jmath}\|\nabla v\|_2^2\left(\frac{\ell\jmath}{kC_{p+\jmath}^{p+\jmath}} - \|\nabla v\|_2^{p+\jmath-2}\right) \\
&> \frac{k}{\jmath e}C_{p+\jmath}^{p+\jmath}\|\nabla v\|_2^2\left(\frac{\ell\jmath}{kC_{p+\jmath}^{p+\jmath}} - \gamma_\jmath^{p+\jmath-2}\right) \\
&= \frac{k}{\jmath e}C_{p+\jmath}^{p+\jmath}\|\nabla v\|_2^2\left(\frac{\ell\jmath}{kC_{p+\jmath}^{p+\jmath}} - \frac{p\jmath\ell}{k(p+\jmath)C_{p+\jmath}^{p+\jmath}}\right) \\
&= \frac{\jmath\ell}{e(p+\jmath)}\|\nabla v\|_2^2 \geq 0.
\end{aligned}
\tag{5.2}
$$

Considering (5.1) and (5.2), we have $J[v](t) > 0$ for each $t \geq 0$ and hence by definition of the energy functional, for each $t \geq 0$, we get $E[v](t) > 0$. Therefore, for each $t \geq 0$, we obtain

$$
\begin{cases}
\|v_t\|_2^2 \leq 2E(t) \leq 2E(0), \\
\|\nabla v\|_2^2 \leq \dfrac{2p}{(p-2)\ell}E(t) \leq \dfrac{2p}{(p-2)\ell}E(0), \\
\|v\|_p^p \leq \dfrac{p^2}{k}E(t) \leq \dfrac{p^2}{k}E(0), \\
(g \diamond \nabla v)(t) \leq \dfrac{2p}{p-2}E(t) \leq \dfrac{2p}{p-2}E(0).
\end{cases}
\tag{5.3}
$$

In fact, (5.3) shows that how the solutions are bounded with respect to time. □

**Lemma 5.2.** *Under the assumptions* ($A_1$), ($A_2$), ($S_2$), *and* ($S_3$), *there exist positive constants* $\beta_i$, $i = 0, 1, 2, 3$ *such that*

$$
\begin{aligned}
\frac{d}{dt}(v, v_t) \leq &- E(t) + \beta_0\|v_t\|_2^2 + \beta_1(g \diamond \nabla v)(t) \\
&+ \beta_2\|\nabla v\|_2^{2\theta}\|v_t\|_2^2 + \beta_3\|\nabla v\|_2^{2\theta}\|v_t\|_{\alpha+2}^{\alpha+2}, \quad \forall t \geq 0.
\end{aligned}
$$

*Proof.* Using (1.1) and with the boundary and initial conditions (1.2)-(1.3) we have

$$
\begin{aligned}
\frac{d}{dt}(v, v_t) =& \|v_t\|_2^2 - \|\nabla v\|_2^2 + \int_0^t g(t-s)\int_\Omega \nabla v(s).\nabla v(t)dxds \\
&+ \int_\Omega |v|^p \ln|v|^k dx - k_1\|\nabla v\|_2^{2\theta}\int_\Omega vv_t dxds \\
&- k_0\|\nabla v\|_2^{2\theta}\int_\Omega vv_t|v_t|^\alpha dxds.
\end{aligned}
\tag{5.4}
$$

Using Young's inequality, for the second integral on the right side of (5.4), for every $\varepsilon > 0$, we obtain

$$\begin{aligned}
&\int_0^t g(t-s)\int_\Omega \nabla v(s).\nabla v(t)dxds \\
&\quad= \left(\int_0^t g(s)ds\right)\|\nabla v\|_2^2+\int_\Omega \nabla v(t)\ .\int_0^t g(t-s)\big(\nabla v(s)-\nabla v(t)\big)dsdx \\
&\quad\le \left(\varepsilon+\int_0^t g(s)ds\right)\|\nabla v\|_2^2+\frac{1}{4\varepsilon}\int_\Omega\left(\int_0^t g(t-s)\big(\nabla v(s)-\nabla v(t)\big)ds\right)^2 dx \\
&\quad\le \left(\varepsilon+\int_0^t g(s)ds\right)\|\nabla v\|_2^2+\frac{1}{4\varepsilon}\ \|g\|_1\ (g\circ\nabla v)(t).
\end{aligned} \tag{5.5}$$

So, using (5.4), (5.5) and the energy identity, we get

$$\begin{aligned}
\frac{d}{dt}(v,v_t)\le &-E(t)+\frac{3}{2}\|v_t\|_2^2-\frac{1}{2}\left(1-\int_0^t g(s)ds-2\varepsilon\right)\|\nabla v\|_2^2 \\
&+\left(\frac{1}{4\varepsilon}\|g\|_1+\frac{1}{2}\right)(g\diamond\nabla v)(t)+\frac{k}{p^2}\|v\|_p^p+\left(\frac{p-1}{p}\right)\int_\Omega |v|^p\ln|v|^k dx \\
&-k_0\|\nabla v\|_2^{2\theta}\int_\Omega v v_t|v_t|^\alpha dxds-k_1\|\nabla v\|_2^{2\theta}\int_\Omega v v_t dxds.
\end{aligned} \tag{5.6}$$

Using ($A_2$) and (5.3), for the fifth term on the right side of (5.6), we have

$$\|v\|_p^p\le C_p^p\|\nabla v\|_2^p\le C_p^p\left(\frac{2p}{(p-2)\ell}E(0)\right)^{\frac{p-2}{2}}\|\nabla v\|_2^2. \tag{5.7}$$

Also, for the sixth term, using the Lemma 2.3 and (5.3), we obtain

$$\begin{aligned}
\int_\Omega |v|^p\ln|v|^k dx &= k\int_{\{x\in\Omega\ :\ |v|\le 1\}}|v|^p\ln|v|dx+k\int_{\{x\in\Omega\ :\ |v|>1\}}|v|^p\ln|v|dx \\
&\le k\int_{\{x\in\Omega\ :\ |v|>1\}}|v|^p\ln|v|dx\le\frac{k}{e\jmath}\int_{\{x\in\Omega\ :\ |v|>1\}}|v|^{p+\jmath}dx \\
&\le\frac{k}{e\jmath}\int_\Omega|v|^{p+\jmath}dx\le\frac{k}{e\jmath}C_{p+\jmath}^{p+\jmath}\|\nabla v\|_2^{p+\jmath} \\
&\le\frac{k}{e\jmath}C_{p+\jmath}^{p+\jmath}\left(\frac{2p}{(p-2)\ell}E(0)\right)^{\frac{p+\jmath-2}{2}}\|\nabla v\|_2^2.
\end{aligned} \tag{5.8}$$

For the seventh term, using Young's inequality, ($S_3$) and (5.3), for every $\varepsilon_1>0$, we have

$$\begin{aligned}
&\|\nabla v\|_2^{2\theta}\int_\Omega v v_t|v_t|^\alpha dx \\
&\quad\le\frac{\varepsilon_1^{\alpha+2}}{\alpha+2}\|\nabla v\|_2^{2\theta}\|v\|_{\alpha+2}^{\alpha+2}+\frac{\alpha+1}{\alpha+2}\varepsilon_1^{-\frac{\alpha+2}{\alpha+1}}\|\nabla v\|_2^{2\theta}\|v_t\|_{\alpha+2}^{\alpha+2} \\
&\quad\le\frac{\varepsilon_1^{\alpha+2}}{\alpha+2}C_{\alpha+2}^{\alpha+2}\|\nabla v\|_2^{\alpha+2\theta+2}+\frac{\alpha+1}{\alpha+2}\varepsilon_1^{-\frac{\alpha+2}{\alpha+1}}\|\nabla v\|_2^{2\theta}\|v_t\|_{\alpha+2}^{\alpha+2} \\
&\quad\le\frac{\varepsilon_1^{\alpha+2}}{\alpha+2}C_{\alpha+2}^{\alpha+2}\left(\frac{2p}{(p-2)\ell}E(0)\right)^{\frac{\alpha+2\theta}{2}}\|\nabla v\|_2^2 \\
&\qquad\qquad+\frac{\alpha+1}{\alpha+2}\varepsilon_1^{-\frac{\alpha+2}{\alpha+1}}\|\nabla v\|_2^{2\theta}\|v_t\|_{\alpha+2}^{\alpha+2}.
\end{aligned} \tag{5.9}$$

Similarly, for the last term on the right hand side of (5.9), for any $\varepsilon_2 > 0$, we have

$$\begin{aligned} \|\nabla v\|_2^{2\theta} \int_\Omega v v_t dx &\leq \frac{\varepsilon_2}{2} B^2 \|\nabla v\|_2^{2\theta+2} + \frac{1}{2\varepsilon_2} \|\nabla v\|_2^{2\theta} \|v_t\|_2^2 \\ &\leq \frac{\varepsilon_2}{2} B^2 \left( \frac{2p}{(p-2)\ell} E(0) \right)^\theta \|\nabla v\|_2^2 + \frac{1}{2\varepsilon_2} \|\nabla v\|_2^{2\theta} \|v_t\|_2^2, \end{aligned} \tag{5.10}$$

where $B$ is the optimal constant of Poincaré's inequality. Now, by choosing $0 < \varepsilon < \frac{\ell}{2}$ in (5.6) and using the inequalities (5.7) - (5.10), the following estimate can be found for (5.6):

$$\begin{aligned} \frac{d}{dt}(v, v_t) \leq & - E(t) + \frac{3}{2}\|v_t\|_2^2 + \left( \frac{-\ell}{4} + \frac{k}{p^2} C_p^p \left( \frac{2p}{(p-2)\ell} E(0) \right)^{\frac{p-2}{2}} \right. \\ & \left. + \frac{k(p-1)}{e\jmath p} C_{p+\jmath}^{p+\jmath} \left( \frac{2p}{(p-2)\ell} E(0) \right)^{\frac{p+\jmath-2}{2}} + \varepsilon_1^{\alpha+2}\Lambda_1 + \varepsilon_2 \Lambda_2 \right) \|\nabla v\|_2^2 \\ & + \left( \frac{1}{4\varepsilon} \|g\|_1 + \frac{1}{2} \right) (g \diamond \nabla v)(t) + \frac{k_1}{2\varepsilon_2} \|\nabla v\|_2^{2\theta} \|v_t\|_2^2 \\ & + \frac{k_0(\alpha+1)}{\alpha+2} \varepsilon_1^{-\frac{\alpha+2}{\alpha+1}} \|\nabla v\|_2^{2\theta} \|v_t\|_{\alpha+2}^{\alpha+2}, \end{aligned} \tag{5.11}$$

where

$$\Lambda_1 = \frac{k_0}{\alpha+2} C_{\alpha+2}^{\alpha+2} \left( \frac{2p}{(p-2)\ell} E(0) \right)^{\frac{\alpha+2\theta}{2}}, \qquad \Lambda_2 = \frac{k_1}{2} B^2 \left( \frac{2p}{(p-2)\ell} E(0) \right)^\theta.$$

Now, according to the condition ($S_2$), we have

$$-\frac{\ell}{8} + \frac{k}{p^2} C_p^p \left( \frac{2p}{(p-2)\ell} E(0) \right)^{\frac{p-2}{2}} < 0, \quad -\frac{\ell}{8} + \frac{k(p-1)}{e\jmath p} C_{p+\jmath}^{p+\jmath} \left( \frac{2p}{(p-2)\ell} E(0) \right)^{\frac{p+\jmath-2}{2}} < 0.$$

Therefore, $\varepsilon_1$ and $\varepsilon_2$ can be chosen so small such that

$$-\frac{\ell}{4} + \frac{k}{p^2} C_p^p \left( \frac{2p}{(p-2)\ell} E(0) \right)^{\frac{p-2}{2}} + \frac{k(p-1)}{e\jmath p} C_{p+\jmath}^{p+\jmath} \left( \frac{2p}{(p-2)\ell} E(0) \right)^{\frac{p+\jmath-2}{2}} + \varepsilon_1^{\alpha+2}\Lambda_1 + \varepsilon_2 \Lambda_2 < 0.$$

Hence, choosing $\beta_0 = \frac{3}{2}$, $\beta_1 = \frac{1}{4\varepsilon}\|g\|_1 + \frac{1}{2}$, $\beta_2 = \frac{k_1}{2\varepsilon_2}$ and $\beta_3 = \frac{k_0(\alpha+1)}{\alpha+2} \varepsilon_1^{-\frac{\alpha+2}{\alpha+1}}$ completes the proof. □

Now, we are in a position to state and proof our main stability result.

**Theorem 5.1.** *Suppose that the conditions* ($A_1$)-($A_3$) *and* ($S_1$)-($S_3$) *hold. Then, there exist the constants* $K > 0$ *and* $\kappa > 0$ *such that*

$$E(t) \leq K E(0) e^{-\kappa \int_0^t \zeta(s) ds}, \qquad \forall t \geq 0.$$

*Proof.* According to (5.3), there are positive constants such as $K_1$ and $K_2$ such that

$$K_1 \leq \|\nabla v(t)\|_2^{2\theta} \leq K_2, \qquad \forall t \geq 0. \tag{5.12}$$

For any $\eta > 0$, define

$$G_\eta(t) = E(t) + \eta \int_\Omega v v_t dx.$$

It is easy to see that for sufficiently small selection of $\eta$, there exist constants $L_1, L_2 > 0$ such that

$$L_1 E(t) \le G_\eta(t) \le L_2 E(t), \qquad \forall t \ge 0. \tag{5.13}$$

Then, by the Lemma 5.2, we have

$$\begin{aligned} G'_\eta(t) \le & E'(t) - \eta E(t) + \eta\beta_0 \|v_t\|_2^2 + \eta\beta_1 (g \diamond \nabla v)(t) \\ & + \eta\beta_2 \|\nabla v\|_2^{2\theta} \|v_t\|_2^2 + \eta\beta_3 \|\nabla v\|_2^{2\theta} \|v_t\|_{\alpha+2}^{\alpha+2}, \quad \forall t \ge 0. \end{aligned} \tag{5.14}$$

By using the Lemma 2.2 and (5.12), the inequality (5.14) can be written as

$$G'_\eta(t) \le \left(1 - \eta\left(\frac{\beta_0}{k_1 K_1} + \frac{\beta_2}{k_1} + \frac{\beta_3}{k_0}\right)\right) E'(t) - \eta E(t) + \eta\beta_1 (g \diamond \nabla v)(t), \quad \forall t \ge 0. \tag{5.15}$$

In (5.15), we choose $\eta$ such that $1 - \eta\left(\frac{\beta_0}{k_1 K_1} + \frac{\beta_2}{k_1} + \frac{\beta_3}{k_0}\right) > 0$. So, we get

$$G'_\eta(t) \le -\eta E(t) + \eta\beta_1 (g \diamond \nabla v)(t), \quad \forall t \ge 0. \tag{5.16}$$

Multiplying $\zeta$ on the both sides of (5.16) and using ($\mathrm{S}_1$) and taking the Lemma 2.2 into account, we have

$$\begin{aligned} \zeta(t) G'_\eta(t) &\le -\eta\zeta(t) E(t) + \eta\beta_1 \zeta(t) (g \diamond \nabla v)(t) \\ &\le -\eta\zeta(t) E(t) - \eta\beta_1 (g' \diamond \nabla v)(t) \\ &\le -\eta\zeta(t) E(t) - 2\eta\beta_1 E'(t), \quad \forall t \ge 0. \end{aligned} \tag{5.17}$$

The inequality (5.17) can also be rewritten as

$$\mathcal{L}'(t) \le \zeta'(t) G_\eta(t) - \eta\zeta(t) E(t), \qquad \forall t \ge 0, \tag{5.18}$$

where

$$\mathcal{L}(t) = \zeta(t) G_\eta(t) + 2\eta\beta_1 E(t). \tag{5.19}$$

Considering (5.13) and the fact that $\zeta(t) \le \zeta(0)$, we have

$$\mathcal{L}(t) \le \big(L_2 \zeta(0) + 2\eta\beta_1\big) E(t). \tag{5.20}$$

Therefore, since $\zeta'(t) \le 0$, by (5.20), from the inequality (5.18), we get

$$\mathcal{L}'(t) \le -\frac{\eta\zeta(t)}{L_2\zeta(0) + 2\eta\beta_1} \mathcal{L}(t). \tag{5.21}$$

By integrating the sides of (5.21) over $(0, t)$, we obtain

$$\mathcal{L}(t) \le \mathcal{L}(0)\ \exp\left(-\frac{\eta}{L_2\zeta(0) + 2\eta\beta_1} \int_0^t \zeta(s) ds\right),$$

Consequently,

$$E(t) \le \frac{\big(L_2\zeta(0) + 2\eta\beta_1\big) E(0)}{2\eta\beta_1}\ \exp\left(-\frac{\eta}{L_2\zeta(0) + 2\eta\beta_1} \int_0^t \zeta(s) ds\right), \quad \forall t \ge 0.$$

## 6 Conclusion

In this work, an investigation has been carried out on the non-existence and asymptotic stability of global solutions to the problem (1.1) - (1.3). By imposing appropriate conditions on the parameters, initial data and the positivity of the initial energy of the model, it has been demonstrated that how the solutions blow up at a finite time. Upon examining the blow-up characteristics of the solutions, the obtained results reveal the predominance of the logarithmic nonlinear source effect over the non-local nonlinear damping effect, under the assumptions ($L_3$) and ($H_1$) which are crucial conditions. Furthermore, some estimates obtained for the upper and lower bounds of the blow-up time. Also, in this work, the stability of the global solutions has been established by determining the general decay rates of the solutions, which are proportional to the decay rates of the function $g$, without interactions between the nonlinearities. The results obtained in this study improve and generalize some recent studies in the literature. For example, in [50] the problem (1.1) - (1.3) considered with the non-local linear damping effect and in the absence of the memory kernel $g$, that is, an equation in the form

$$v_{tt} - \Delta v + \sigma(\|\nabla v\|_2^2)v_t = |v|^{p-2}v \ln |v|^k.$$

The authors studied blow-up properties of solutions when $E(0) < 0$. As another example, we can refer to [20], where the considered nonlinear source has a polynomial structure, and the existence of blowing up solutions has been studied only in the case that the model under consideration is affected by the weak non-local linear damping effect.

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