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On the well-posedness and stability analysis of standing waves for a 1D-Benney-Roskes system

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Abstract. In this paper, we revisit the well-posedness for the Benney-Roskes system (also known as Zakharov-Rubenchik systems) for $N = 1, 2, 3$, and establish the nonlinear orbital stability of ground state standing waves in the case $N = 1$, by using the variational approach induced by the Hamiltonian structure and the Liapunov method.

Keywords. Well-posedness, standing waves, orbital stability

1 Introduction

In this work we consider the following system of differential equations

$$\left\{\begin{array}{ll} i\partial_t\psi + \epsilon\partial_z^2\psi & = -\sigma_1\Delta_\perp\psi + \left(\sigma|\psi|^2 + W(\rho + D\partial_z\varphi)\right)\psi, \\ \partial_t\rho + \sigma_2\partial_z\rho & = -\Delta_\perp\varphi - \partial_z^2\varphi - D\partial_z(|\psi|^2), \\ \partial_t\varphi + \sigma_2\partial_z\varphi & = -\frac{1}{M^2}\rho - |\psi|^2, \end{array}\right. \tag{1.1}$$

which describes the interaction of high-frequency and low-frequency waves in plasmas and magnetohydrodynamics, where we are using the notation $\mathbf{x} = (x, y, z)$ for $N = 3$, $\mathbf{x} = (x, z)$ for $N = 2$, $\Delta_\perp = \partial_x^2 + \partial_y^2$ for $N = 3$, and $\Delta_\perp = \partial_x^2$ for $N = 2$. The model was first derived for D. Benney and G. Roskes in the context of gravity waves [1] and also for A. Rubenchik and V. Zakharov in the context of the interaction of spectral narrow high frequency wave packet of small amplitude with low-frequency acoustic type oscillations [19]. The system of differential equations (1.1) is written in nondimensional form according to the parameters and rescaling used by T. Passot, C. Sulem and P. Sulem [17], G. Ponce and J. Saut [18], J. Ghidaglia and J. Saut [6], and J. Cordero [4], after considering a reference frame moving with the group velocity. In the Benney-Roskes system, the function $\psi = \psi(\mathbf{x}, t) \in \mathbb{C}$ denotes the complex amplitude of the high frequency, $\rho = \rho(\mathbf{x}, t) \in \mathbb{R}$ denotes the density fluctuation and $\varphi = \varphi(\mathbf{x}, t) \in \mathbb{R}$ is the hydrodynamic potential. The parameter σ measures the self-interaction of the carrying wave, D is a proportional constant to the Doppler shift α, ϵ denotes the constant dispersion, $W = \frac{c\beta^2}{v_g^2} > 0$, and $M = \frac{|v_g|}{c_s} > 0$ is the Mach number due to the group velocity v_g (only in the direction of the z-axis) and c_s is the sound velocity. Constants σ_1, σ_2 ($\sigma_2^2 = 1$) are parameters depending on the

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group velocity. For more details on the physical background and a complete description of the system, we refer to the following works [19, 10, 11].

Regarding the well posedness problem in the case $N = 2, 3$ and $s > \frac{N}{2}$, G. Ponce and J-C. Saut in [18] established a local well posedness results for the system (1.1) in the space $H^s(\mathbb{R}^N) \times H^{s-\frac{1}{2}}(\mathbb{R}^N) \times H^{s+\frac{1}{2}}(\mathbb{R}^N)$, after reducing the system to a nonlinear Schrödinger equation coupled with two wave equations and using smoothing effects associated to the Schrödinger group. They also obtained existence of a global weak solution for initial data in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, by using that the Hamiltonian and the charge are conserved quantities on solutions for the system (1.1) and the positiveness of a special quadratic form. It is important to point out that G. Ponce and J. C. Saut obtained only weak global solutions because their energy space included the norm $||\varphi||_{H^1(\mathbb{R}^N)}$, but the conserved quantities control only $||\nabla\varphi||_{L^2(\mathbb{R}^N)}$. Local well-posedness of the Benney-Roskes (Zakharov-Rubenchik) system was also obtained by C. Obrecht in [15], for $s > 2$ in the elliptic case ($\epsilon\sigma_1 > 0$), using an energy method as done by H. Schochet and M. Weinstein in [20] on the nonlinear Schrödinger limit of the Zakharov system. In this previous two works, the Benney-Roskes system is rewritten as a dispersive perturbation of a symmetric nonlinear hyperbolic system. On the other hand, H. Luong, N. Mauser and J. C. Saut in [13] used the Schochet-Weinstein method to prove a local existence for the Benney-Roskes (Zakharov-Rubenchik) system, keeping a small parameter which is relevant for deep water waves. They also studied the Cauchy problem in the background of a line solitary wave, in order to establish the transverse stability/instability of the one-dimensional solitary wave (line solitary).

On the other hand, in the case $N = 1$, F. Linares and C. Matheus [12] and F. Oliveira [16] established well-posedness for a modified system in the variable (ψ, ρ, u) where $u = \varphi_x$. In Oliveira's work, the global well posedness result for the modified system was obtained in the space $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$, which is only contained in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$, and in F. Linares and C. Matheus work, the local well posedness result was obtained in the space $H^k(\mathbb{R}) \times H^r(\mathbb{R}) \times H^s(\mathbb{R})$, where

$$-\frac{1}{2} < k - l \leq 1, \quad 0 \leq l + \frac{1}{2} \leq 2k, \ \frac{1}{2} < k - s \leq 1, \quad 0 \leq s + \frac{1}{2} \leq 2k.$$

Moreover, F. Linares and C. Matheus in [12] obtained a global well posedness result for the modified system in the space $H^k(\mathbb{R}) \times H^r(\mathbb{R}) \times H^l(\mathbb{R})$, where $0 \leq k = l + \frac{1}{2}$, which includes the energy space. It is important to mention that those two results are not applicable to our system for $N = 1$ in the sense that the last component u should have the mean zero property to make possible to recover the original φ as $\varphi = \partial_x^{-1} u$, something that was no included in the functional space for well posedness.

Regarding the stability of standing waves for the system (1.1) in the case $N = 2, 3$, J. Cordero and J. Quintero in [5] established the instability of ground state standing waves of the form $(e^{i\omega t}u(\mathbf{x}, t), \rho(\mathbf{x}), \varphi(\mathbf{x}))$ for the Benney-Roskes system. On the other hand, in the case of the stability of standing waves for the modified system (1.1) ($u = \varphi_x$) for $N = 1$, F. Oliveira showed the orbital stability of the standing wave solutions for the modified system using the method of Liapunov, but in a smaller than the energy space. In this case, the stability result seems to be incomplete, since the norm N_E used to measure the deviation of a solution from the orbit of a standing wave only controls the first of three components.

In this work, we establish the existence of global solutions for the system (1.1) using the appropriate energy space dictated by the Hamiltonian energy. In contrast with the existence results given by F. Oliveira for $N = 1$ (for the modified system) and G. Ponce and J-C. Saut for

$N = 2, 3$, we did not modify the Benney-Roskes system by taking the t-derivative in the last two nonlinear transport equations, reducing the system to a nonlinear Schrödinger equation coupled with two wave equations. Instated of this approach, we derive the whole group for the system and without modifying the variables, we proceed to analyze the well-posedness of the Cauchy problem. In the case $N = 1$, we prove that the standing waves of the Benney-Roskes system are orbitally stable, by using the Lyapunov method, as done by M. Weinstein in [23] in the case of the nonlinear Schrödinger equation (NLS) and the generalization of the Kortewegde Vries equation (GKdV).

As happens for the NLS, the Benney-Roskes system has phase and translation symmetries, meaning that if $\Psi = (\psi(t,\cdot), \rho(t,\cdot), \varphi(t,\cdot))$ is solution, so is $\Psi_{\gamma,x_0}(t,\cdot) = (e^{\gamma}\psi(t,\cdot + x_0), \rho(t,\cdot + x_0), \varphi(t,\cdot + x_0))$, for any x_0 and $\gamma \in [0, 2\pi)$. Due to this fact, orbital stability means stability modulo these symmetries. In order to be more precise, we define the orbit of a function χ as

$$\mathcal{G}_\chi = \{\chi_{\gamma,x_0} : \ (x_0, \gamma) \in \mathbb{R} \times [0, 2\pi)\} . \tag{1.2}$$

In this context, a ground state is orbitally stable, if for an initial data being near the ground state orbit, the solution remains near the ground state orbit at all later times. In order to measure the deviation of the solution $\Phi(t,\cdot)$ from the orbit $\mathcal{G}_\Psi$, we use the following metric:

$$d_E(\Phi(t,\cdot), \mathcal{G}_\Psi) = \inf \mathcal{N}_E(\Phi_{\gamma,x_0}(t,\cdot), \Psi) \tag{1.3}$$

where the infimum is taken over all $x_0 \in \mathbb{R}$ and $\gamma \in [0, 2\pi)$ and a metric $\mathcal{N}_E$ defined in the space $X_{1,\frac{1}{2}}$.

This paper is organized as follows. In section 2, we revisit the well posedness associated with the Benney-Roskes system (1.1) analyzing the complete group in the energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times H_1(\mathbb{R}^N)$, where the (quotient) space $H_s(\mathbb{R}^N)$ is given by

$$H_s(\mathbb{R}^N) = \left\{ f \in \mathcal{S}'(\mathbb{R}^N) : ||f||_{H_s(\mathbb{R}^N)} = ||\nabla_{\mathbf{x}} f||_{H^{s-1}(\mathbb{R}^N)} < +\infty \right\},$$

identifying the constant functions with the zero function. In section 3, we present some preliminaries related with the existence of standing waves for the system (1.1) of the form

$$\Psi_{\omega,c}(x,t) = \left(e^{i\omega t} e^{\frac{ic}{2\epsilon}(x-ct)} u(x-ct), -M^2 u^2(x-ct), 0 \right),$$

where u is a real positive function. Following M. Weinstein approach in [23], we study the deviation from a solution $\Phi(t,\cdot)$ and $\Psi_{\omega,c}$ using an appropriate action functional $\mathcal{F}_{\omega,c}$ which is conserved in time. Using that the minimum in (1.3) is attained, this defines $x_0(t)$ and $\gamma_0(t)$, choice which is clever in the analysis. The stability is based on a suitable lower bound on the second variation of the energy functional $\mathcal{F}_{\omega,c}$.

2 Local and global well posedness

In this section, we describe completely the group $(T(t))_{t\in\mathbb{R}}$ associated with the linear part of the Benney-Roskes system (1.1). We recall the Benney-Roskes system (1.1) can be written for $\Psi = (\psi, \rho, \varphi)^t$ as

$$\partial_t \Psi = \mathcal{B}\Psi + \mathcal{C}(\Psi),$$

where the operator $\mathcal{B}$ is given by

$$\mathcal{B} = \begin{pmatrix} i\epsilon\partial_z^2 + i\sigma_1\Delta_\perp & 0 & 0 \\ 0 & -\sigma_2\partial_z & -\Delta_\perp - \partial_z^2 \\ 0 & -\frac{1}{M^2} & -\sigma_2\partial_z \end{pmatrix}$$

with domain $Dom(\mathcal{B}) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times H_2(\mathbb{R}^N) \subset H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times H_1(\mathbb{R}^N)$, and the nonlinear term $\mathcal{C}$ is given by

$$\mathcal{C}(\Psi) = \begin{pmatrix} -i(\sigma|\psi|^2\psi + W\rho\psi + WD\partial_z\varphi\psi) \\ -D\partial_z(|\psi|^2) \\ -|\psi|^2 \end{pmatrix}. \tag{2.1}$$

We start the discussion by describing the group associated with the linear system

$$\partial_t\Psi = \mathcal{B}\Psi. \tag{2.2}$$

We note that the system (2.2) can be uncoupled in a linear like Schrödinger equation and a 2×2 linear system. By taking the Fourier transform in the last two equations, we get that

$$\partial_t \begin{pmatrix} \widehat{\rho} \\ \widehat{\varphi} \end{pmatrix} = \begin{pmatrix} -i\sigma_2\xi_N & |\xi|^2 \\ -\frac{1}{M^2} & -i\sigma_2\xi_N \end{pmatrix} \begin{pmatrix} \widehat{\rho} \\ \widehat{\varphi} \end{pmatrix} := A(\xi) \begin{pmatrix} \widehat{\rho} \\ \widehat{\varphi} \end{pmatrix},$$

where we are using the following notation: $\xi = (\xi_1, \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, and $|\xi|^2 = |\xi_1|^2 + |\xi_N|^2$ (for $N = 1$, we use $\xi_N = \xi$). On the other hand, we know that if A is a 2×2 matrix of the form

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

then its eigenvalues are distinct for $a \neq 0$ and that the exponential of A has the form

$$e^A = e^a \begin{pmatrix} \cosh(\sqrt{bc}) & b\frac{\sinh(\sqrt{bc})}{\sqrt{bc}} \\ c\frac{\sinh(\sqrt{bc})}{\sqrt{bc}} & \cosh(\sqrt{bc}) \end{pmatrix}.$$

In particular, we also have that

$$e^{tA} = e^{ta} \begin{pmatrix} \cosh(t\sqrt{bc}) & b\frac{\sinh(t\sqrt{bc})}{\sqrt{bc}} \\ c\frac{\sinh(t\sqrt{bc})}{\sqrt{bc}} & \cosh(t\sqrt{bc}) \end{pmatrix}.$$

In our case, we have that

$$a = -i\sigma_2\xi_N, \quad b = |\xi|^2, \quad c = -\frac{1}{M^2}, \quad \sqrt{bc} = i\frac{|\xi|}{M}.$$

So, replacing these into the exponential matrix and using that

$$\cosh(i\alpha) = \cos(\alpha), \quad \sinh(i\alpha) = i\sin(\alpha),$$

we get that

$$e^{tA(\xi)} = e^{-i\sigma_2\xi_N t} \begin{pmatrix} \cos\left(\frac{|\xi|}{M}t\right) & M|\xi|\sin\left(\frac{|\xi|}{M}t\right) \\ -\frac{\sin\left(\frac{|\xi|}{M}t\right)}{M|\xi|} & \cos\left(\frac{|\xi|}{M}t\right) \end{pmatrix}.$$

Before we go further, we use the notation $\nabla_{\mathbf{x}} = (\partial_x, \partial_y, \partial_z)^t$ for $N = 3$, $\nabla_{\mathbf{x}} = (\partial_x, \partial_z)^t$ for $N = 2$ and $\nabla_{\mathbf{x}} = \partial_z$ for $N = 1$. Now, for $r, s \in \mathbb{R}$, we set the space

$$X_{r,s} := H^r(\mathbb{R}^N) \times H^{s-\frac{1}{2}}(\mathbb{R}^N) \times H_{s+\frac{1}{2}}(\mathbb{R}^N).$$

We see that the group $(T(t))_{t\in\mathbb{R}}$ defined in $X_{r,s}$ associated with the linear system (2.2) has the form

$$T(t) = \mathcal{F}^{-1} \begin{pmatrix} e^{-i(\epsilon|\xi_N|^2+\sigma_1|\xi_1|^2)t} & 0 \\ 0 & e^{tA(\xi)} \end{pmatrix} \mathcal{F}, \tag{2.3}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and its inverse, respectively.

Hereafter for a generic space H, we say that a triplet $\Psi = (\psi, \rho, \varphi)^t \in C^0(\mathbb{R}_t; H)$ is a mild solution of the system (1.1) with initial data $\Psi_0 = (\psi_0, \rho_0, \varphi_0)^t \in H$, if Ψ satisfies the fixed point integral equation $\mathcal{A}(\Psi) = \Psi$, where the functional operator $\mathcal{A}$ is given by

$$\mathcal{A}(\Psi)(t) = T(t)\Psi_0 + \int_0^t T(t-y)\mathcal{C}(\Psi)(y)\, dy. \tag{2.4}$$

Now, for a given $\Psi = (\psi, \rho, \varphi)^t$, we set

$$Q_1(t)(\widehat{\Psi})(\xi) = e^{-i(\epsilon|\xi_N|^2+\sigma_1|\xi_1|^2)t}\widehat{\psi}(\xi), \tag{2.5}$$

$$Q_2(t)(\widehat{\Psi})(\xi) = e^{-i\sigma_2\xi_N t} \cos\left(\frac{|\xi|}{M}t\right) \widehat{\rho}(\xi) + Me^{-i\sigma_2\xi_N t}|\xi| \sin\left(\frac{|\xi|}{M}t\right) \widehat{\varphi}(\xi), \tag{2.6}$$

$$Q_3(t)(\widehat{\Psi})(\xi) = -e^{-i\sigma_2\xi_N t} \frac{\sin\left(\frac{|\xi|}{M}t\right)}{M|\xi|} \widehat{\rho}(\xi) + e^{-i\sigma_2\xi_N t} \cos\left(\frac{|\xi|}{M}t\right) \widehat{\varphi}(\xi). \tag{2.7}$$

From this notation, we see that

$$T(t)(\Psi) = \left(\mathcal{F}^{-1}\left(Q_1(t)(\widehat{\Psi})\right), \mathcal{F}^{-1}\left(Q_2(t)(\widehat{\Psi})\right), \mathcal{F}^{-1}\left(Q_3(t)(\widehat{\Psi})\right)\right)^t.$$

Now, if we set $\Phi(\Psi) = (\Theta_1, \Theta_2, \Theta_3)^t$ and set $U_1(t)$ and $U_2(t)$ in terms of the Fourier transform as

$$\widehat{U_1(t)f}(\xi) = \widehat{e^{i\Delta_{\epsilon,\sigma_1}t}f}(\xi) = e^{-i(\epsilon|\xi_N|^2+\sigma_1|\xi_1|^2)t}\widehat{f}(\xi), \ (\Delta_{\epsilon,\sigma_1} = \epsilon\partial_z^2 + \sigma_1\Delta_\perp)$$

$$\widehat{U_2(t)f}(\xi) = \frac{M\sin\left(\frac{|\xi|}{M}t\right)}{|\xi|}\widehat{f}(\xi),$$

then we see directly that

$$\begin{aligned}
\Theta_1(t) &= U_1(t)\psi_0 - i\int_0^t U_1(t-s)L(\psi,\rho,\varphi)(s)\, ds \\
\Theta_2(t) &= U_2'(t)\rho_0(\cdot - (0,\sigma_2 t)) - U_2(t)(\vec{\nabla}_{\mathbf{x}} \cdot \nabla_{\mathbf{x}}\varphi_0)(\cdot - (0,\sigma_2 t)) \\
&\quad - \int_0^t \left(DU_2'(t-s)\partial_z(|\psi|^2) - U_2(t-s)(\vec{\nabla}_{\mathbf{x}} \cdot \nabla_{\mathbf{x}}|\psi|^2)\right)(\cdot - (0,\sigma_2(t-s)), s)\, ds, \\
\Theta_3(t) &= -U_2(t)\rho_0(\cdot - (0,\sigma_2 t)) + U_2'(t)\varphi_0(\cdot - (0,\sigma_2 t)) \\
&\quad + \int_0^t \left(DU_2(t-s)\partial_z(|\psi|^2) - U_2'(t-s)(|\psi|^2)\right)(\cdot - (0,\sigma_2(t-s)), s)\, ds,
\end{aligned}$$

where are using that $\widehat{f(\cdot-(0,r))}(\xi)=e^{-ir\xi_N}\hat{f}(\xi)$ and that L is given by

$$L(\psi,\rho,\varphi)=\sigma|\psi|^2\psi+W\rho\psi+WD\partial_z\varphi\psi.$$

Now, a direct computation shows that

$$\begin{aligned}||U_2(t)f||_2&\le|t|||f||_2,\\ ||\nabla_{\mathbf{x}}U_2(t)f||_2&\le M||f||_2,\\ ||U_2'(t)f||_2&\le||f||_2,\end{aligned}$$

where $||\cdot||_2=||\cdot||_{L^2}$. The first result is related with the group estimate,

Lemma 2.1. *Let $r,s\in\mathbb{R}$ and $t\in\mathbb{R}$ be given, then $T(t)$ is a linear bounded operator from $X_{r,s}$ to $X_{r,s}$. Moreover, there is $K>0$ (independent of t) such that we have*

$$\|T(t)\Psi\|_{X_{r,s}}\le K\|\Psi\|_{X_{r,s}}.$$

Proof. Let $r\in\mathbb{R}$. Then we see that

$$\left\|\mathcal{F}^{-1}(Q_1(t)(\widehat{\Psi}))\right\|_{H^r}^2=\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^r|\widehat{\psi}(\xi)|^2d\xi\le K_1\|\Psi\|_{X_{r,s}}^2.$$

On the other hand, we also have that

$$\begin{aligned}\left\|\mathcal{F}^{-1}(Q_2(t)(\widehat{\Psi}))\right\|_{H^{s-\frac12}}^2&\le K_2(M)\int_{\mathbb{R}^N}(1+|\xi|^2)^{s-\frac12}\left(|\widehat{\rho}(\xi)|^2+|\xi|^2|\widehat{\varphi}(\xi)|^2\right)d\xi\\ &\le K_2(M)\left(\|\rho\|_{H_{s-\frac12}}^2+\|\varphi\|_{H_{s+\frac12}}^2\right)\\ &\le K_2(M)\|\Psi\|_{X_{r,s}}^2.\end{aligned}$$

In a similar fashion, we have that

$$\left\|\nabla_{\mathbf{x}}\mathcal{F}^{-1}(Q_3(t)(\widehat{U}))\right\|_{H^{s-\frac12}}\le K_3(M)\|\Psi\|_{X_{r,s}}.$$

□

In order to perform the computations for the group $(T(t))_{t\ge0}$, we introduce the following notation (see Constantin [3], Ponce-Saut [18] and Ghidaglia-Saut [9]):

$$||f||_{l_\mu^\infty L_T^2L_{\mathbf{x}}^2}=\sup_{\mu\in\mathbb{Z}^N}\left(\int_{Q_\mu\times[0,T]}|f(\mathbf{x},t)|^2\,d\mathbf{x}\,dt\right)^{\frac12},\tag{2.8}$$

$$||f||_{l_\mu^1 L_T^2L_{\mathbf{x}}^2}=\sum_{\mu\in\mathbb{Z}^N}\left(\int_{Q_\mu\times[0,T]}|f(\mathbf{x},t)|^2\,d\mathbf{x}\,dt\right)^{\frac12},\tag{2.9}$$

$$||f||_{l_\mu^2 L_T^2L_{\mathbf{x}}^2}=\left(\sum_{\mu\in\mathbb{Z}^N}\int_{Q_\mu\times[0,T]}|f(\mathbf{x},t)|^2\,d\mathbf{x}\,dt\right)^{\frac12},\tag{2.10}$$

$$||f||_{l_\mu^2 L_T^1L_{\mathbf{x}}^2}=\left(\sum_{\mu\in\mathbb{Z}^N}\left(\int_0^T\left(\int_{Q_\mu}|f(\mathbf{x},t)|^2\,d\mathbf{x}\right)^{\frac12}dt\right)^2\right)^{\frac12},\tag{2.11}$$

$$||f||_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} = \left(\sum_{\mu \in \mathbb{Z}^N} \sup_{t\in[0,T]} \int_{Q_\mu} |f(\mathbf{x},t)|^2 \, d\mathbf{x} \right)^{\frac{1}{2}}. \tag{2.12}$$

where $(Q_\mu)_\mu$ is a family of unit cubes parallel to the coordinates axis with disjoint interiors covering $\mathbb{R}^N$.

From works by P. Constantin [3], P. Sölin [21], T. Kato [7], L. Vega [22], C. Kenig, G. Ponce and L. Vega [8]-[9], we have the following estimates

$$||I^{\frac{1}{2}}_{\mathbf{x}} U_1(t) f||_{l^\infty_\mu L^2_T L^2_{\mathbf{x}}} \le C ||f||_2, \tag{2.13}$$

$$\sup_{0\le t\le T} \left\| I^{\frac{1}{2}}_{\mathbf{x}} \int_0^t U_1(t-s) f(s)\, ds \right\|_2 \le C ||f||_{l^1_\mu L^2_T L^2_{\mathbf{x}}}, \tag{2.14}$$

$$\left\| \nabla_{\mathbf{x}} \int_0^t U_1(t-s) f(s)\, ds \right\|_{l^\infty_\mu L^2_T L^2_x} \le C ||f||_{l^1_\mu L^2_T L^2_{\mathbf{x}}}, \tag{2.15}$$

$$\sup_{0\le t\le T} \left\| J^{s+\frac{1}{2}} \int_0^t U_1(t-s) f(s)\, ds \right\|_2 \le C |||J^s f||_{l^1_\mu L^2_T L^2_{\mathbf{x}}}, \tag{2.16}$$

where $\widehat{I^{\frac{1}{2}}_{\mathbf{x}} f} = |\xi|^{\frac{1}{2}} \widehat{f}$, $\widehat{J} = (1+|\xi|^2)^{\frac{1}{2}}$ and C is a positive constant independent of T.

Hereafter, K is a generic constant independent of functions or the time variable, and so it could be up dated at any step. From these estimates, we have that

Lemma 2.2. *Let $N = 2, 3$ and $r > \frac{N}{2}$. For $\Psi = (\psi, \rho, \varphi)^t \in C([0,T]; X_r)$ with $|||\Psi|||_T < \infty$, we have that*

$$\left\| \int_0^t U_1(t-s) L(\Psi)(s)\, ds \right\|_{H^r} \le KT \left(|||\Psi(t)|||_T^2 + |||\Psi(t)|||_T^3 \right), \tag{2.17}$$

$$\left\| J^{r+\frac{1}{2}} \int_0^t U_1(t-s) L(\Psi)(s)\, ds \right\|_{l^\infty_\mu L^2_T L^2_x} \le KT \left(|||\Psi(t)|||_T^2 + |||\Psi(t)|||_T^3 \right), \tag{2.18}$$

where $X_r := X_{r,r}$ and

$$|||\Psi|||_T := \sup_{0\le t\le T} \|\Psi(t)\|_{X_r} + ||J^{r+\frac{1}{2}}\psi||_{l^\infty_\mu L^2_T L^2_{\mathbf{x}}}.$$

Proof. The first remark is that

$$||J^{r+\frac{1}{2}}\psi||_{l^\infty_\mu L^2_T L^2_{\mathbf{x}}} \sim \sum_{|\alpha|=r+\frac{1}{2}} ||\partial^\alpha_x \psi||_{l^\infty_\mu L^2_T L^2_{\mathbf{x}}},$$

for $r+\frac{1}{2} \in \mathbb{N}$. To simplify the computations, we only consider the case $N = 3$ and $r = 2+\frac{1}{2}$. As pointed out in Ponce-Saut's paper, the general case follows by inequalities involving fractional derivatives. Now, assuming the proper conditions on f and g, a direct computation shows that

$$||fg||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} \le \sup_{0\le t\le T} ||f(t)||^{\frac{1}{2}}_{H^1} \sup_{0\le t\le T} ||g(t)||^{\frac{1}{2}}_{H^1} ||f||^{\frac{1}{2}}_{l^2_\mu L^2_T L^2_{\mathbf{x}}} ||g||^{\frac{1}{2}}_{l^2_\mu L^2_T L^2_{\mathbf{x}}}$$

Now, for $\eta \in C([0,T]; H^l)$ we have that

$$||\partial^l_j \eta||_{l^2_\mu L^2_T L^2_{\mathbf{x}}} \le KT^{\frac{1}{2}} \sup_{0\le t\le T} ||\eta(t)||_{H^l}, \tag{2.19}$$

From previous estimates, for $\eta \in C([0,T];H^l)$ and $\rho \in C([0,T];H^m)$ we have that

$$||\partial_j^m \rho \partial_k^l \eta||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} \leq KT \sup_{0\leq t\leq T} ||\rho(t)||_{H^m} \sup_{0\leq t\leq T} ||\eta(t)||_{H^l}.$$

On the other hand, we have that

$$\begin{aligned} ||J^{r-\frac{1}{2}}(\eta)\psi||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} &\leq KT \sup_{0\leq t\leq T} ||\eta||_{H^{r-\frac{1}{2}}} \sup_{0\leq t\leq T} ||\psi||_2 \\ ||\eta J^{r-\frac{1}{2}}(\psi)||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} &\leq KT \sup_{0\leq t\leq T} ||\eta(t)||_2 \sup_{0\leq t\leq T} ||\psi(t)||_{H^{r-\frac{1}{2}}} \\ \sum_{0<|l|+|m|<r-\frac{1}{2}} ||\partial^l \eta \partial^m \psi||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} &\leq KT \sup_{0\leq t\leq T} ||\eta(t)||_{H^l} \sup_{0\leq t\leq T} ||\psi(t)||_{H^m}) \end{aligned}$$

From these facts and (2.14), we see in the case $r - \frac{1}{2} \in \mathbb{N}$ (the other case can be treated using interpolation) that

$$\begin{aligned} \left\| \int_0^t U_1(t-s)\eta(s)\psi(s)\,ds \right\|_{H^r} &= \left\| J^r \int_0^t U_1(t-s)\eta(s)\psi(s)\,ds \right\|_2 \\ &\leq C||J^{r-\frac{1}{2}}(\eta\psi)||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} \\ &\leq ||J^{r-\frac{1}{2}}(\eta)\psi||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} + ||\eta J^{r-\frac{1}{2}}(\psi)||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} \\ &+ \sum_{0<|l|+|m|<r-\frac{1}{2}} ||\partial^l \eta \partial^m \psi||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} \\ &\leq KT \sup_{0\leq t\leq T} ||\eta(t)||_{H^{r-\frac{1}{2}}} \sup_{0\leq t\leq T} ||\psi(t)||_{H^r}. \end{aligned}$$

Using this estimate, we have that

$$\begin{aligned} \left\| \int_0^t U_1(t-s)L(\Psi)(s)\,ds \right\|_{H^r} &\leq KT \Big(\sup_{0\leq t\leq T} \|\psi(t)\|^3_{H^r} + \sup_{0\leq t\leq T} ||\rho(t)||_{H^{r-\frac{1}{2}}} \times \\ &\sup_{0\leq t\leq T} ||\psi||_{H^r} + \sup_{0\leq t\leq T} ||\partial_z \varphi(t)||_{H^{r-\frac{1}{2}}} \sup_{0\leq t\leq T} ||\psi(t)||_{H^r} \Big) \\ &\leq KT \left(|||\Psi(t)|||^2_T + |||\Psi(t)|||^3_T \right). \end{aligned}$$

On the other hand, using the estimate (2.15) and following similar estimates, we have that

$$\begin{aligned} \left\| J^{r+\frac{1}{2}} \int_0^t U_1(t-s)L(\Psi)(s)\,ds \right\|_{l^\infty_\mu L^2_T L^2_x} &\leq ||J^{r-\frac{1}{2}} L(\Psi)||_{l^1_\mu L^2_T L^2_{\mathbf{x}}} \\ &\leq KT \left(|||\Psi(t)|||^2_T + |||\Psi(t)|||^3_T \right). \end{aligned} \tag{2.20}$$

□

In order to finish the nonlinear estimates, we need to establish analogous estimates to those obtained by G. Ponce and J. C. Saut in Lemma 2.3 of [18] for $N = 3$. We point out that Lemma 2.3 in [18] holds for $N = 2, 3$ with the power $\frac{N}{2}$ instead of 3. We obtain such estimates by adapting Lemma 2.3 in [18] in order to include the effect of the transport group.

Theorem 2.1. *Let $N = 2, 3$. Then we have the following estimates:*

$$||U_2'(t)f(\cdot + (0, -\sigma_2 t)||_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} \leq K(1+T)^N ||f||_2, \tag{2.21}$$

$$||U_2(t)f(\cdot+(0,-\sigma_2 t)||_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} \le KT(1+T)^N ||f||_2, \tag{2.22}$$

$$||U_2(t)\partial_j f(\cdot+(0,-\sigma_2 t)||_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} \le K(1+T)^N ||f||_2, \tag{2.23}$$

$$\left\|\nabla_{\mathbf{x}}\int_0^t U_2(t-s)f(\cdot+(0,-\sigma_2(t-s)),s)\,ds\right\|_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} \le K(1+T)^N ||f||_{l^2_\mu L^1_T L^2_{\mathbf{x}}}, \tag{2.24}$$

$$\left\|\int_0^t U_2'(t-s)f(\cdot+(0,-\sigma_2(t-s)),s)\,ds\right\|_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} \le K(1+T)^N ||f||_{l^2_\mu L^1_T L^2_{\mathbf{x}}}. \tag{2.25}$$

Proof. We first establish the estimate (2.21). We note that

$$\begin{aligned}\int_{Q_\mu} |U_2'(t)f(\mathbf{x}+(0,-\sigma_2 t))|^2\,d\mathbf{x} &\le \int_{Q_\mu+(0,\sigma_2 t)} |U_2'(t)f(y)|^2\,dy\\ &\le \sum_\nu \int_{Q_\nu} |U_2'(t)f(y)|^2\,dy,\end{aligned}$$

where for each $\mu>0$, the sum has $N(T)$ summands and $\bigcup_\nu Q_\nu \subset (1+|\sigma_2|T)Q_\mu$ with βQ_μ denoting a cube with the same center of Q_μ and size $\beta>0$. Then, adding up on μ, we conclude that

$$\begin{aligned}\|U_2'(t)f(\cdot+(0,-\sigma_2 t))\|_{l^2_\mu L^\infty_T L^2_x} &\le C(1+T)^{\frac{N}{2}}\|U_2'(t)f\|_{l^2_\mu L^\infty_T L^2_x}\\ &\le C(1+T)^N\|f\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

where we are using estimate (2-15) in [18] to get the last conclusion. In a similar fashion, we get estimates (2.22) and (2.23). On the other hand, to get the estimate (2.24), we set the function

$$v(\mathbf{x},t)=\int_0^t U_2(t-s)f(\mathbf{x}+(0,-\sigma_2(t-s),s)\,ds.$$

We see directly that v satisfies the equation

$$v_{tt}+2\sigma_2 v_{tz}-\left(\frac{1}{M}\Delta-\partial^2_{zz}\right)v=f(\mathbf{x},t)$$

with conditions $v(\mathbf{x},0)=v_t(\mathbf{x},0)=v_z(\mathbf{x},0)=0$ (recall that $\sigma_2^2=1$). From this fact, we see that $w(x,y,z,t)=v(x,y,z+\sigma_2 t,t)$ satisfies the wave equation

$$w_{tt}-\frac{1}{M}\Delta w=\tilde{f}(\mathbf{x},t),$$

with conditions $w(\mathbf{x},0)=w_t(\mathbf{x},0)=0$ and $\tilde{f}(\mathbf{x},t)=f(x,y,z+\sigma_2 t,t)$. Using the energy estimates for w (see estimate (2-24) in [18]), we have that

$$\sup_{t\in[0,T]}||\partial_j w(\cdot,t)||_{L^2(Q_\mu)}+\sup_{t\in[0,T]}||\partial_t w(\cdot,t)||_{L^2(Q_\mu)} \le K\int_0^T ||\tilde{f}(\cdot,t)||_{L^2((1+MT)Q_\mu)}\,dt. \tag{2.26}$$

Form this, we conclude that

$$\begin{aligned}\sup_{t\in[0,T]}||\partial_j v(\cdot,t)||_{L^2(Q_\mu)} &\le \sup_{t\in[0,T]}||\partial_j w(\cdot,t)||_{L^2(Q_\mu)}\\ &\le K\int_0^T ||\tilde{f}(\cdot,t)||_{L^2((1+MT)Q_\mu)}\,dt\end{aligned}$$

$$\leq K\int_0^T ||f(\cdot,t)||_{L^2((1+MT)(1+|\sigma_2|T)Q_\mu)}\,dt$$

Adding on μ, we get the desired estimate

$$\left\|\nabla_{\mathbf{x}}\int_0^t U_2(t-s)f(\cdot+(0,-\sigma_2 t),s)\,ds\right\|_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} \leq K(1+T)^N||f||_{l^2_\mu L^1_T L^2_{\mathbf{x}}},$$

since for any cube Q_μ we are adding $(1+MT)^N(1+|\sigma_2|T)^N$ cubes in $\mathbb{R}^N$. To get the estimate (2.25), we follow a similar argument as above. In fact, we set

$$v_1(\mathbf{x},t)=\int_0^t U_2'(t-s)f(\mathbf{x}+(0,-\sigma_2(t-s),s)\,ds.$$

From the definition of v and w above, we see that

$$\partial_t w(\mathbf{x},t)=\partial_t v(x,y,z+\sigma_2 t,t)+\sigma_2\partial_z v(x,y,z+\sigma_2 t,t)=v_1(x,y,z+\sigma_2 t,t).$$

Using the estimate (2.26), we conclude that

$$\begin{aligned}\sup_{t\in[0,T]}||v_1(\cdot,t)||_{L^2(Q_\mu)} &= \sup_{t\in[0,T]}||\partial_t w(\cdot,t)||_{L^2(Q_\mu)}\\ &\leq K\int_0^T ||\tilde{f}(\cdot,t)||_{L^2((1+MT)Q_\mu)}\,dt\\ &\leq K\int_0^T ||f(\cdot,t)||_{L^2((1+MT)(1+|\sigma_2|T)Q_\mu)}\,dt.\end{aligned}$$

Adding on μ, we get the desired estimate

$$\left\|\int_0^t U_2'(t-s)f(\cdot+(0,-\sigma_2 t),s)\,ds\right\|_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} \leq K(1+T)^N||f||_{l^2_\mu L^1_T L^2_{\mathbf{x}}},$$

□

On the other hand, a direct computation shows that

$$||g||_{l^2_\mu L^1_T L^2_{\mathbf{x}}} \leq T^{\frac{1}{2}}||g||_{l^2_\mu L^2_T L^2_{\mathbf{x}}}$$

$$\sup_{t\in[0,T]}||g(t)||_{H^{r-\frac{1}{2}}} \leq \left(\sum_\mu \sup_{t\in[0,T]}||J^{r-\frac{1}{2}}g(t)||^2_{L^2(Q_\mu)}\right)^{\frac{1}{2}} = ||J^{r-\frac{1}{2}}g||_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}},$$

If we set the functions

$$g_1(t,s)=\partial_z(|\psi|^2)(\cdot-(0,\sigma_2 t)),s),\quad g_2(t,s)=(\vec{\nabla}\cdot\nabla_{\mathbf{x}}(|\psi|^2)(\cdot-(0,\sigma_2 t),s),$$

then from previous facts, if $|||\Psi|||_T<\infty$ we obtain the following estimates,

$$\begin{aligned}\left\|J^{r-\frac{1}{2}}\int_0^t U_2'(t-s)g_1(t-s,s)\,ds\right\|_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} &\leq K(1+T)^N||J^{r-\frac{1}{2}}g_3||_{l^2_\mu L^1_T L^2_{\mathbf{x}}}\\ &\leq KT^{\frac{1}{2}}(1+T)^N||J^{r+\frac{1}{2}}(|\psi|^2)||_{l^2_\mu L^2_T L^2_{\mathbf{x}}},\end{aligned} \tag{2.27}$$

$$\left\| J^{r-\frac{1}{2}} \int_0^t U_2(t-s) g_2(t-s,s)\, ds \right\|_{l^2_\mu L^\infty_T L^2_{\mathbf{x}}} \leq K(1+T)^N \| J^{r-\frac{3}{2}} g_4 \|_{l^2_\mu L^1_T L^2_{\mathbf{x}}}$$
$$\leq K T^{\frac{1}{2}} (1+T)^N \| J^{r+\frac{1}{2}} (|\psi|^2) \|_{l^2_\mu L^2_T L^2_{\mathbf{x}}}. \tag{2.28}$$

where $g_3 = \partial_z(|\psi|^2)$ and $g_4 = \vec{\nabla}_{\mathbf{x}} \cdot \nabla_{\mathbf{x}}(|\psi|^2)$.

Now, we are in position to establish the local existence and uniqueness result for the Cauchy problem associated to the system (1.1) in the space X_r.

Theorem 2.2. *Let $r \geq 0$ for $N = 1$ and $r > \frac{N}{2}$ for $N = 2, 3$. For a given $\Psi_0 = (\psi_0, \rho_0, \varphi_0)^t \in X_r$, there exist $T(\|\Psi_0\|_{X_r}) > 0$ and a unique solution $\Psi(t)$ of the integral equation (2.4) such that $\Psi \in C([0,T]; X_r)$ with*

$$\left\| J^{r+\frac{1}{2}} \psi \right\|_{l^\infty_\mu L^2_T L^2_{\mathbf{x}}} < \infty.$$

Moreover, the mapping $\Psi_0 \mapsto \Psi$ from X_r in the class $C([0,T]; X_r)$ is locally Lipschitz.

Proof. We first consider $N = 1$. In this case, the existence result follows by the work of F. Linares and C. Matheus in [12]. The only remark is that the variable $u_x = \varphi$ and the coefficients ω, ν, β, γ and θ in F. Linares and C. Matheus work are related with the coefficients ϵ, σ, W, D, M and σ_2 in the present work in the following way:

$$\omega = \epsilon, \quad -\frac{\nu}{\theta} = \sigma_2, \quad \gamma = D, \quad -\frac{\gamma\nu}{2} = W, \quad \frac{\beta}{\theta} = \frac{1}{M^2}, \quad -\frac{\gamma}{2\theta} = 1,$$

under the restrictions: $\omega > 0$, $\beta < 0$, $\nu < 0$, $\theta < 0$, $\gamma > 0$. If we choose $\Phi_0 = (\psi_0, \rho_0, \varphi_0)^t \in X_r$ for $r \geq 0$ and define $B_0 = \psi_0$, and $u_0 = \partial_x \varphi_0$. From Theorem 1.2 in F. Linares and C. Matheus work in [12], there is a unique local (which is in fact global) solution $(B, \rho, u) \in H^r \times H^{r-\frac{1}{2}} \times H^{r-\frac{1}{2}}$. Moreover, since u_0 has the mean zero property, so does u, because we have that

$$\int_{\mathbb{R}} u(t)\, dx = \int_{\mathbb{R}} u_0\, dx = 0.$$

In this case, we are allowed to define $\varphi(t) = \partial_x^{-1} u(t)$ in such a way that $u(t) = \varphi_x(t)$. So, we have that $(\psi, \rho, \varphi)^t \in X_r$ with $\psi = B$.

Now in the case $N = 2, 3$, for $a > 0$ and $T > 0$, we define,

$$X^a_T := \Big\{ \Psi : \mathbb{R}^N \times [0,T] \to \mathbb{C} \times \mathbb{R} \times \mathbb{R} \mid \Psi \in C([0,T]; X_r), \quad |||\Psi|||_T \leq a \Big\}.$$

For a given $\Psi_0 \in X_r$, we consider the operator

$$\mathcal{A}(\Psi)(t) := T(t)\Psi_0 + \int_0^t T(t-s)\mathcal{C}(\Psi)(s)\, ds.$$

defined in X^a_T. We will see that $\mathcal{A}$ defines a contraction in the closed ball X^a_T for appropriate values of $a, T > 0$, therefore $\mathcal{A}$ has a fixed point $\Psi \in X^a_T$, which is the solution of the integral equation (2.4).

First, we set the notation $w_j(t) := (\mathcal{A}(\Psi)(t))_j$ for $1 \leq j \leq 3$. We note that

$$w_1(t) = U_1(t)\psi_0 - i \int_0^t U_1(t-l) L(\Psi)(l)\, dl.$$

From the estimate (2.17), we have that

$$||w_1(t)||_{H^r} \le K||\psi_0||_{H^r} + KT(|||\Psi|||_T^2 + |||\Psi|||_T^3).$$

Moreover, from the estimates (2.13), (2.18) and (2.20), we have that

$$||J^{r+\frac{1}{2}} w_1(t)||_{l^\infty_\mu L^2_T L^2_{\mathbf{x}}} \le K||\psi_0||_{H^r} + ||J^{r+\frac{1}{2}} \int_0^t U_1(t-l)L(\Psi)(l)\,dl||_{l^\infty_\mu L^2_T L^2_{\mathbf{x}}} \tag{2.29}$$

$$\le K||\psi_0||_{H^r} + KT^{\frac{1}{2}}\left(|||\Psi(t)|||_T^2 + |||\Psi(t)|||_T^3\right). \tag{2.30}$$

To illustrate the estimates for $||w_2(t)||_{H^{r-\frac{1}{2}}}$ and $||w_3(t)||_{H_{r+\frac{1}{2}}}$, we consider the case $N=3$ to simplify the computations and we only consider the case $r+\frac{1}{2} \in \mathbb{N}$ (we use an interpolation argument in the other case). In this case, for $0<\alpha+\beta<r-\frac{1}{2}$ we need to compute terms like

$$\begin{aligned}\|\partial_j^\beta \psi(t)\partial_j^\alpha \bar{\psi}(t)\|^2_{L^2(Q_\mu)} &\le \|\partial_j^\beta \psi(t)\|^2_{L^4(Q_\mu)}\|\partial_j^\alpha \bar{\psi}\|^2_{L^4(Q_\mu)}\\ &\le K\|J^{r-\frac{1}{2}}\psi(t)\|^4_{L^2(Q_\mu)}\\ &\le K\|J^{r-\frac{1}{2}}\psi(t)\|^2_{L^2(Q_\mu)} \sup_{0\le t\le T}||\psi(t)||^2_{H^{r-\frac{1}{2}}},\end{aligned}$$

where we are using the Hölder inequality, the Rellich-Kondrachov Compactness theorem with $p=2$ and $N=3$, and the estimates (2.19). Moreover, we also have that

$$\|\partial_j^{r+\frac{1}{2}}\psi(t)\bar{\psi}(t)\|^2_{L^2(Q_\mu)} + \|\psi(t)\partial_j^{r+\frac{1}{2}}\bar{\psi}(t)\|^2_{L^2(Q_\mu)} \le ||J^{r+\frac{1}{2}}(\psi)(t)||^2_{L^2(Q_\mu)} \sup_{0\le t\le T}||\psi(t)||^2_{H^{r-\frac{1}{2}}}.$$

In other words, we have that

$$||\partial_j^{r+\frac{1}{2}}(|\psi|^2)||_{l^2_\mu L^2_T L^2_{\mathbf{x}}} \le K \sup_{0\le t\le T}||\psi(t)||_{H^{r-\frac{1}{2}}}||J^{r+\frac{1}{2}}(|\psi|^2)||_{l^2_\mu L^2_T L^2_{\mathbf{x}}} \le K|||\Psi|||_T^2,$$

which implies that

$$||J^{r+\frac{1}{2}}(|\psi|^2)||_{l^2_\mu L^2_T L^2_{\mathbf{x}}} \le K|||\Psi|||_T^2.$$

From the semigroup and the estimates (2.27) and (2.28), we have that

$$||w_2(t)||_{H^{r-\frac{1}{2}}} \le K\left(||\rho_0||_{H^{r-\frac{1}{2}}} + ||\varphi_0||_{H_{r+\frac{1}{2}}} + T^{\frac{1}{2}}(1+T)^N|||\Psi|||_T^2\right).$$

In a similar fashion, we see that

$$||w_3(t)||_{H^{r-\frac{1}{2}}} \le K\left(||\rho_0||_{H^{r-\frac{1}{2}}} + ||\varphi_0||_{H_{r+\frac{1}{2}}} + T^{\frac{1}{2}}(1+T)^N|||\Psi|||_T^2\right).$$

Putting together previous estimates, we conclude for $\Psi \in X_T^a$ that

$$|||\mathcal{A}(\Psi)|||_T \le K||\Psi_0||_{X^r} + KT|||\Psi|||_T^3 + K(T^{\frac{1}{2}}(1+T)^N + T)|||\Psi|||_T^2.$$

If we choose $2K||\Psi_0||_{X_r} = a$ and take $T>0$ small enough such that

$$2K(Ta^2 + (T^{\frac{1}{2}}(1+T)^N + T)a) < 1,$$

we have that $\mathcal{A}(X_T^a) \subset X_T^a$. Now, from the same arguments as above, we have for $\Psi, \widetilde{\Psi} \in X_T^a$ that

$$|||\mathcal{A}(\Psi)-\mathcal{A}(\tilde{\Psi})|||_T$$
$$\leq K(T+T(1+T)^N+T)\left(|||\Psi|||_T^2+|||\Psi|||_T+|||\tilde{\Psi}|||_T^2+|||\tilde{\Psi}|||_T\right)|||\Psi-\tilde{\Psi}|||_T,$$

meaning that if we choose $T>0$ small enough such that

$$2K(T+T^{\frac{1}{2}}(1+T)^N+T)(a^2+a^3)<1,$$

then $\mathcal{A}$ is a contraction on X_T^a, as desired. □

2.1 Conserved quantities and global solutions

In this section, we discuss properties of the Benney-Roskes system that will be used in the stability analysis. The first remark is that there exists a Hamiltonian structure which provides relevant information to determine the stability of standing waves. In this case, the Hamiltonian structure is given by

$$\partial_t\begin{pmatrix}\psi\\ \rho\\ \varphi\end{pmatrix}=\mathcal{J}\mathcal{H}'\begin{pmatrix}\psi\\ \rho\\ \varphi\end{pmatrix},\qquad \mathcal{J}=\begin{pmatrix}-i&0&0\\ 0&0&\frac{2}{W}\\ 0&-\frac{2}{W}&0\end{pmatrix}, \tag{2.31}$$

where the Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H}\begin{pmatrix}\psi\\ \rho\\ \varphi\end{pmatrix}=\frac{1}{2}\int_{\mathbb{R}^N}\Big(\sigma_1|\nabla_\perp\psi|^2+\epsilon|\partial_z\psi|^2+\frac{\sigma}{2}|\psi|^4+W\rho|\psi|^2$$
$$+W\sigma_2\rho\partial_z\varphi+\frac{W}{2}|\nabla\varphi|^2+\frac{W}{2M^2}|\rho|^2+DW|\psi|^2\partial_z\varphi\Big)\,d\mathbf{x}. \tag{2.32}$$

We note that the Hamiltonian is conserved in time on solutions Ψ since,

$$\frac{d}{dt}\mathcal{H}(\Psi)=\langle\mathcal{H}'(\Psi),\Psi'\rangle=\langle\mathcal{H}'(\Psi),\mathcal{J}\mathcal{H}'(\Psi)\rangle=0,$$

using that $\mathcal{J}$ is a skew-adjoint operator.

We use the conserved integrals I_1, I_2 and I_3, to construct a Lyapunov function.

Proposition 2.1. *The Benny-Roskes system (1.1) has the following conserved quantities with respect to time,*

$$I_1(t)=\int_{\mathbb{R}^N}\Big(\sigma_1|\nabla_\perp\psi|^2+\epsilon|\partial_z\psi|^2+\frac{\sigma}{2}|\psi|^4+W\rho|\psi|^2+W\sigma_2\rho\partial_z\varphi$$
$$+\frac{W}{2}|\nabla\varphi|^2+\frac{W}{2M^2}|\rho|^2+DW|\psi|^2\partial_z\varphi\Big)\,d\mathbf{x},$$
$$I_2(t)=\int_{\mathbb{R}^N}|\psi|^2\,d\mathbf{x},$$
$$I_3(t)=\int_{\mathbb{R}^N}\left(W\rho\partial_z\varphi+\frac{i}{2}(\psi\partial_z\bar{\psi}-\partial_z\psi\bar{\psi})\right)d\mathbf{x},$$
$$I_4(t)=\int_{\mathbb{R}^N}\Big(\sigma_1|\nabla_\perp\psi|^2+\epsilon|\partial_z\psi|^2+\frac{\sigma}{2}|\psi|^4+W(\rho+D\partial_z\varphi)|\psi|^2+\frac{W}{2}|\nabla\varphi|^2$$
$$+\frac{W}{2M^2}|\rho|^2-\frac{i\sigma_2}{2}(\psi\partial_z\bar{\psi}-\partial_z\psi\bar{\psi})\Big)\,d\mathbf{x}.$$

Proof. The first quantity corresponds to the Hamiltonian $\mathcal{H}$. For the second one, we note directly that

$$\frac{d}{dt}\int_{\mathbb{R}^N}|\psi|^2\,\mathbf{x} = \int_{\mathbb{R}^N}(\psi_t\bar{\psi}+\psi\bar{\psi}_t)\,d\mathbf{x} = 0.$$

On the other hand, I_4 is a combination of I_1 and I_3. So, we only need to establish the result for I_3. First, a direct computation shows that

$$\frac{d}{dt}\int_{\mathbb{R}^N}(\psi\partial_z\bar{\psi}-\partial_z\psi\bar{\psi})\,d\mathbf{x} = 2i\int_{\mathbb{R}^N}(W(\rho_z+D\varphi_{zz})|\psi|^2\,d\mathbf{x}.$$

On the other hand, we also have that

$$\frac{d}{dt}\int_{\mathbb{R}^N}\rho\partial_z\varphi\,d\mathbf{x} = \int_{\mathbb{R}^N}(\rho_z+D\varphi_{zz})|\psi|^2\,d\mathbf{x},$$

which implies that

$$\frac{d}{dt}\int_{\mathbb{R}^N}(\psi\partial_z\bar{\psi}-\partial_z\psi\bar{\psi}-2iW\partial_z\varphi\rho)\,d\mathbf{x} = 0,$$

or also that

$$\frac{d}{dt}\int_{\mathbb{R}^N}\left(W\rho\partial_z\varphi+\frac{i}{2}(\psi\partial_z\bar{\psi}-\partial_z\psi\bar{\psi})\right)\,d\mathbf{x} = 0.$$

So, we also have that $I_3(t) = I_3(0)$. $\square$

Remark 1. On the global existence result.

As we mention above, G. Ponce and J. C. Saut in [18] obtained weak global solutions due to the fact that their energy space included the norm $||\varphi||_{H^1(\mathbb{R}^N)}$, but the conserved quantities control only the term $||\nabla\varphi||_{L^2(\mathbb{R}^N)}$. The first remark is that we have the following estimate:

$$\int_{\mathbb{R}^N}\epsilon|\partial_z\psi|^2\,d\mathbf{x} - i\sigma_2\int_{\mathbb{R}^N}\left(\psi\partial_z\bar{\psi}-\partial_z\psi\bar{\psi}\right)\,d\mathbf{x} \geq -CI_2(0).$$

In fact, from Young's inequality, we have for any $\alpha > 0$ that

$$\left|i\sigma_2\int_{\mathbb{R}^N}\left(\psi\partial_z\bar{\psi}-\partial_z\psi\bar{\psi}\right)\,d\mathbf{x}\right| \leq |\sigma_2|\left(\frac{\alpha}{2}||\psi||_2^2+\frac{1}{2\alpha}||\partial_z\psi||_2^2\right)$$

If we take $\epsilon > \frac{|\sigma_2|}{2\alpha}$ and use that $||\psi(t)||_2^2 = ||\psi(0)||_2^2 = I_2(0)$, then we get the conclusion with $C = \frac{\alpha|\sigma_2|}{2}$. From this and previous conserved quantities, we see that

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\sigma_1|\nabla_\perp\psi|^2+\frac{\epsilon}{2}|\partial_z\psi|^2+\frac{\sigma}{2}|\psi|^4+W(\rho+D\partial_z\varphi)|\psi|^2+\frac{W}{2}|\nabla\varphi|^2\\ +\frac{W}{2M^2}|\rho|^2\Big)\,d\mathbf{x} \leq I_4(0)+C_1I_2(0).\end{aligned}$$

On the other hand, for $\beta > 0$, $\theta > 0$ and $\gamma > 0$ we also have that

$$W(\rho+D\partial_z\varphi)|\psi|^2 \leq \frac{W\beta}{2}|\rho|^2+\frac{D\theta}{2}|\partial_z\varphi|^2+\left(\frac{W}{2\beta}+\frac{D}{2\theta}\right)|\psi|^4,$$

If we choose $\beta > 0$ and $\theta > 0$ such $\beta = \frac{1}{2M^2}$ and $\theta = \frac{W}{2D}$, then we have that

$$\int_{\mathbb{R}^N}\left(\sigma_1|\nabla_\perp\psi|^2+\frac{\epsilon}{2}|\partial_z\psi|^2+\left(\frac{\sigma}{2}-\left(WM^2+\frac{D^2}{W}\right)\right)|\psi|^4\right.$$
$$\left.+\frac{W}{2}|\nabla_\perp\varphi|^2+\frac{W}{4}|\partial_z\varphi|^2+\frac{W}{4M^2}|\rho|^2\right)d\mathbf{x}\le I_4(0)+C_1I_2(0).$$

Clearly, in the case $\frac{\sigma}{2}>WM^2+\frac{D^2}{W}$, we have that

$$||\Psi||^2_{X_{1,\frac{1}{2}}}\le C(\epsilon,W,M,D,\sigma_1,\sigma)(I_4(0)+C_1I_2(0)),$$

which implies the existence of global solutions. In particular, for $N=1$, F. Linares and C. Matheus in [12] obtained global solutions under Oliveira's assumption in [16]: $\omega>4\epsilon$.

We will establish the existence of global solutions for the Benney-Roskes system (1.1), by imposing the minimum set of restrictions on the parameters, as done by G. Ponce and J. C. Saut in [18]. Hereafter, we define the quadratic form

$$C(p,q,r)=\frac{\sigma}{2}p^2+\frac{W}{2M^2}q^2+\frac{W}{2}r^2+Wpq+DWpr+W\sigma_2qr. \tag{2.33}$$

We note that under the assumption that the quadratic form C given by (2.33) is positive definite, we have for any local solution Ψ that

$$C_0||\Psi(t)||^2_{X_{1,\frac{1}{2}}}\le I_1(t)=I_1(0),$$

which using the Theorem 4.2 in Ponce-Saut's paper [18], guarantees the following result.

Theorem 2.3. *Assume that $\epsilon>0$, $\sigma_1>0$ and $\sigma>0$ and that the quadratic form C given by (2.33) is positive definite. Then, for $\left(\psi_0,\rho_0,\frac{\partial\varphi_0}{\partial x_j}\right)\in H^1(\mathbb{R}^N))\times L^2(\mathbb{R}^N))\times L^2(\mathbb{R}^N)$ with $1\le j\le N$, there is a global solution $(\psi(t),\rho(t),\varphi(t))$ such that*

$$\psi(t)\in L^\infty((0,\infty);H^1(\mathbb{R}^N)),\ \ \rho(t),\frac{\partial\varphi(t)}{\partial x_j}\qquad\in L^\infty((0,\infty);L^2(\mathbb{R}^N))$$
$$\psi'(t),\rho'(t),\frac{\partial\varphi'(t)}{\partial x_j}\in L^\infty((0,\infty);H^{-1}(\mathbb{R}^N)).$$

3 Stability of standing waves

In this section we establish the existence and orbital stability of nontrivial standing waves to the system (1.1) for $N=1$, which takes the form,

$$\begin{cases} i\partial_t\psi+\epsilon\partial_x^2\psi &=\left(\sigma|\psi|^2+W(\rho+D\partial_x\varphi)\right)\psi,\\ \partial_t\rho+\sigma_2\partial_x\rho &=-\partial_x^2\varphi-D\partial_x(|\psi|^2),\\ \partial_t\varphi+\sigma_2\partial_x\varphi &=-\frac{1}{M^2}\rho-|\psi|^2,\end{cases} \tag{3.1}$$

For this system, we look for solutions of the form

$$\psi(x,t)=e^{i\omega t}e^{\frac{ic}{2\epsilon}(x-ct)}u(x-ct),\quad \rho(x,t)=-M^2u^2(x-ct)\quad \varphi(x,t)=0, \tag{3.2}$$

where $\omega>\frac{c^2}{4\epsilon}$ and $c\in\mathbb{R}$. We see directly that $D=M^2(\sigma_2-c)$ and that u satisfies the cubic Schrödinger equation

$$\epsilon\partial_x^2u-\left(\omega-\frac{c^2}{4\epsilon}\right)u+(M^2W-\sigma)|u|^2u=0,\qquad u\in H^1(\mathbb{R})\setminus\{0\}. \tag{3.3}$$

From the work by K. McLeod and J. Serrin in [14], we have the following result related with the decaying positive solutions for (3.3) and also the uniqueness of the positive ground state in $H^1(\mathbb{R})$.

Theorem 3.1. *Let $\epsilon > 0$, $\omega > \frac{c^2}{4\epsilon}$, $D = M^2(\sigma_2 - c)$, $E = \frac{1}{\epsilon}\left(\omega - \frac{c^2}{4\epsilon}\right)$ and $M^2W > \sigma$. Then the triplet $\Psi_{\omega,c} = (\psi, \rho, \varphi)$ with*

$$\psi(x,t) = e^{i\omega t} e^{\frac{ic}{2\epsilon}(x-ct)} u(x-ct), \quad \rho(x,t) = -M^2u^2(x-ct) \quad \varphi(x,t) = 0$$

is a standing wave to the 1D-Benney-Roskes system (3.1), where u is the unique positive solution (3.3), which is even and exponential decreasing for $x > 0$. Moreover, u is given by

$$u_{\omega,c}(x) = \sqrt{\frac{2\epsilon E}{(M^2W - \sigma)}}\, sech\left(\sqrt{E}x\right) \tag{3.4}$$

As we discuss above, the Benny-Roskes system has the phase and translation symmetries. So, the orbit $\mathcal{G}_\omega$ associated with a ground state $\Psi_\omega = (\psi, \rho, \varphi)$ reduces, due to the uniqueness, to the set

$$\begin{aligned} \mathcal{G}_\omega &= \left\{ \left(e^{i\gamma}\psi(\cdot + x_0), \rho(\cdot + x_0), \varphi(\cdot + x_0)\right) : (x_0, \gamma) \in \mathbb{R} \times [0, 2\pi) \right\}, \\ &= \left\{ (\Psi_\omega)_{\gamma, x_0} : (x_0, \gamma) \in \mathbb{R} \times [0, 2\pi) \right\}. \end{aligned}$$

In particular, we have that if $\Psi(x,t) = (\psi(x,t), \rho(x,t), \varphi(x,t))$ is a solution for the system (3.1), then for $(x_0, \gamma) \in \mathbb{R} \times [0, 2\pi)$ the function $\Psi_{x_0,\gamma}(t, \cdot)$ is also a solution. Finally, we define the distance between $\Phi(t, \cdot)$ and the orbit of the ground state $\mathcal{G}_\omega$ as

$$d_E(\Phi, \Psi_\omega) = \inf_{(x_0,\gamma) \in \mathbb{R} \times [0,2\pi)} \mathcal{N}_E(\Phi_{\gamma,x_0}(t, \cdot), \Psi_\omega),$$

where the metric $\mathcal{N}_E : X_{1,\frac{1}{2}} \times X_{1,\frac{1}{2}} \to \mathbb{R}$ is given by

$$\mathcal{N}_E(\Psi, \widetilde{\Psi}) = \sqrt{\|\psi' - \widetilde{\psi}'\|_{L^2(\mathbb{R})} + E\|\psi - \widetilde{\psi}\|_{L^2(\mathbb{R})} + \|\rho - \widetilde{\rho}\|_{L^2(\mathbb{R})} + \|\varphi_x - \widetilde{\varphi}_x\|_{L^2(\mathbb{R})}},$$

with $E = \frac{1}{\epsilon}\left(\omega - \frac{c^2}{4\epsilon}\right)$. We note that the norm defined by the metric $\mathcal{N}_E$ is equivalent to the norm in the space $X_{1,\frac{1}{2}}$.

Definition 1. (Orbital Stability) We say that standing wave Ψ_{ω_0,c_0} of (3.1) with frequency $\omega_0 > \frac{c^2}{4\epsilon}$ and wave speed $c_0 \in \mathbb{R}$ is orbitally stable in the following sense: for given $\eta > 0$, there exists $\delta(\eta) > 0$ such that if $\Psi_0 \in X_{1,\frac{1}{2}}$ satisfies that

$$\mathcal{N}_E(\Psi_0, \Psi_{\omega_0,c_0}) < \delta(\eta),$$

then the Cauchy problem associated with the system (3.1) has a unique weak solution $\Psi(t) \in C\left(\mathbb{R}, X_{1,\frac{1}{2}}\right)$ such that $\Psi(0) = \Psi_0$ and for all $t \in \mathbb{R}$

$$d_E(\Psi(t, \cdot), \Psi_{\omega_0,c_0}) < \eta.$$

Before going further, we consider the linear operators L_+ and L_- corresponding to the real part and imaginary part, respectively, of the Nonlinear Schrödinger type system (3.1) linearized operator about the ground state u,

$$L_+ = -\partial_{xx}^2 + EI + \frac{3}{\epsilon}(\sigma - M^2W)u^2, \quad L_- = -\partial_{xx}^2 + EI + \frac{1}{\epsilon}(\sigma - M^2W)u^2.$$

We point out that this linear operator are related with the second variation of the action functional associated with the standing waves,

$$\mathcal{F}_{\omega,c}(\Phi) = \frac{1}{2\epsilon}\left(I_1(\Phi) + \omega I_2(\Phi) - cI_3(\Phi)\right), \tag{3.5}$$

which according with Proposition (2.1) is a conserved quantity in time on solutions for the Benny-Roskes system. As done in the case of the NLS in [23], $\mathcal{F}_{\omega,c}$ will be estimated in terms of ρ_E and will be used to measure the deviation of $\Psi(\cdot,t)$ from the ground state orbit. We write

$$\mathcal{W}(t,\cdot) = \Phi_{x_0,\gamma}(t,\cdot) - \Psi_{\omega,c}(t,\cdot) = (\upsilon_1(t,\cdot), \upsilon_2(t,\cdot), \upsilon_3(t,\cdot)), \tag{3.6}$$

Then, if $\Delta\mathcal{F}_{\omega.c}(t)$ denotes the deviation of $\Psi(\cdot,t)$ from the orbit of the ground state orbit of $\Psi_{\omega,c}$, then

$$\begin{aligned}\Delta\mathcal{F}_{\omega.c}(t) &= \mathcal{F}_{\omega.c}(\Psi_0(\cdot)) - \mathcal{F}_{\omega.c}(\Psi_{\omega,c})\\ &= \mathcal{F}_{\omega.c}(\Psi(\cdot,t)) - \mathcal{F}_{\omega,c}(\Psi_{\omega,c})\\ &= \mathcal{F}_{\omega.c}(\Psi_{\gamma,x_0}(\cdot,t)) - \mathcal{F}_{\omega,c}(\Psi_{\omega,c})\\ &= \mathcal{F}_{\omega.c}(\Psi_{\omega,c} + \mathcal{W}) - \mathcal{F}_{\omega.c}(\Psi_{\omega,c}),\end{aligned}$$

where we are using that $\mathcal{F}_{\omega,c}$ is conserved in time on solutions and the scale invariance. So, the main goal is to establish for some positive constants A, B, C that

$$\Delta\mathcal{F}_{\omega,c}(t) \geq A||\mathcal{W}(t,\cdot)||^2_{X_{1,\frac{1}{2}}}\left(1 - B||\mathcal{W}(t,\cdot)||_{X_{1,\frac{1}{2}}} - C||\mathcal{W}(t,\cdot)||^2_{X_{1,\frac{1}{2}}}\right),$$

where $||\mathcal{W}(t,\cdot)||^2_{X_{1,\frac{1}{2}}} = ||\upsilon_1(t,\cdot)||^2_{H^1} + ||\upsilon_2(t,\cdot)||^2_{L^2} + ||\upsilon_3'(t,\cdot)||^2_{L^2}$.

Lemma 3.1. *Let $\omega > \frac{c^2}{4\epsilon}$ and $\Psi(t,\cdot) \in X_{1,\frac{1}{2}}$ be a solution of the Benny-Roskes system with initial condition $\Psi_0 \in X_{1,\frac{1}{2}}$. Then for $c > 0$ large enough there are positive constants A_1, A_2, A_3 and A_4 such that,*

$$\begin{aligned}\Delta\mathcal{F}_{\omega,c}(t) \geq \frac{1}{2}((L_+h_1, h_1) + (L_-h_2, h_2)) + A_1||\upsilon_2||^2_{L^2} + A_2||\upsilon_3'||^2_{L^2}\\ - A_3||\upsilon_1||^3_{H^1} - A_4||\upsilon_1||^4_{H^1},\end{aligned} \tag{3.7}$$

where $h_1 = \Re(\upsilon_1)$ and $h_2 = \Im(\upsilon_1)$.

Proof. As done by M. Weinstein in [23] (see also the work by F. Oliveira [16]), to estimate the deviation of a solution $\Psi(t,\cdot)$ and the ground state orbit of $\Psi_{\omega,c}$, we consider the perturbation variable $\mathcal{W}(t,\cdot) = \Psi_{x_0,\gamma}(t,\cdot) - \Psi_{\omega,c}(t,\cdot) = (\upsilon_1(t,\cdot), \upsilon_2(t,\cdot), \upsilon_3(t,\cdot))$. Now, we may assume without losing generality that $\upsilon_1(t,)$ can be replaced by $e^{i\omega t}e^{\frac{ic}{2\epsilon}(x-ct)}\upsilon_1(t,\cdot)$, since the deviation depends on the $||\upsilon_1(t,\cdot)||_{H^1}$ and that

$$||e^{i\omega t}e^{\frac{ic}{2\epsilon}(x-ct)}\upsilon_1(t,\cdot)||^2_{H^1} \sim ||\upsilon_1(t,\cdot)||^2_{H^1},$$

for $c > 0$ large enough. From this fact, we may assume that

$$\begin{aligned}e^{i\gamma}\psi(t, x + x_0) &= e^{i\omega t}e^{\frac{ic}{2\epsilon}(x-ct)}(\upsilon_1(t,x) + u(x-ct)),\\ \rho(t, x + x_0) &= \upsilon_2(t,x) - M^2u^2(x-ct),\end{aligned}$$

$$\varphi(t, x + x_0) = \upsilon_3(t, x).$$

The first observation is that

$$\mathcal{F}_{\omega,c}(\Psi_{\omega,c}) = \frac{1}{2}\int_{\mathbb{R}}\left(\epsilon(u')^2 + \left(\omega - \frac{c^2}{4\epsilon}\right)u^2 + \frac{1}{2}\left(\sigma - M^2W\right)u^4\right)dx.$$

Now, a direct computation shows that

$$\begin{aligned}
|\psi'(t, x + x_0)|^2 &= |\upsilon_1'|^2 + 2h_1'u' + (u')^2 + \frac{ic}{2\epsilon}(\upsilon_1\overline{\upsilon_1'} - \overline{\upsilon_1}\upsilon_1') - \frac{c}{\epsilon}h_2u' + \frac{c}{\epsilon}h_2'u,\\
&\quad + \frac{c^2}{4\epsilon^2}(|\upsilon_1|^2 + 2h_1u + (u)^2),\\
|\psi(t, x + x_0)|^4 &= |\upsilon_1|^4 + u^4 + 4h_1^2u^2 + 4|\upsilon_1|^2h_1u + 2|\upsilon_1|^2u^2 + 4h_1u^3,\\
|\rho(t, x + x_0)|^2 &= |\upsilon_2|^2 - 2M^2\upsilon_2u^2 + M^4u^4,\\
(\rho\varphi')(t, x + x_0) &= \upsilon_2\upsilon_3' - M^2u^2\upsilon_3',\\
(\rho + D\varphi')|\psi|^2(t, x + x_0) &= \upsilon_2|\upsilon_1|^2 + 2h_1\upsilon_2u + \upsilon_2u^2 - M^2|\upsilon_1|^2u^2 - 2M^2h_1u^3 - M^2u^4\\
&\quad + D(|\upsilon_1|^2\upsilon_3' + 2h_1u\upsilon_3' + u^2\upsilon_3'),\\
\frac{i}{2}(\psi\overline{\psi'} - \psi'\overline{\psi}) &= \frac{c}{2\epsilon}(|\upsilon_1|^2 + 2h_1u + u^2) + \frac{i}{2}(\upsilon_1\overline{\upsilon_1'} - \upsilon_1'\overline{\upsilon_1}) - h_2u' + h_2'u.
\end{aligned}$$

Now, using previous formulas and that $D = M^2(\sigma_2 - c)$, we see that

$$\begin{aligned}
\mathcal{F}_{\omega,c}(\Psi_{x_0,\gamma}(t,\cdot)) &= \frac{1}{2}\int_{\mathbb{R}}\Bigg((\epsilon|\upsilon_1'|^2 + |\upsilon_1|^2\left(\omega - \frac{c^2}{4\epsilon} + (\sigma - WM^2)u^2\right) +\\
&2h_1\left(-\epsilon u'' + \left(\omega - \frac{c^2}{4\epsilon}\right)u + (\sigma - WM^2)u^3\right) + \left(\epsilon(u')^2 + \left(\omega - \frac{c^2}{4\epsilon}\right)u^2 + \frac{1}{2}(\sigma - WM^2)u^4\right)\\
&+ \frac{W}{2}(\upsilon_3')^2 + \frac{W}{2M^2}\upsilon_2^2 + \frac{\sigma}{2}|\upsilon_1|^4 + 2\sigma h_1^2u^2 + 2\sigma|\upsilon_1|^2h_1u + \frac{WD}{M^2}\upsilon_2\upsilon_3' + DW|\upsilon_2|^2\upsilon_3'\\
&+ 2DWh_1u\upsilon_3' + W|\upsilon_1|^2\upsilon_2 + 2Wh_1u\upsilon_2\Bigg)dx. \qquad (3.8)
\end{aligned}$$

Using that $\Delta\mathcal{F}_{\omega,c}(t) = \mathcal{F}_{\omega,c}(\Psi_{x_0,\gamma}(t,\cdot)) - \mathcal{F}_{\omega,c}(\Psi_{\omega,c})$ and the equation for u, we get that

$$\begin{aligned}
\Delta\mathcal{F}_{\omega,c}(t) &= \frac{1}{2}\int_{\mathbb{R}}\Bigg(|\upsilon_1'|^2 + \frac{|\upsilon_1|^2}{\epsilon}\left(\omega - \frac{c^2}{4\epsilon} + (\sigma - WM^2)u^2\right)\\
&+ \frac{1}{2\epsilon}(WM^2 - \sigma)u^4 + \frac{W}{4\epsilon M^2}K + \frac{W}{4\epsilon}(\upsilon_3')^2 + \frac{W}{4\epsilon M^2}\upsilon_2^2\Bigg)dx. \qquad (3.9)
\end{aligned}$$

where the function K is defined as,

$$\begin{aligned}
K &= \upsilon_2^2 + M^2(\upsilon_3')^2 + \frac{2\sigma M^2}{W}|\upsilon_1|^4 + \frac{8\sigma M^2}{W}h_1^2u^2 + \frac{8\sigma M^2}{W}|\upsilon_1|^2h_1u + 4D\upsilon_2\upsilon_3'\\
&+ 4M^2D|\upsilon_1|^2\upsilon_3' + 8M^2Dh_1u\upsilon_3' + 4M^2|\upsilon_1|^2\upsilon_2 + 8M^2h_1u\upsilon_2. \qquad (3.10)
\end{aligned}$$

Now, we see directly that,

$$(\upsilon_2 + 2D\upsilon_3' + 4M^2h_1u + 2M^2|\upsilon_1|^2)^2 + \Bigg(\sqrt{M^2 - 4D^2}\,\upsilon_3' - \frac{4DM^2}{\sqrt{M^2 - 4D^2}}h_1u$$

$$-\frac{2DM^2}{\sqrt{M^2-4D^2}}|\upsilon_1|^2\Big)^2 = K + \frac{2M^2}{W}N(\sigma,M,W,D)(|\upsilon_1|^4 + 4u^2h_1^2 + 4|\upsilon_1|^2h_1u), \quad (3.11)$$

where $N(\sigma,M,W,D) = \frac{\left(\sigma(M^2-4D^2)-2M^2(M^2-3D^2)W\right)}{(M^2-4D^2)}$. On the other hand, we also have that,

$$\int_{\mathbb{R}}\Big(|\upsilon_1'|^2 + \frac{|\upsilon_1|^2}{\epsilon}\Big(\omega - \frac{c^2}{4\epsilon} + (\sigma - WM^2)u^2\Big)\Big)\,dx = (L_+h_1,h_1) + (L_-h_2,h_2) + \frac{2(WM^2-\sigma)}{\epsilon}\int_R u^2h_1^2\,dx. \quad (3.12)$$

Putting this estimates together, we conclude that,

$$\Delta\mathcal{F}_{\omega,c}(t) = \frac{1}{2}((L_+h_1,h_1)+(L_-h_2,h_2)) - \frac{N}{2\epsilon}\int_{\mathbb{R}}\left(|\upsilon_1|^4 + 4|\upsilon_1|^2h_1u\right)\,dx$$
$$+\int_{\mathbb{R}}\Big(\frac{2}{\epsilon}((WM^2-\sigma)-N(\sigma,M,W,D))u^2h_1^2 + \frac{1}{2\epsilon}(WM^2-\sigma)u^4 + \frac{W}{4\epsilon}(\upsilon_3')^2 + \frac{W}{4\epsilon M^2}\upsilon_2^2\Big)\,dx.$$

Now, for $M^2-4D^2>0$ and $WM^2-\sigma>0$, we have that,

$$(WM^2-\sigma) - N(\sigma,M,W,D) = \frac{(3M^2W-2\sigma)(M^2-4D^2)+2M^2D^2W}{(M^2-4D^2)} > 0.$$

From these fact and the Young inequality, we conclude for positive constants A_3, A_4 such that

$$\Delta\mathcal{F}_{\omega,c}(t) \geq \frac{1}{2}((L_+h_1,h_1)+(L_-h_2,h_2)) + \frac{W}{4\epsilon M^2}||\upsilon_2||^2_{L^2} + \frac{W}{4\epsilon}||\upsilon_3'||^2_{L^2} - A_3||\upsilon_1||^4 - A_4||\upsilon_1||^3_{H^1},$$

where we are using that $H^1 \hookrightarrow L^4$ and $H^1 \hookrightarrow L^\infty$. □

From the work by J. Bona in [2], it is possible to obtain the following technical result,

Lemma 3.2. *If $x_0 = x_0(t)$ and $\gamma_0 = \gamma(t)$ are chosen to minimize*

$$\mathcal{N}(\Psi_{x_0(t),\gamma_0(t)},\Psi_\omega), \quad (3.13)$$

then there are positive constants D_1, D_2 and D_3 such that,

$$(L_+h_1,h_1)+(L_-h_2,h_2) \geq D_1||\upsilon_1||^2_{H^1} - D_2||\upsilon_1||^3_{H^1} - D_3||\upsilon_1||^4_{H^1}, \quad (3.14)$$

in the case

$$\int_{\mathbb{R}}|\psi(t,x)|^2\,dx = \int_{\mathbb{R}}u^2(x)\,dx.$$

Proof. That the minimum is attained at finite values x_0 and γ and that $\mathcal{W}(t,\cdot)$ as defined in (3.6) has a continuous H^1 norm can be obtained by performing the approach used by J. Bona in [2]. We note that the minimization of (3.13) over x_0 and γ implies that

$$\int_{\mathbb{R}}\left(3(M^2W-\sigma)u^2u'h_1(t,x) + M^2u^2\phi_2'(t,x)\right)\,dx = 0, \quad (3.15)$$

$$\int_{\mathbb{R}}u^3h_2(t,x)\,dx = 0, \quad (3.16)$$

after differentiating with respect to x_0 and γ, respectively.

Now, from the fact that L_- is non-degenerate and that $L_- u = 0$ where $u > 0$ is the ground state of L_-, we have that L_- is a non-negative operator. If we consider the infimum of $(L_- v, v)/(v, v)$ subject to (3.16), we have that is non zero. In fact, if this were zero, then it is attained at u, contradicting the restriction (3.16), meaning that the minimum is positive. Therefore, there is a positive constant $A_3 > 0$ such that for any $v \in H^1$,

$$(L_- v, v) \geq A_3(v, v).$$

Taking $\delta > 0$ in such a way that $A_3 > 3\delta(M^2W - \sigma)||u||^2_{L^\infty}$, we have that

$$\begin{aligned}(1+\delta)(L_- v, v) &\geq \delta||v'||^2_{L^2} + (A_3 + \delta\omega)||v||^2_{L^2} - 3\delta(M^2W - \sigma)\int u^2 v^2\, dx,\\ &\geq \delta||v'||^2_{L^2} + (A_3 + \delta\omega - 3\delta(M^2W - \sigma)||u||^2_{L^\infty})||v||^2_{L^2}.\end{aligned}$$

In other words, there is $A_0 > 0$ such that

$$(L_- v, v) \geq A_0||v||^2_{H^1}.$$

On the other hand, From Lemma 4.2 in the work by M. Weinstein [23], L_+ has exactly one negative eigenvalue, but (3.15) is not enough to assure the positivity of $(L_+ z, z)$. From Proposition 3.1 in the work by M. Weinstein in [23], we have that $(L_+ z, z) \geq 0$ for any $z \in H^1$ such that $(z, u) = 0$ and that

$$\inf_{(f,u)=0}(L_+ f, f) = 0.$$

So, to obtain a lower bound on $(L_+ f, f)$, it is necessary to assume that the perturbed solution have the same square integral as the first component of the ground state,

$$\int_{\mathbb{R}} |\psi(t,x)|^2\, dx = \int_{\mathbb{R}} u(x)^2\, dx.$$

In this setting, we have that

$$(\Re v_1, u) = (h_1, u) = -\frac{1}{2}(||h_1||^2_{L^2} + ||h_2|||^2_{L^2}) = -\frac{1}{2}||v_1||^2_{L^2}.$$

In this case, we assume that $||u||_{L^2} = 1$ and decompose $h_1 = \Re v_1 \in H^1$ by $h_1 = f_1 + f_2$ in such a way that $(f_2, u) = 0$, meaning that

$$\begin{aligned}f_1 &= (h_1, u)u = -\frac{1}{2}||v_1|||^2_{L^2} u,\\ f_2 &= h_1 - (h_1, u)u = h_1 + \frac{1}{2}||v_1|||^2_{L^2} u.\end{aligned}$$

Moreover, we also have that

$$(L_+ h_1, h_1) = (L_+ f_1, f_1) + 2(L_+ f_1, f_2) + (L_+ f_2, f_2)$$

From the discussion above, we have that $(L_+ f_2, f_2) \geq 0$. Now, if we consider the infimun of $(L_+ f_2, f_2)/(f_2, f_2)$ subject to (3.15), we see that, if this were zero, it is attained at cu', but this contradicts (3.15), since the second component of Ψ_ω is $-M^2u^2$ and (3.15) reads

$$0 = \int_{\mathbb{R}} \left(3(M^2W - \sigma)u^2(u')^2 + M^2u^2(-M^2u^2)'\right)\, dx = 3(M^2W - \sigma)\int_{\mathbb{R}} u^2(u')^2\, dx.$$

As a consequence of this, we have that

$$
\begin{aligned}
(L_+f_2, f_2) &\geq C_3(f_2, f_2) = C_3((f, f) - (f_1, f_1)) \\
&\geq C_3\left(||h_1||_{L^2}^2 - \frac{1}{4}||\upsilon_1||_{L^2}^4\right), \\
(L_+f_1, f_1) &= \frac{1}{4}(||h_1||_{L^2}^2 + ||h_2||_{L^2}^2)^2(L_+u, u) = -\frac{1}{2}(M^2W - \sigma)||u||_{L^4}^4||\upsilon_1||_{L^4}^4, \\
(L_+f_1, f_2) &= -\frac{1}{2}(||h_1||_{L^2}^2 + ||h_2||_{L^2}^2)(L_+f_2, u) \\
&\geq -C_4||\upsilon_1||_{L^2}^2||\upsilon_1||_{H^1} \geq -C_4||\upsilon_1||_{H^1}^3.
\end{aligned}
$$

From this estimates, we have that

$$(L_+h_1, h_1) \geq D_1||h_1||_{H^1}^2 - D_2||\upsilon_1||_{H^1}^3 - D_3||\upsilon_1||_{H^1}^4.$$

for some positive constants $D_1, D_2, D_3 > 0$. So, we have shown that the estimate (3.14) holds, under the assumption $||\psi||_{L^2} = ||u||_{L^2}$. □

Now, we are in position to establish the stability result to the standing waves.

Theorem 3.2. *Let $\omega > \frac{c^2}{4\epsilon}$, $M^2 > 4D^2$ and $M^2W > \sigma$. Then for $c > 0$ large enough, the standing wave $\Psi_{\omega,c}(t, x) = (\psi(t, x), \rho(t, x), \varphi(t, x))$ where*

$$\psi(x, t) = e^{i\omega t}e^{\frac{ic}{2\epsilon}(x-ct)}u_{\omega,c}(x - ct), \quad \rho(x, t) = -M^2u_{\omega,c}^2(x - ct) \quad \varphi(x, t) = 0,$$

is orbitally stable, $u_{\omega,c}$ is the unique positive solution (3.3).

Proof. From estimates (3.7) and (3.14), in the case $||\psi|_{L^2} = ||u||_{L^2}$, we have the estimate

$$
\begin{aligned}
\Delta\mathcal{F}_{\omega,c}(t) \geq \frac{1}{2}(A_0 + D_1)||\upsilon_1||_{H^1}^2 + A_1||\upsilon_2||_{L^2}^2 + A_2||\upsilon_3'||_{L^2}^2 \\
- (D_2 + A_3)||\upsilon_1||_{H^1}^3 - (D_3 + A_4)||\upsilon_1||_{H^1}^4.
\end{aligned}
$$

Moreover, we also have for some positive constants A, B, C that

$$\Delta\mathcal{F}_{\omega,c}(t) \geq A||\mathcal{W}(t,\cdot)||_{X_{1,\frac{1}{2}}}^2\left(1 - B||\mathcal{W}(t,\cdot)||_{X_{1,\frac{1}{2}}} - C||\mathcal{W}(t,\cdot)||_{X_{1,\frac{1}{2}}}^2\right), \tag{3.17}$$

since we have that

$$||\mathcal{W}(t,\cdot)||_{X_{1,\frac{1}{2}}}^2 = ||\upsilon_1||_{H^1}^2 + ||\upsilon_2||_{L^2}^2 + ||\upsilon_3'||_{L^2}^2, \quad ||\upsilon_1||_{H^1} \leq ||\Upsilon(t,\cdot)||_{X_{1,\frac{1}{2}}}.$$

Now, let $\eta > 0$ be given. To remove the restriction on the L^2 norm, we take a ground state $\Psi_{\omega,\widetilde{c}}$ for $\widetilde{c}$ near c in such a way that $||\Psi_{\omega,c} - \Psi_{\omega,\widetilde{c}}||_{H^1} \leq \frac{\eta}{2}$ and that $||\psi||_{L^2} = ||u_{\omega,\widetilde{c}}||_{L^2}$. In fact, a direct computation shows that

$$||u_{\omega,c}||_{L^2}^2 = \frac{2\sqrt{\epsilon}\sqrt{\omega - \frac{c^2}{4\epsilon}}}{M^2W - \sigma}\int_{\mathbb{R}} sech^2(y)\, dy,$$

which implies that

$$\frac{||u_{\omega,c}||_{L^2}^2}{||u_{\omega,\widetilde{c}}||_{L^2}^2} = \frac{\sqrt{\omega - \frac{c^2}{4\epsilon}}}{\sqrt{\omega - \frac{\widetilde{c}^2}{4\epsilon}}}.$$

So, for $\delta(\eta) > 0$ small enough, we can choose $\widetilde{c}$ near c such that

$$\frac{||u_{\omega,c}||^2_{L^2}}{||\psi||^2_{L^2}} = \frac{\sqrt{\omega - \frac{c^2}{4\epsilon}}}{\sqrt{\omega - \frac{\widetilde{c}^2}{4\epsilon}}},$$

meaning that $||u_{\omega,\widetilde{c}}||^2_{L^2} = ||\psi||^2_{L^2}$. Therefore, we have that

$$||\Psi_{x_0,\gamma_0}(t,\cdot) - \Psi_{\omega,c}||_{H^1} \leq ||\Psi_{x_0,\gamma_0}(t,\cdot) - \Psi_{\omega,\tilde{c}}||_{H^1} + ||\Psi_{\omega,\tilde{c}} - \Psi_{\omega,c}||_{H^1},$$

which implies that the estimate (3.17) holds also in this case using the continuity of $\mathcal{F}_{\omega,c}$ and using the estimate of the deviation from $\Psi(t,\cdot)$ and $\Psi_{\omega,\tilde{c}}$. So, under previous estimates we have that

$$\Delta\mathcal{F}_{\omega,c}(t) \geq \nu(d_E(\Phi(t,\cdot),\mathcal{G}_\omega)),$$

where $\nu(y) = Ay^2(1 - By - Cy^2)$.

We see directly that the function ν is such that $\nu(0) = 0$, $\nu(y) > 0$ for $0 < y \ll 1$ and ν is an increasing function near zero. Now, by the continuity in $X_{1,\frac{1}{2}}$ of $\mathcal{F}_{\omega,c}$ near $\Psi_{\omega,c}$, we have for this $\eta > 0$, that there exists $\delta > 0$ such that

$$d_E(\Phi_0,\mathcal{G}_\omega) < \delta \quad \Rightarrow \quad \Delta\mathcal{F}_{\omega,c}(0) < \nu(\eta).$$

which implies for all $t > 0$ that

$$\nu(\eta) > \Delta\mathcal{F}_{\omega,c}(0) = \Delta\mathcal{F}_{\omega,c}(t) \geq \nu(d_E(\Phi(t,\cdot),\mathcal{G}_\omega)),$$

where we are using that $\Delta\mathcal{F}_{\omega,c}(t)$ is conserved in time. From this estimate and the properties of the function ν near zero, we conclude for all $t > 0$ that

$$d_E(\Psi(t,\cdot),\mathcal{G}_\omega)) < \eta,$$

as desired. □

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