TAMKANG JOURNAL OF MATHEMATICS
Volume 56, Number 1, 93-121, 2025
DOI:10.5556/j.tkjm.56.2025.5269

Homogenization of partial differential equations with Preisach operators

Achille Landri Pokam Kakeu

Abstract. The current work deals with initial boundary value parabolic problems with Preisach hysteresis whose the density functions are allowed to depend on the variable of space. The model contains nonlinear monotone operators in the diffusion term, arising from an energy. Thanks to the properties of Preisach hysteresis operators and to the sigma-convergence method, we obtain the convergence of the microscopic solutions to the solution of the homogenized problem. The effective operator is obtained in terms of a solution of a nonlinear corrector equation addressed in the usual sense of distributions, leading in an approximate scheme for the homogenized coefficient which is an important step towards the numerical implementation of the results from the homogenization theory beyond the periodic setting.

Keywords. Nonlinear problem; homogenization; Preisach Hysteresis; corrector

1 Introduction

1.1 Setting of the problem

It is widely known that materials whose properties are spatially inhomogeneous differ at different points. The objective of our work is to carry out the asymptotic behavior of the following problem modeling a diffusion process with spatially inhomogeneous Preisach hysteresis:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(c^\varepsilon u_\varepsilon + \mathcal{P}[u_\varepsilon(x,\cdot);x](t)\right) - \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon) = g \text{ in } Q = \Omega\times(0,T) \\ u_\varepsilon = 0 \text{ on } \partial\Omega\times(0,T) \text{ and } u_\varepsilon(x,0) = u^0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ approaching zero is a small parameter representing the scale of the inhomogeneities which are small compared with the global size of the material Ω, a bounded open set in $\mathbb{R}^d$ (integer $d \geq 1$). T is a given positive real number and the operator ∇ denotes the usual gradient, i.e. $\nabla = \left(\frac{\partial}{\partial x_i}\right)_{1\leq i\leq d}$. The operator div stands for the divergence operator with respect to the variable x, and $\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon)$ denotes the function defined on Q by $\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon)(x,t) = \boldsymbol{a}(x/\varepsilon,\nabla u_\varepsilon(x,t))$.

We suppose that the coefficients in (1.1) satisfy the following hypotheses:

Received date: January 6, 2024; Published online: June 6, 2024.
2010 *Mathematics Subject Classification*. 35B40, 46J10.
Corresponding author: Achille Landri Pokam Kakeu.

(**A1**) $\boldsymbol{a} = (a_i)_{1\le i\le d}$ is a function defined by $a_i(y,\lambda) = \frac{\partial J}{\partial \lambda_i}(y,\lambda)$ where the function $J : \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$ verifies the conditions below:

(i) $J(\cdot,\lambda)$ is measurable $\forall \lambda \in \mathbb{R}^d$,

(ii) $J(y,\cdot)$ is strictly convex for $a.a \quad y \in \mathbb{R}^d$,

(iii) There exist three constants $p \ge 2$, $\alpha_1 > 0$ and $\alpha_2 > 0$ verifying

$$\alpha_1 |\lambda|^p \le J(y,\lambda) \le \alpha_2(1 + |\lambda|^p) \tag{1.2}$$

for all $\lambda \in \mathbb{R}^d$ and for $a.a \quad y \in \mathbb{R}^d$.

(**A2**) $c^\varepsilon(x) = c(\frac{x}{\varepsilon})$ is a function, with $0 < \alpha \le c \in L^\infty(\mathbb{R}^d)$, where α is a constant not dependent on $y \in \mathbb{R}^d$.

(**A3**) $g \in L^2(Q)$ and $u^0 \in W_0^{1,p}(\Omega)$.

(**A4**) For any $x \in \Omega$, the Preisach operator $\mathcal{P}[\cdot;x]$ is continuous on $\mathcal{C}([0,T])$ and piecewise increasing. Furthermore $\mathcal{P}$ is affine bounded and there exist a function $\kappa_0 \in L^2(\Omega)$ and a constant γ_0 which is positive and such that, for all $\ell \in \mathbb{N}$, the parameterized final value mapping

$$(s,x) \mapsto \mathcal{P}_f(s;x), \;\; s = (v_0, ..., v_\ell) \in S$$

is measurable and verifies

$$|\mathcal{P}_f(s;x)| \le \kappa_0(x) + \gamma_0 \|s\|_\infty . \tag{1.3}$$

Here, $\mathcal{P}_f$ denotes the generating functional of the Preisach operator $\mathcal{P}$ and S stands for the set of all finite strings of real numbers, a string being as usual a vector having either finitely or countably infinitely many real components. In the sequel we will set

$$w_\varepsilon(x,t) = \mathcal{P}[u_\varepsilon(x,\cdot);x](t) \text{ in } Q.$$

Some information concerning the operator $\mathcal{P}$, useful for our work, are given in Section 2 below.

Apart from piecewise monotonicity and continuity, we need a further hypotheses on the Preisach operator $\mathcal{P}[\cdot;x]$, which will guarantee the uniqueness of the solution of (1.1) (see [21, Theorem 5.1 and Corollary 5.1]):

(**A5**) For every $x \in \Omega$, the Preisach operator $\mathcal{P}[\cdot;x]$ maps $W^{1,1}(0,T)$ into itself, and there exist $\gamma_1 > 0$ and $\kappa_1 \in L^2(\Omega)$ such that the condition

$$|(\mathcal{P}[v;x])'(t)| \le \kappa_1(x) + \gamma_1 |v'(t)| \quad \forall x \in \Omega, \text{ for a.e. } t \in (0,T)$$

holds for every $v \in W^{1,1}(0,T)$.

In (**A5**), v' stands for the time derivative (in the classical sense of distributions) of a function $v \in W^{1,1}(0,T)$.

1.2 Motivation

The hysteresis phenomenon can be defined as a rate independent memory effect. It is a nonlinear and natural phenomenon which occurs in lots of constructed systems (see e.g. [3, 14]).

The main questions arising in the asymptotic analysis are:

- Does the solution of the P.D.E converge to some limit function?
- If that is true, does the limit function solve some limit boundary value problem and can we describe it explicitly?

Answering these questions is the aim of the mathematical theory of homogenization. Let us point out that since the coefficients of the P.D.E describe the characteristic of the material at the microscale, it is not realistic to suppose that the coefficients are smooth, for instance, continuous. Consequently, in general, the suitable framework is that of weak solutions in Sobolev type spaces and variational formulations. Hence, the main challenge when passing to the limit is how to deal with products of two (or more) weakly convergent functions, which do not converge to the product of their weak limit. That is why we make use of the sigma-convergence which generalizes the well known two-scale convergence concept, and which is suited for our study. Let us mention that the goal of the mathematical homogenization is to model microscopically heterogeneous media, providing macroscopic models that describe their effective behavior. As an example of these media, we can consider composite materials, which are characterized by the fact that they contain two or more finely mixed constituents. These materials are widely used nowadays in industries, due to their properties. Indeed, they have in general a better behavior than the average behavior of its constituents. Well-known examples are the multifilamentary superconducting composites that are used in the composition of optical fibers.

1.3 Main contribution.

The heat conduction model (1.1) above describes unsaturated flow of a compressible fluid through a porous medium with hysteresis effect taken into account where the effect of gravitation is neglected. A great particularity lies in the existence issue of the corrector problem which will be addressed for this type of problem and will be for the first time determined as a solution of a P.D.E. in the usual sense of distribution. We will state the Sobolev type space $L^p_{uloc}(\mathbb{R}^d)$ which is actually the Wiener amalgam space $(L^p, \ell^\infty)(\mathbb{R}^d)$ introduced by Wiener [26] and we will also define the Sobolev type space $W^{1,p}_{uloc}(\mathbb{R}^d)$ accordingly. The nonlinear corrector associated to nonlinear monotone operators is nothing else but the distributional solution in the class of functions $u \in W^{1,p}_{loc}(\mathbb{R}^d)$ with $\nabla u \in L^p_{uloc}(\mathbb{R}^d)^d$, of the corrector problem given by

$$-\operatorname{div} \boldsymbol{a}(\cdot, r + \nabla u) = 0 \text{ in } \mathbb{R}^d. \tag{1.4}$$

The main contribution at this level is to solve (1.4) in the usual sense of distributions in $\mathbb{R}^d$, by looking for solutions of (1.4) verifying further $\nabla u \in B^p_A(\mathbb{R}^d)^d$, where B^p_A is the generalized Besicovitch space associated to the algebra with mean value A (see Section 4 below).

It is an urgent matter to highlight that solving the corrector problem in the sense of distribution lays the basics to the study of regularity results in the general deterministic setting beyond the periodic framework. We will go through a detailed proof to the existence of the corrector, based on the existence of the local weak solutions to nonlinear monotone equations. Making use of the Caccioppoli inequality, we will explicitly build an important estimate, which is sharp compared to its counterpart in [4, 30]. Therefore after proving the existence of a distributional corrector, we will look for an approximate scheme for the homogenized coefficient. Applications of our homogenization results are multiple, e.g. in understanding biological tissue behaviour, oil and gas extraction and geothermal energy systems. The obtained results will represent a crucial step as far as the numerical implementation of the results from the deterministic homogenization theory beyond the periodic setting is concerned.

Moreover, the homogenization theory connected to operators with hysteresis have already been investigated in the literature. We may cite a few of them ([7, 8, 9, 12, 13]). In the work [7], the author considered a Prandtl-Ishlinskii hysteresis operator of play type that is characterized by a distribution function, and the diffusion term of his problem is a linear operator. That author derived the homogenized equation by making use of the two-scale convergence concept introduced in [16]. In [9], a special attention has been paid to spatially inhomogeneous Prandtl-Ishlinskii operators by the authors. Those operators have then been homogenized by dealing with a sequence of equations of the above type with spatially periodic data. In reference [13], some properties of Preisach operators and the concept of two-scale convergence have been used by the author to obtain the effective operator. Concerning the homogenization of parabolic equations involving monotone operators, but without hysteresis operators, see e.g. [19, 20].

In the current work we generalize the results in the previous references by considering in the diffusion term a nonlinear monotone operator arising from a convex energy functional. Moreover, instead of solving the problem in the periodic framework, we deal with the general deterministic setting which includes as special cases, the periodic one, the almost periodic one, and others. Another great improvement is the possibility to compute numerically the effective parameters of the problem since the existence issue of the corrector equation is addressed here in the classical sense of distributions in $\mathbb{R}^d$. .

1.4 The plan

The layout of the paper is organised as follows. Section 2 is devoted to a description of the Preisach operator and its important properties. We also provide there a result on continuity of Preisach operators with respect to convergence of density functions. Section 3 is mainly concerned with some preliminaries in which we state a well-posedness result for (1.1) (for each freely fixed $\varepsilon > 0$) and we establish therein some useful uniform estimates. Details of the existence and uniqueness result can be found in [21] and thus are omitted here. Fundamental of the Σ-convergence concept are gathered in Section 4 while Section 5 deals with the existence of distributional corrector which is based on the existence of local weak solutions to nonlinear monotone problems. It is worth noticing that the linear counterpart of this result has been addressed in [11]. Next, Section 6 is concerned with the deterministic homogenization process, and we prove therein the main homogenization result for (1.1). Finally in Section 7, we draw a scheme showing us how the homogenized coefficient can be approximated by finite integral means.

2 The Preisach operator

The Preisach model is often applied to represent the magnetization of magnetic field relation in a ferromagnetic body formed by an aggregate of single-domain particles. Although the properties that can be derived from the Preisach model are in good qualitative agreement with the physical evidence, for several ferromagnetic materials there are quantitative discrepancies. This also applies to the vector Preisach model. Physicists and engineers proposed several variants of the originary Preisach model, in order to provide a more adequate model of ferromagnetic hysteresis.

The Preisach operator and its derivatives have been widely successfully used in the modeling of physical system with hysteresis. When dealing with the Preisach operator to model a physical system, it is really a density function that models this physical phenomenon. First of all, it is necessary to find this density function. In an application, we have to determine a density function for the Preisach operator using for example the input-output behaviour of the system at hand.

We recall that $BV(0,T)$ denotes the Banach space of functions $[0,T] \to \mathbb{R}$ having finite total variation. The simplest example of a hysteresis nonlinearity is given by a switch or relay with hysteresis

$$h_{v,r} : \mathcal{C}([0,T]) \times \{-1;1\} \to BV(0,T),$$

with input u seen as magnetic field and output $h_{v,r}$ as magnetization. The relay is characterized by two parameters $v \in \mathbb{R}$ as interaction field and $r > 0$ the critical field of coercivity and is defined formally as follows: Let $\mathbb{R}^2_+$ denote the set $\{(v,r) \in \mathbb{R}^2, r > 0\}$. For given parameters $(v,r) \in \mathbb{R}^2_+$ input $u \in \mathcal{C}([0,T])$, initial magnetization $\zeta \in \{-1;1\}$ and any time $t \in (0,T)$, we set

$$B_t = \{\tau \in]0,T] : u(\tau) = v - r \text{ or } u(\tau) = v + r\}. \tag{2.1}$$

Thus the function $h_{v,r}(u,\zeta) : [0,T] \to \{-1;1\}$ can be defined as follows:

$$h_{v,r}(u,\zeta)(0) = \begin{cases} -1 \text{ if } u(0) \le v - r \\ \zeta \text{ if } v - r < u(0) < v + r, \\ 1 \text{ if } u(0) \ge v + r \end{cases}$$

and

$$h_{v,r}(u,\zeta)(t) = \begin{cases} h_{v,r}(u,\zeta)(0) \text{ if } B_t = \emptyset \\ -1 \text{ if } B_t \ne \emptyset \text{ and } u(\max B_t) = v - r, \\ 1 \text{ if } B_t \ne \emptyset \text{ and } u(\max B_t) = v + r \end{cases}$$

Lemma 2.1. *Let $u \in \mathcal{C}([0,T])$ be given. For every $(v,r) \in \mathbb{R}^2_+$, put $\zeta := -1$ if $v \ge 0$, $\zeta = 1$ if $v < 0$. Then for all $t \in (0,T)$ and $(v,r) \in \mathbb{R}^2_+$, $v \ne \mathcal{E}_r[u](t)$ we have*

$$h_{v,r}(u,\zeta)(t) = \begin{cases} -1 \text{ if } v > \mathcal{E}_r[u](t) \\ 1 \text{ if } v < \mathcal{E}_r[u](t). \end{cases}$$

Therefore the output of the Preisach model is formally defined as an average over all elementary switches with a given density function $\psi \in L^1_{loc}(\mathbb{R}^2_+)$ by the following formula

$$\mathcal{P}[u](t) = \int_0^\infty \int_{-\infty}^\infty \psi_\varepsilon(v,r) h_{v,r}(u,\zeta)(t) dv dr, \tag{2.2}$$

where the initial values of the relays are taken as -1 if $v > 0$ and $+1$ otherwise. In order to justify the integration in (2.2) we can suppose that the antisymmetric part $\psi_a(v,r) = \frac{1}{2}(\psi(v,r) - \psi(-v,r))$ of ψ satisfies $\psi_a \in L^1(\mathbb{R}^2_+)$ and we can consider the integral in the sense of principal value. According to Lemma 2.1 on the representation of the relay by a system of plays, the output of the Preisach operator can be expressed as follows:

$$\mathcal{P}[u](t) = C' + \int_0^\infty g(\mathcal{E}_r[u](t), r) dr, \tag{2.3}$$

where

$$g(v,r) = \int_0^v \psi(z,r) dz, \tag{2.4}$$

C' is a constant and $\mathcal{E}_r[u](t)$ denotes the play operator.

Remark 1. It is important to note that the integral in (2.3) only make sense if $u \in \mathcal{C}([0,T])$ since $\mathcal{E}_r[u](t) = 0$ for r sufficiently large and $g(0,r) = 0$ for all $r > 0$.

Next, the following assumptions will be used:

(H1) There exists $\beta \in L^1_{loc}(0,\infty)$, $\beta \geq 0$ a.e. such that

$$0 \leq \psi(z,r) \leq \beta(r) \text{ for a.e. } (z,r) \in \mathbb{R}^2_+. \tag{2.5}$$

For $R > 0$ put $b(R) = \int_0^R \beta(r)dr$.

(H2) We have

$$\frac{d\psi}{dz} \in L^\infty(\mathbb{R}^2_+). \tag{2.6}$$

The following result presents the conditions under which the Preisach operator is Lipschitz continuous on $\mathcal{C}([0,T])$. In [25], we can find the proof of a more general version of this theorem.

Theorem 2.1. *Assume that* (H2) *holds. Then for every $u, v \in \mathcal{C}([0,T])$ verifying the inequalities*

$$\|u\|_{\mathcal{C}([0,T])} \leq R \quad \text{and} \quad \|v\|_{\mathcal{C}([0,T])} \leq R, \qquad \text{for some } R > 0.$$

The Preisach operator (2.3) *maps $\mathcal{C}([0,T]) \to \mathcal{C}([0,T])$ and verifies*

$$\|\mathcal{P}[u] - \mathcal{P}[v]\|_{\mathcal{C}([0,T])} \leq b(R)\, \|u - v\|_{\mathcal{C}([0,T])}\,. \tag{2.7}$$

Lemma 2.2. *Suppose that* (H1) *and* (H2) *hold. Then for $u \in W^{1,1}(0,T)$, $r > 0$ and $t \in (0,T)$, we have $\mathcal{P} \in W^{1,1}(0,T)$ and for a.e. $t \in (0,T)$, we have*

$$\dot{\mathcal{P}}[u](t) = \int_0^\infty \dot{\mathcal{E}}_r \psi(\mathcal{E}_r[u](t), r)dr, \tag{2.8}$$

where $\dot{\mathcal{P}} = \frac{\partial \mathcal{P}}{\partial t}$ and $\dot{\mathcal{E}}_r = \frac{\partial \mathcal{E}_r}{\partial t}$.

It emerges from Lemma 2.2 and from the definition of the play operator that the Preisach operator is piecewise monotone.

We have the following important result which will be useful for our purposes.

Theorem 2.2. *Suppose that* (H1) *and* (H2) *hold. Then the Preisach operator is piecewise monotone, i.e., for all $u \in W^{1,1}(0,T)$,*

$$\dot{\mathcal{P}}[u](t)\dot{u}(t) \geq 0 \text{ for a.e. } t \in (0,T) \tag{2.9}$$

Next we present a convergence result of spatially dependent Preisach operators exactly like in [12]. Consider the spatially dependent constitutive relation that can be described by the Preisach operator with a spatially dependent density function $\psi(x,y,z) \in L^1_{loc}(\Omega \times \mathbb{R}^2_+)$.

Theorem 2.3. *Let ψ_n be a sequence of space dependent density functions in $L^\infty(\Omega \times \mathbb{R}^2_+)$, satisfying the assumption $(H1)$ for a.e. $x \in \Omega$. We will assume that ψ_n converge to ψ in $L^\infty(\Omega \times \mathbb{R}^2_+)$ weakly star. Let us denote by $\mathcal{P}_n$ and $\mathcal{P}$ the Preisach operators corresponding to ψ_n and ψ respectively. Let u_n be a sequence in $L^2(\Omega, \mathcal{C}(0,T))$ and $\|u_n - u\|_{L^2(\Omega,C(0,T))} \to 0$ as $n \to \infty$.*

Then $\mathcal{P}_n[u_n](\cdot,t)$ converge to $\mathcal{P}[u](\cdot,t)$ for every $t \in (0,T)$ in $L^\infty(\Omega)$ weakly star.

Proof. We have for $a.e.$ $x \in \Omega$ and every $t \in (0,T)$

$$\begin{aligned}
&\int_0^\infty \int_0^{\mathcal{E}_r[u_n](t)} \psi_n(x,z,r)dzdr - \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} \psi(x,z,r)dzdr \\
&= \int_0^\infty \int_0^{\mathcal{E}_r[u_n](t)} \psi_n(x,z,r)dzdr - \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} \psi_n(x,z,r)dzdr \\
&\quad + \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr \\
&= \int_0^\infty \int_{\mathcal{E}_r[u](t)}^{\mathcal{E}[u_n](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr \\
&\quad + \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr \\
&= \int_0^\infty \int_{\mathcal{E}_r[u](t)}^{\mathcal{E}[u_n](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr \\
&\quad + \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr
\end{aligned}$$

The first integral on the right hand side of the last expression can be estimated by using the assumption $(H1)$ as follows

$$\int_0^\infty \int_{\mathcal{E}_r[u](t)}^{\mathcal{E}[u_n](t)} \psi_n(x,z,r)dzdr \leq \int_0^\infty \int_{\mathcal{E}_r[u](t)}^{\mathcal{E}[u_n](t)} \beta(r)dzdr$$

$$= \int_0^\infty \beta(r)\,[\mathcal{E}_r[u_n](t) - \mathcal{E}_r[u](t)]dr \leq \int_0^R \beta(r)\,|\mathcal{E}_r[u_n](t) - \mathcal{E}_r[u](t)|dr$$

for some $R > 0$. The later term can be further estimated by using the Lipschitz continuity of the play operator in $\mathcal{C}([0,T])$ as follows:

$$\int_0^R \beta(r)\,|\mathcal{E}_r[u_n](t) - \mathcal{E}_r[u](t)|dr \leq b(R)\,\|u_n - u\|_{C([0,T])}$$

where $b(R)$ is defined in $(H1)$. The estimates above imply that for every $t \in (0,T)$.

$$\begin{aligned}
&\|\mathcal{P}_n[u_n](t) - \mathcal{P}[u](t)\|_{L^p(\Omega)} \leq b(R)\,\|u_n - u\|_{L^p(\Omega,\mathcal{C}([0,T]))} + \\
&\left\| \int_0^\infty \int_0^{\mathcal{E}_r[u_n](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr \right\|.
\end{aligned}$$

The first term on the right hand side of the last inequality converges by assumptions to 0. To estimate the second term, $\mathcal{E}_r[u_n](t) = 0$ for r sufficiently large, and if $u(x,\cdot) \in \mathcal{C}([0,T])$ for *a.e.* $x \in \Omega$, $\mathcal{E}_r[u_n](t) \in \mathcal{C}([0,T])$, so the integral over r is on a finite interval, and converges to zero because of the assumption on the convergence of ψ_n . The statement follows. □

Remark 2. The convergence of ψ_n to ψ in $L^p(\Omega, L^1_{loc}(\mathbb{R}^2_+))$ can be easily replaced by the convergence in $L^\infty(\Omega, (\mathbb{R}^2_+))$ weakly star, as is typically the case we get in homogenization arguments, getting the weak star convergence of the Preisach operators in $L^\infty(\Omega)$.

3 Existence result and uniform estimates

3.1 Existence and uniqueness result

Let $\varepsilon > 0$ be fixed. Owing to assumption (**A1**), we can observe that the function

$$\boldsymbol{a}^{\varepsilon} : (x, \lambda) \mapsto \boldsymbol{a}^{\varepsilon}(x, \lambda) := \boldsymbol{a}(x/\varepsilon, \lambda) \tag{3.1}$$

from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}^d$ verifies the following well-known hypotheses:

$(\mathrm{H})_1$ For each $\lambda \in \mathbb{R}^d$, the function $x \mapsto \boldsymbol{a}^{\varepsilon}(x, \lambda)$ is measurable from Ω into $\mathbb{R}^d$.

$(\mathrm{H})_2$ There exists a positive constant α_3 such that $\boldsymbol{a}^{\varepsilon}(x, \lambda) \cdot \lambda \geq \alpha_1 |\lambda|^p - \alpha_3$.

$(\mathrm{H})_3$ There is a constant $C_2 > 0$, such that, a.e. in $x \in \Omega$, for $\lambda_1, \lambda_2 \in \mathbb{R}^d$,

$$(\boldsymbol{a}^{\varepsilon}(x, \lambda_1) - \boldsymbol{a}^{\varepsilon}(x, \lambda_2)) \cdot (\lambda_1 - \lambda_2) \geq 0,$$

$$|\boldsymbol{a}^{\varepsilon}(x, \lambda_1) - \boldsymbol{a}^{\varepsilon}(x, \lambda_2)| \leq C_2(1 + |\lambda_1| + |\lambda_2|)^{p-2} |\lambda_1 - \lambda_2|,$$

where the dot stands for the usual Euclidean inner product in $\mathbb{R}^d$, and $|\cdot|$ the associated norm.

Thanks to [19, Proposition 2.1] and [2], the diffusion term of the differential operator in (1.1) is well defined. More precisely, let $u \in L^p(0, T; W^{1,p}(\Omega))$; then $\boldsymbol{a}^{\varepsilon}(\cdot, \nabla u) \in L^{p'}(Q)^d$, as pointed out above. But we may as well see $\boldsymbol{a}^{\varepsilon}(\cdot, \nabla u)$ as a function in $L^{p'}(0, T; L^{p'}(\Omega)^d)$. Hence, $\operatorname{div} \boldsymbol{a}^{\varepsilon}(\cdot, \nabla u)$ turns out to rigorously represent the function $t \mapsto \operatorname{div} \boldsymbol{a}^{\varepsilon}(\cdot, \nabla u(\cdot, t))$ of $(0, T)$ into $W^{-1,p'}(\Omega)$, which lies in $L^{p'}(0, T; W^{-1,p'}(\Omega))$.

We are now able to define the notion of weak solution we will deal with in the sequel.

Definition 1. Let us assume that Assumptions (**A1**)-(**A4**) hold. Then, a function $u_{\varepsilon} : Q \to \mathbb{R}$ is said to be a weak solution of (1.1) if

$$\begin{cases} u_{\varepsilon} \in L^{\infty}(0, T; W_0^{1,p}(\Omega)) \text{ with } u_{\varepsilon}' \in L^2(0, T; L^2(\Omega)), \\ w_{\varepsilon} \in L^2(Q) \cap L^2(\Omega; \mathcal{C}([0, T])) \text{ with } w_{\varepsilon}' \in L^{p'}(0, T; W^{-1,p'}(\Omega)) \end{cases}$$

and u_{ε} verifies Eq. (3.2) below

$$\begin{cases} \int_Q c^{\varepsilon} u_{\varepsilon}'(x, t)\varphi(x, t) dx dt + \int_0^T \langle w_{\varepsilon}'(\cdot, t), \varphi(\cdot, t)\rangle \, dt \\ \quad + \int_Q \boldsymbol{a}^{\varepsilon}(x, \nabla u_{\varepsilon}(x, t)) \cdot \nabla\varphi(x, t) dx dt = \int_Q g(x, t)\varphi(x, t) dx dt \\ \varphi \in L^p(0, T; W_0^{1,p}(\Omega)). \end{cases} \tag{3.2}$$

The next result will be of interest in the work.

Lemma 3.1. *Consider the function* $\boldsymbol{a}$ *defined by* (**A1**), *i.e.* $\boldsymbol{a}(y, \lambda) = \nabla_{\lambda} J(y, \lambda)$ *where* ∇_{λ} *stands for the gradient with respect to* λ *of the function* J *defined in* (**A1**) *and verifying* (i)-(iii) *therein. For a freely fixed* $\varepsilon > 0$, *let* $\boldsymbol{a}^{\varepsilon}$ *be defined by* (3.1) *and verifying* $(\mathrm{H})_1$-$(\mathrm{H})_3$ *above. Suppose that* $u_{\varepsilon} \in L^{\infty}(0, T; W_0^{1,p}(\Omega))$, $u_{\varepsilon}' \in L^2(0, T; H_0^1(\Omega))$ *and* $\operatorname{div} \boldsymbol{a}^{\varepsilon}(\cdot, \nabla u_{\varepsilon}) \in L^{p'}(0, T; W^{-1,p'}(\Omega))$. *Then the function*

$$t \mapsto \sigma(u_{\varepsilon}(t)) := \int_{\Omega} J(\cdot, \nabla u_{\varepsilon}(t)) dx \tag{3.3}$$

is absolutely continuous on $(0,T)$ *and*

$$\frac{d}{dt}\sigma(u_\varepsilon(t)) = -\left\langle \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon(t)), u_\varepsilon'(t)\right\rangle \text{ for a.e. } t \in [0,T] \tag{3.4}$$

where $u_\varepsilon(t) = u_\varepsilon(\cdot, t)$ *and* $\langle\cdot,\cdot\rangle$ *denotes the duality pairings between* $L^{p'}(0,T;W^{-1,p'}(\Omega))$ *and* $L^p(0,T;W_0^{1,p}(\Omega))$.

The detailed proof of Lemma 3.1 can be found in [21].

Remark 3. It is worth noticing that Lemma 3.1 remains true if the assumptions

$$u_\varepsilon' \in L^2(0,T;H_0^1(\Omega)) \text{ and } \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon) \in L^{p'}(0,T;W^{-1,p'}(\Omega))$$

are replaced by the following ones therein:

$$u_\varepsilon' \in L^2(0,T;L^2(\Omega)) \text{ and } \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon) \in L^2(0,T;L^2(\Omega)), \tag{3.5}$$

the other ones remaining unchanged. In that case, the duality pairings $\langle\cdot,\cdot\rangle$ can be replaced by the inner product in $L^2(0,T;L^2(\Omega))$ and we may proceed by approximation like in [2, Proposition 2.11] to get (3.4) for the approximating sequence and then conclude like in [2, Lemma 3.3] for the passage to limit.

Theorem 3.1. *Let assumptions* (**A1**)-(**A4**) *be in force. Then for each* $\varepsilon > 0$ *there exists at least one solution* u_ε *in the sense of Definition 1. Moreover if assumption* (**A5**) *holds, then* u_ε *is unique and the following estimate is satified:*

$$\alpha \int_{t_1}^{t_2}\int_\Omega |u_\varepsilon'(x,t)|^2\,dxdt + 2\sigma(u_\varepsilon(t_2)) - 2\sigma(u_\varepsilon(t_1)) \le \frac{1}{\alpha}\int_{t_1}^{t_2}\int_\Omega |g(x,t)|^2\,dxdt \tag{3.6}$$

for all $0 \le t_1 \le t_2 \le T$. *Here* $\alpha > 0$ *is the same as in assumption* (**A2**) *and* $\sigma(\cdot)$ *is given by* (3.3).

This theorem has been established in [21] in which an existence and uniqueness result is stated and proved by using an implicit time discretization scheme together with a fundamental inequality due to M. Hilpert [10].

Remark 4. Specifically, u_ε lies in

$$V^p = \{v \in L^p(0,T;W_0^{1,p}(\Omega)) : v' = \frac{\partial v}{\partial t} \in L^2(0,T;L^2(\Omega))\}.$$

Endowed with the norm

$$\|v\|_{V^p} = \|v\|_{L^p(0,T;W^{1,p}(\Omega))} + \|v'\|_{L^2(0,T;L^2(\Omega))},$$

V^p is a Banach space. For further needs it is worth remarking that, since $p \ge 2$, the space $W_0^{1,p}(\Omega)$ is densely and continuously embedded in $L^2(\Omega)$. Consequently, identifying $L^2(\Omega)$ with his dual, it readily follows

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$$

with continuous embeddings. This has two important consequences:

1. We will use the same symbol, denoting both the inner product in $L^2(\Omega)$ and the duality pairing between the space $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.
2. The space V^p is continuously embedded in $\mathcal{C}([0,T];L^2(\Omega))$ (this is a well-known result). Hence, we can define $v(t)$ for $v \in V^p$ and $0 \le t \le T$, and further the mapping $v \to v(t)$ sends continuously V^p *into* $L^2(\Omega)$. Thus, we can consider the space $V_0^p = \{v \in V^p : v(0) = u^0\}$, a closed convex hull, which turns out to contain the solution u_ε of (1.1).

3.2 A priori estimates

Lemma 3.2. *Let assumptions* (**A1**)-(**A4**) *be satisfied. Then the solution* u_ε *of the problem* (1.1) *verifies the following estimates:*

$$\int_0^T\int_\Omega |u'_\varepsilon|^2 dxdt + \sup_{0\le t\le T}\Big(\|\nabla u_\varepsilon(t)\|^p_{L^p(\Omega)} + \|\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon))\|^p_{L^{p'}(0,T,W^{-1,p'}(\Omega))}\Big) \le C, \tag{3.7}$$

$$\|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le C, \quad \|\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon)\|_{L^{p'}(Q)} \le C, \quad \|w_\varepsilon\|_{L^2(Q)} \le C, \tag{3.8}$$

and

$$\left\|\frac{\partial}{\partial t}\left(c^\varepsilon u_\varepsilon + w_\varepsilon\right)\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C$$

where the constant C *depends on the domain* Ω*, the norm of* g *in* $L^{p'}(0,T,W^{-1,p'}(\Omega))$*,* $u^0 \in W_0^{1,p}(\Omega)$ *and the constant* α*.*

Proof. Let us test (1.1) by $u'_\varepsilon = \frac{\partial u_\varepsilon}{\partial t}$ and integrate over Ω to get, for $t>0$,

$$\int_\Omega c^\varepsilon \left|u_\varepsilon{}'(t)\right|^2 dx + \int_\Omega u_\varepsilon{}'(t)\frac{\partial w_\varepsilon}{\partial t}(t)dx - \langle \operatorname{div}\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon(t)), u'_\varepsilon(t)\rangle = \int_\Omega u_\varepsilon{}'(t)g(t)dx \tag{3.9}$$

where we have considered the abbreviation $u_\varepsilon(t) = u_\varepsilon(\cdot,t)$. Assuming that $\mathcal{P}[\cdot;x]$ is piecewise monotone for every $x\in\Omega$, we have $u'_\varepsilon\frac{\partial w_\varepsilon}{\partial t} \ge 0$, so that the second term of the left-hand side of (3.9) becomes non-negative, *i.e.* $\int_\Omega u_\varepsilon{}'(t)\frac{\partial w_\varepsilon}{\partial t}(t)dt \ge 0$. Since (see (3.4))

$$-\langle \operatorname{div}\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon(t)), u'_\varepsilon(t)\rangle = \frac{d}{dt}\sigma(u_\varepsilon(t))$$

where

$$\sigma(u_\varepsilon(t)) = \int_\Omega J(x,\nabla u_\varepsilon(x,t))dx, \tag{3.10}$$

we integrate (3.9) with respect to t and apply suitable Young's inequality to its right-hand side to get

$$\alpha\int_0^t\int_\Omega \left|u'_\varepsilon(\tau)\right|^2 dxd\tau + \sigma(u_\varepsilon(t)) - \sigma(u^0) \le \frac{1}{2\alpha}\int_0^t\int_\Omega |g|^2\,dxd\tau + \frac{\alpha}{2}\int_0^t\int_\Omega \left|u'_\varepsilon(\tau)\right|^2 dxd\tau,$$

where we have also considered the inequality $c^\varepsilon \ge \alpha$ from assumption (**A2**). We utilize the left-hand side of inequality (1.2), we infer

$$\int_0^t\int_\Omega \left|u'_\varepsilon(\tau)\right|^2 dxd\tau + \|\nabla u_\varepsilon(t)\|^p_{L^p(\Omega)} \le C\int_0^T\int_\Omega |g|^2\,dxdt + C\left\|\nabla u^0\right\|^p_{L^p(\Omega)}$$

where C depends only on α, α_1 and p. Hence we find the a priori estimate

$$\int_0^t\int_\Omega \left|u'_\varepsilon(\tau)\right|^2 dxd\tau + \sup_{0\le t\le T}\|\nabla u_\varepsilon(t)\|^p_{L^p(\Omega)} \le C\int_0^T\int_\Omega |g|^2\,dxdt + C\left\|\nabla u^0\right\|^p_{L^p(\Omega)} \tag{3.11}$$

where the constant C in (3.11) depends only on α, α_1 and p. Using the equality

$$\int_Q |u'_\varepsilon|^2 dxdt = \int_0^T\int_\Omega \left|u'_\varepsilon(t)\right|^2 dxdt,$$

we infer from (3.11) that

$$\int_Q |u_\varepsilon'|^2 dxdt + \sup_{0\leq t\leq T} \|\nabla u_\varepsilon(t)\|_{L^p(\Omega)}^p \leq C \tag{3.12}$$

where the positive constant C in (3.12) depends only on α, α_1, p, u^0 and g. Now, from the second inequality in $(\mathrm{H})_3$ we get

$$\begin{aligned}|\boldsymbol{a}^\varepsilon(x,\lambda)| &\leq C_2(1+|\lambda|)^{p-2}|\lambda| + \sup_{y\in\mathbb{R}^d}|\boldsymbol{a}(y,0)| \\ &\leq C(1+|\lambda|)^{p-1}\end{aligned}$$

where $C = \max(C_2, \sup_{y\in\mathbb{R}^d}|\boldsymbol{a}(y,0)|)$, and where we recall that $\sup_{y\in\mathbb{R}^d}|\boldsymbol{a}(y,0)| < \infty$ since J satisfies (1.2). Hence

$$|\boldsymbol{a}^\varepsilon(x,\lambda)|^{p'} \leq C(1+|\lambda|)^p \leq C(1+|\lambda|^p),$$

the constant in the last inequality above being depending on C_2, $\boldsymbol{a}$, and p. It follows readily from (3.12) that

$$\|\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon)\|_{L^{p'}(Q)}^{p'} \leq C(1+\|\nabla u_\varepsilon\|_{L^p(Q)}^p) \leq C \tag{3.13}$$

where C is a positive constant depending on the measure of Ω, C_2, $\boldsymbol{a}$, p and T.

Also

$$\|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))\cap H^1(0,T;L^2(\Omega))} \leq C. \tag{3.14}$$

The properties of the monotone operator $\boldsymbol{a}^\varepsilon$ together with (3.14) yield

$$\|\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon)\|_{L^{p'}(Q)} \leq C.$$

Finally we find from (3.2) and (3.7) that for any $\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$,

$$\left|\int_Q \frac{\partial w_\varepsilon}{\partial t}\varphi dxdt\right| \leq C\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))}, \tag{3.15}$$

so that

$$\left\|\frac{\partial w_\varepsilon}{\partial t}\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C. \tag{3.16}$$

According to assumption (**A4**), $\mathcal{P}$ is affine bounded, i.e. there exist $L > 0$ and $\upsilon \in L^2(\Omega)$ such that for every measurable function $u : \Omega \to \mathcal{C}([0,T])$ we have

$$\|\mathcal{P}(u_\varepsilon)(x,\cdot)\|_{\mathcal{C}([0,T])} \leq L\|u_\varepsilon(x,\cdot)\|_{\mathcal{C}([0,T])} + \upsilon(x) \text{ a.e. in } \Omega, \tag{3.17}$$

and using (3.14) and (3.17), we get

$$\|w_\varepsilon\|_{L^2(Q)} \leq \sqrt{T}\|w_\varepsilon\|_{L^2(\Omega;\mathcal{C}([0,T]))} \leq \sqrt{T}L\|u_\varepsilon\|_{L^2(\Omega;\mathcal{C}([0,T]))} + \sqrt{T}\|\upsilon\|_{L^2(\Omega)} \leq C.$$

So we obtain

$$\|w_\varepsilon\|_{L^2(Q)} \leq C.$$

The same reasoning as in (3.15) yields

$$\left\|\frac{\partial}{\partial t}(c^\varepsilon u_\varepsilon + w_\varepsilon)\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C.$$

The last point is to check that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in V^p. To this end, observe that

$$\int_0^T \left(c^\varepsilon {u_\varepsilon}'(t), v(t)\right) dt + \int_\Omega \left(\frac{\partial w_\varepsilon}{\partial t}(t), v(t)\right) dx + \int_Q \boldsymbol{a}^\varepsilon(x, \nabla u_\varepsilon(x,t)) \cdot \nabla v(x,t) dx dt = \int_0^T (g(t), v(t)) dt \tag{3.18}$$

for all $v \in V^p$, where $\varepsilon > 0$ is arbitrarily fixed. Taking in particular $v = u_\varepsilon$ and using the series of inequalities

$$0 \leq \frac{1}{2}\alpha \left\| u_\varepsilon(T) \right\|^2_{L^2(\Omega)} = \alpha \int_0^T \left({u_\varepsilon}'(t), u_\varepsilon(t)\right) dt \leq \int_0^T \left(c^\varepsilon {u_\varepsilon}'(t), u_\varepsilon(t)\right) dt \tag{3.19}$$

and the properties of $\boldsymbol{a}$, we obtain by mere routine

$$\sup_{\varepsilon>0} \left\| u_\varepsilon \right\|_{L^p(0,T,W_0^{1,p}(\Omega))} < \infty. \tag{3.20}$$

Using the hypothesis $(\mathrm{H})_1$-$(\mathrm{H})_3$, it follows

$$\sup_{\varepsilon>0} \left\| \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon) \right\|_{L^{p'}(Q)^d} < \infty, \tag{3.21}$$

hence $\sup_{\varepsilon>0} \left\| \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon) \right\|_{L^{p'}(0,T,W^{-1,p'}(\Omega))} < \infty$. We deduce by (1.1) that

$$\sup_{\varepsilon>0} \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} < \infty, \tag{3.22}$$

which combines with (3.20) to show that the sequence $(u_\varepsilon)_{\varepsilon\in E}$ is bounded in V^p. □

4 Sigma-convergence

We recall in this section the main properties and some basic facts about the concept of sigma-convergence. We refer the reader to [22, 23, 28, 29] for the details regarding most of the results of this section.

4.1 Algebra with mean value

Let A be an algebra with mean value on $\mathbb{R}^d$, that is, a closed subalgebra of the Banach algebra $\mathrm{BUC}(\mathbb{R}^d)$ (of bounded uniformly continuous real-valued functions on $\mathbb{R}^d$) that contains the constants, is translation invariant ($\tau_a u = u(\cdot + a) \in A$ for any $u \in A$ and $a \in \mathbb{R}^d$) and is such that any of its elements possesses a mean value in the following sense: for every $u \in A$,

$$M(u) = \lim_{R\to\infty} \fint_{B_R} u(y) dy \tag{4.1}$$

where B_R stands for the open ball in $\mathbb{R}^d$ of radius R centered at the origin and $\fint_{B_R} = \frac{1}{|B_R|}\int_{B_R}$.

Let $u \in \mathrm{BUC}(\mathbb{R}^d)$ and assume that $M(u)$ exists. Then defining the sequence $(u^\varepsilon)_{\varepsilon>0} \subset \mathrm{BUC}(\mathbb{R}^d)$ by $u^\varepsilon(x) = u(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}^d$, we have

$$u^\varepsilon \to M(u) \text{ in } L^\infty(\mathbb{R}^d)\text{-weak} * \text{ as } \varepsilon \to 0.$$

This is an easy consequence of the fact that the set of finite linear combinations of the characteristic functions of open balls in $\mathbb{R}^d$ is dense in $L^1(\mathbb{R}^d)$.

Let A be an algebra with mean value. Define the space A^∞ by

$$A^\infty = \left\{u \in A : D_y^\alpha u \in A \text{ for every } \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d\right\}.$$

Then endowed with the family of norms $|||\cdot|||_m$ defined by $|||u|||_m = \sup_{|\alpha|\leq m} \sup_{y\in\mathbb{R}^d} |D_y^\alpha u|$ where $D_y^\alpha = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1}...\partial y_d^{\alpha_d}}$, A^∞ is a Fréchet space.

In order to define the generalized Besicovitch space ([6, 18]), we first need to define the Marcinkiewicz space $\mathfrak{M}^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$, which is the space of functions $u \in L^p_{loc}(\mathbb{R}^d)$ satisfying $\limsup_{R\to\infty} \fint_{B_R} |u(y)|^p \, dy < \infty$. Endowed with the seminorm

$$\|u\|_p = \limsup_{R\to\infty} \left(\fint_{B_R} |u(y)|^p \, dy\right)^{\frac{1}{p}},$$

$\mathfrak{M}^p(\mathbb{R}^d)$ is a complete seminormed space. Next we define the generalized Besicovitch space $B_A^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$ associated to the algebra with mean value A as the closure in $\mathfrak{M}^p(\mathbb{R}^d)$ of A with respect to $\|\cdot\|_p$. It is easy to see that for $f \in A$ and $0 < p < \infty$, $|f|^p \in A$, so that

$$\|f\|_p = \left(\lim_{R\to\infty} \fint_{B_R} |f(y)|^p\right)^{\frac{1}{p}} \equiv (M(|f|^p))^{\frac{1}{p}}. \tag{4.2}$$

The equality (4.2) extends by continuity to any $f \in B_A^p(\mathbb{R}^d)$. Equipped with the seminorm (4.2), $B_A^p(\mathbb{R}^d)$ is a complete seminormed space. We refer the reader to [18, 28, 29] for further details about these spaces. Namely, the following holds true:

(1) The space $\mathcal{B}_A^p(\mathbb{R}^d) = B_A^p(\mathbb{R}^d)/\mathcal{N}$, (where $\mathcal{N} = \{u \in B_A^p(\mathbb{R}^d) : \|u\|_p = 0\}$) is a Banach space under the norm $\|u + \mathcal{N}\|_p = \|u\|_p$ for $u \in B_A^p(\mathbb{R}^d)$.

(2) The mean value $M : A \to \mathbb{R}$ extends by continuity to a continuous linear mapping (still denoted by M) on $B_A^p(\mathbb{R}^d)$. Furthermore, considered as defined on $B_A^p(\mathbb{R}^d)$, M extends in a natural way to $\mathcal{B}_A^p(\mathbb{R}^d)$ as follows: for $u = v + \mathcal{N} \in \mathcal{B}_A^p(\mathbb{R}^d)$, we set $M(u) := M(v)$; this is well defined since $M(v) = 0$ for any $v \in \mathcal{N}$.

To the space $B_A^p(\mathbb{R}^d)$ we attach the corrector space defined as follows:

$$B_{\#A}^{1,p}(\mathbb{R}^d) = \{u \in W_{loc}^{1,p}(\mathbb{R}^d) : \nabla u \in \left(B_A^p(\mathbb{R}^d)\right)^d \text{ and } M(\nabla u) = 0\}.$$

In $B_{\#A}^{1,p}(\mathbb{R}^d)$ we identify two elements by their gradients: $u = v$ in $B_{\#A}^{1,p}(\mathbb{R}^d)$ if and only if $\nabla(u - v) = 0$, i.e. $\|\nabla(u - v)\|_p = 0$. We may therefore equip $B_{\#A}^{1,p}(\mathbb{R}^d)$ with the gradient norm $\|u\|_{\#,p} = \|\nabla u\|_p$, which makes it a Banach space [6, Theorem 3.12] (or [5, Section 3]).

In the current work, we will deal with the concept of *ergodic* algebras with mean value. A function $u \in \mathcal{B}_A^1(\mathbb{R}^d)$ is said to be *invariant* if for any $y \in \mathbb{R}^d$, $\|u(\cdot + y) - u\|_1 = 0$. This being so, an algebra with mean value A is ergodic if every invariant function u is constant in $\mathcal{B}_A^1(\mathbb{R}^d)$, i.e. if $\|u(\cdot + y) - u\|_1 = 0$ for any $y \in \mathbb{R}^d$, then $\|u - c\|_1 = 0$ where c is a constant. We assume that all the algebras with mean value used in the sequel are ergodic.

4.2 Sigma-convergence

We begin with one underlying notion. By a *fundamental sequence* is meant any ordinary sequence of real numbers $0 < \varepsilon_n \leq 1$ such that $\varepsilon_n \to 0$ as $n \to \infty$. From now on, the letter E will stand for any subset of positive real numbers admitting 0 as accumulation point. We will always write $\varepsilon \to 0$ instead of $\varepsilon_n \to 0$ as $n \to \infty$. Let Ω be an open bounded set in $\mathbb{R}^d$, $1 \leq p < \infty$ and p' defined by $\frac{1}{p} + \frac{1}{p'} = 1$. We set $Q = \Omega \times (0,T)$, where $T > 0$ is fixed. Let A be an algebra with mean value on $\mathbb{R}^d$. In what follows, we keep using the same notations as in the preceding subsection.

Definition 2. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(Q)$ $(1 \leq p < \infty)$ is said to weakly Σ-converge in $L^p(Q)$ to some $u_0 \in L^p(Q; \mathcal{B}^p_A(\mathbb{R}^d))$ if as $E \ni \varepsilon \to 0$, we have

$$\int_Q u_\varepsilon(x,t) v\left(x, t, \frac{x}{\varepsilon}\right) dxdt \to \int_Q M\left(u_0(x,t,\cdot) v(x,t,\cdot)\right) dxdt \tag{4.3}$$

for any $v \in L^{p'}(Q; A)$, for a.e. $(x,t) \in Q$.

We express this by writing "$u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ."

Remark 5. The convergence (4.3) still holds true for $v \in \mathcal{C}(\overline{Q}; B^{p',\infty}_A(\mathbb{R}^d))$, where $B^{p',\infty}_A(\mathbb{R}^d) = B^{p'}_A(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$; see [17] for the justification.

Moreover the uniqueness of the limit u_0 is ensured, and it is also a fact that the weak Σ-convergence in L^p implies the weak convergence in L^p; see e.g. [17, 19, 28].

Definition 3. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(Q)$ $(1 \leq p < \infty)$ is said to strongly Σ-converge in $L^p(Q)$ to $u_0 \in L^p\left(Q; \mathcal{B}^p_A(\mathbb{R}^d)\right)$ if Definition 2 holds true and further $\|u_\varepsilon\|_{L^p(Q)} \to \|u_0\|_{L^p(Q;\mathcal{B}^p_A(\mathbb{R}^d))}$ as $E \ni \varepsilon \to 0$.

We denote it by "$u_\varepsilon \to u_0$ in $L^p(Q)$-strong Σ".

The following are the main properties of the concept of Σ-convergence; they are of utmost importance in the forthcoming homogenization process. We refer the reader to [18, 28, 29] for their proofs.

Theorem 4.1. *Let $(u_\varepsilon)_{\varepsilon \in E}$ (where E is a fundamental sequence) be a bounded sequence in $L^p(Q)$, $(1 < p < \infty)$. Then there exists a subsequence E' from E and a function $u \in L^p\left(Q, \mathcal{B}^p_A(\mathbb{R}^d)\right)$ such that the sequence $(u_\varepsilon)_{\varepsilon \in E'}$ weakly Σ-converges in $L^p(Q)$ to u.*

Theorem 4.2. *Let $(u_\varepsilon)_{\varepsilon \in E}$ (E a fundamental sequence) be a bounded ordinary sequence in $V^p = \{v \in L^p(0,T; W^{1,p}_0(\Omega)) : v' = \frac{\partial v}{\partial t} \in L^2(0,T; L^2(\Omega))\}$. Then there exist a subsequence E' of E and a couple $(u_0, u_1) \in V^p \times L^p(Q, B^{1,p}_{\#A}(\mathbb{R}^d))$ such that as $E' \ni \varepsilon \to 0$,*

- *$u_\varepsilon \to u_0$ in V^p-weak*
- *$u_\varepsilon \to u_0$ in $L^2(Q)$-strong*
- *$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}$ in $L^p(Q)$-weak Σ $(1 \leq j \leq d)$.*

Theorem 4.3. *Let $1 \leq p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ and $v_\varepsilon \to v_0$ in $L^q(Q)$-strong Σ, then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^r(Q)$-weak Σ.*

5 Corrector problem

The main objective of this section is to solve the corrector equation in the classical sense of distributions in $\mathbb{R}^d$. It reads as

$$-\operatorname{div} \boldsymbol{a}(\cdot, r + \nabla u_r) = 0 \text{ in } \mathbb{R}^d \tag{5.1}$$

where $r \in \mathbb{R}^d$ is a fixed parameter. Our aim is to prove the existence of solutions u_r of (5.1) in the space

$$B^{1,p}_{\#A}(\mathbb{R}^d) = \{v \in W^{1,p}_{loc}(\mathbb{R}^d) : \nabla v \in B^p_A(\mathbb{R}^d)^d \text{ and } M(\nabla v) = 0\}.$$

The resolution of (5.1) in the usual sense of distributions has the advantage of making posssible the computation of the homogenized coefficients (that depend on the corrector u_r) and thus to find a numerical scheme to approximate the solution of the homogenized problem. This is an advance concerning the deterministic homogenization beyond the periodic setting. The existence of u_r is based on the existence of approximate correctors which are weak solutions to Eq (5.5) below, in the locally uniform space $L^p_{uloc}(\mathbb{R}^d)$. Let us define the space $L^p_{uloc}(\mathbb{R}^d)$ $(1 \leq p < \infty)$ as the subspace of $L^p_{loc}(\mathbb{R}^d)$ which consists of functions u verifying

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} |u|^p \, dy < \infty$$

where $B(x, 1)$ is the unit ball in $\mathbb{R}^d$ centered at x. Endowed with the norm

$$\|u\|_{L^p_{uloc}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \left(\int_{B(x,1)} |u|^p \, dy \right)^{\frac{1}{p}}, \tag{5.2}$$

$L^p_{uloc}(\mathbb{R}^d)$ is a Banach space. We can also define the Sobolev-type space $W^{1,p}_{uloc}(\mathbb{R}^d)$ accordingly. The norm (5.2) may be replaced by any of the following equivalent ones:

$$\|u\|_{L^p_{uloc}(\mathbb{R}^d)} \approx \sup_{\ell\in\mathbb{Z}^d} \left(\int_{\ell+(0,1)^d} |u|^p \, dy \right)^{\frac{1}{p}} \approx \sup_{\ell\in\mathbb{Z}^d} \left(\int_{\mathbb{R}^d} \varphi(y-\ell)^p \, |u(y)|^p \, dy \right)^{\frac{1}{p}} \tag{5.3}$$

where φ is any nonnegative function in $\mathcal{C}^\infty_0(\mathbb{R}^d)$ such that $\sum_{k\in\mathbb{Z}^d} \varphi(y-k) \geq c_0 > 0$ for all $y \in \mathbb{R}^d$.

If we have to solve (5.1), we need to proceed in several steps. Firstly we solve a general problem posed as follows. Let us suppose $(h, H) \in L^{p'}_{uloc}(\mathbb{R}^d) \times L^{p'}_{uloc}(\mathbb{R}^d)^d$ $(p' = p/(p-1))$, and let us fix $r \in \mathbb{R}^d$. We consider the following equation

$$-\operatorname{div} \boldsymbol{a}(\cdot, r + \nabla u) + |u|^{p-2}\, u = h + \operatorname{div} H \text{ in } \mathbb{R}^d. \tag{5.4}$$

We are able to show that it possesses a solution in $W^{1,p}_{uloc}(\mathbb{R}^d)$. The second step consists of scaling Eq. (5.4) in order to obtain the equation for the general approximate corrector, viz.

$$-\operatorname{div} \boldsymbol{a}(\cdot, r + \nabla u_T) + T^{-p}\, |u_T|^{p-2}\, u_T = h + \operatorname{div} H \text{ in } \mathbb{R}^d. \tag{5.5}$$

The final step is to make use of the sequence $(u_T)_{T\geq 1}$ of solutions of (5.5) in order to show that Eq (5.1) possesses at least a distributional solution whose gradient is unique.

To begin with, the following holds true.

Proposition 5.1. *Let us assume that assumptions* (H)$_1$, (H)$_2$ *and* (H)$_3$ *hold. Then there exists at least a function* $u \in W^{1,p}_{uloc}(\mathbb{R}^d)$ *solution to* (5.4)*. Moreover* u *verifies the uniform local estimate which reads as:*

$$\sup_{x\in\mathbb{R}^d}\int_{B(x,1)}(|\nabla u|^p+|u|^p)\,dy \le C + C\sup_{x\in\mathbb{R}^d}\int_{B(x,1)}\left(|h|^{p'}+|H|^{p'}\right)dy \tag{5.6}$$

where $C = C(r,\alpha_1,\alpha_3,c_2,d) > 0$.

Proof. For a freely fixed $R \ge 1$, let us uniquely define $u_R \in W^{1,p}_0(B_R)$ by

$$-\nabla\cdot\boldsymbol{a}(\cdot, r+\nabla u_R) + |u_R|^{p-2}\,u_R = h + \nabla\cdot H \text{ in } B_R. \tag{5.7}$$

The existence of u_R is given by [15] and its uniqueness is ensured by [1]. We then extend u_R by 0 outside B_R to get a sequence $(u_R)_R$ in $W^{1,p}_{loc}(\mathbb{R}^d)$. Now we can show that the sequence $(u_R)_R$ is bounded in $W^{1,p}_{uloc}(\mathbb{R}^d)$. Consider $0 \le \varphi_0 \in \mathcal{C}^\infty_0(B(0,1))$ be such that $\sum_{\ell\in\mathbb{Z}^d}\varphi_0(\cdot-\ell)$ is bounded from below by a non negative constant and $|\nabla\varphi_0| \le \kappa_0$ where the constant κ_0 will be correctly chosen in the body of the proof. within the variational form of (5.7) we may consider the test function $\varphi^p_{x_0}u_R$ where $\varphi_{x_0} = \varphi_0(\cdot - x_0)$ with $x_0 \in \mathbb{R}^d$ being fixed. Then

$$\begin{aligned}
&\int_{B_R}\varphi^p_{x_0}\boldsymbol{a}(\cdot,r+\nabla u_R)\cdot(r+\nabla u_R) + \int_{B_R}\varphi^p_{x_0}|u_R|^p \\
&= -p\int_{B_R}\varphi^{p-1}_{x_0}u_R\nabla\varphi_{x_0}\cdot\boldsymbol{a}(\cdot,r+\nabla u_R) + \int_{B_R}\varphi^p_{x_0}\boldsymbol{a}(\cdot,r+\nabla u_R)\cdot r + \int_{B_R}\varphi^p_{x_0}hu_R \\
&\quad - p\int_{B_R}(H\cdot\nabla\varphi_{x_0})u_R\varphi^{p-1}_{x_0} - \int_{B_R}\varphi^p_{x_0}H\cdot\nabla u_R.
\end{aligned}$$

Thanks to the properties of the function $\boldsymbol{a}$ (see (H)$_2$-(H)$_3$) we get

$$\begin{aligned}
&\alpha_1\int_{B_R}\varphi^p_{x_0}|r+\nabla u_R|^p + \int_{B_R}\varphi^p_{x_0}|u_R|^p \\
&\le pc_2\int_{B_R}\varphi^{p-1}_{x_0}|u_R|\,|\nabla\varphi_{x_0}| + pc_2\int_{B_R}\varphi^{p-1}_{x_0}|u_R|\,|\nabla\varphi_{x_0}|\,|r+\nabla u_R|^{p-1} \\
&+ c_2\int_{B_R}\varphi^p_{x_0}|r| + c_2\int_{B_R}\varphi^p_{x_0}|r|\,|r+\nabla u_R|^{p-1} + p\int_{B_R}|H|\,|\nabla\varphi_{x_0}|\,|u_R|\,\varphi^{p-1}_{x_0} \\
&+ \int_{B_R}\varphi^p_{x_0}|H|\,|\nabla u_R| + \int_{B_R}\varphi^p_{x_0}|h|\,|u_R| + \alpha_3\int_{B_R}\varphi^p_{x_0} = \sum_{i=1}^{8}I_i.
\end{aligned} \tag{5.8}$$

We have to estimate each term above separately. To go on with, let us fix a constant $k > 0$ that will be determined later. Consider $\delta > 0$ an universal constant such that

$$\sup_{x\in\mathbb{R}^d}\int_{B(x,1)}|v|^p\,dy \le \delta\sup_{x\in\mathbb{R}^d}\int\varphi^p_{x_0}|v|^p\,dy \text{ for all } v\in L^p_{uloc}(\mathbb{R}^d). \tag{5.9}$$

Young's Inequality gives rise to the ensuing estimates.

For I_2, we get

$$I_2 = p\int_{B_R}\left[\left(\frac{\alpha_1c_2}{k}\right)^{\frac{1}{p}}|u_R|\,|\nabla\varphi_{x_0}|\right]\left[c_2^{\frac{1}{p'}}\left(\frac{k}{\alpha_1}\right)^{\frac{1}{p}}\varphi^{p-1}_{x_0}|r+\nabla u_R|^{p-1}\right]$$

$$
\begin{aligned}
&\leq \frac{\alpha_1 c_2}{k} \int_{B_R} |u_R|^p \, |\nabla \varphi_{x_0}|^p + (p-1) c_2 \left(\frac{k}{\alpha_1}\right)^{\frac{1}{p-1}} \int_{B_R} \varphi_{x_0}^p \, |r + \nabla u_R|^p \\
&\leq \frac{\alpha_1 c_2}{k} \kappa_0^p \int_{B(x_0,1)} 1_{B_R} \, |u_R|^p + (p-1) c_2 \left(\frac{k}{\alpha_1}\right)^{\frac{1}{p-1}} \int_{B_R} \varphi_{x_0}^p \, |r + \nabla u_R|^p .
\end{aligned}
$$

As far as I_4 is concerned,

$$
\begin{aligned}
I_4 &= \int_{B_R} \left([c_2 p]^{\frac{1}{p'}} \left(\frac{k}{\alpha_1}\right)^{\frac{1}{p}} \varphi_{x_0}^{p-1} \, |r + \nabla u_R|^{p-1} \right) \left(\frac{c_2^{\frac{1}{p}}}{p^{\frac{1}{p'}}} \, |r| \, \varphi_{x_0} \right) \\
&\leq (p-1) c_2 \left(\frac{k}{\alpha_1}\right)^{\frac{1}{p-1}} \int_{B_R} \varphi_{x_0}^p \, |r + \nabla u_R|^p + \frac{c_2}{p^p} \, |r|^p \int_{B_R} \varphi_{x_0}^p .
\end{aligned}
$$

Next, for I_1 and I_3 we easily obtain

$$
I_3 \leq c_2 \, |r| \int_{B_R} \varphi_{x_0}^p
$$

and

$$
\begin{aligned}
I_1 &= p \int_{B_R} \left[\left(\frac{\alpha_1 c_2}{k} \right)^{\frac{1}{p}} |u_R| \, |\nabla \varphi_{x_0}| \right] \left(c_2^{\frac{1}{p'}} \varphi_{x_0}^{p-1} \right) \\
&\leq \frac{\alpha_1 c_2}{k} \int_{B_R} |u_R|^p \, |\nabla \varphi_{x_0}|^p + (p-1) c_2 \int_{B_R} \varphi_{x_0}^p \\
&\leq \frac{\alpha_1 c_2}{k} \kappa_0^p \int_{B(x_0,1)} 1_{B_R} \, |u_R|^p + (p-1) c_2 \int_{B_R} \varphi_{x_0}^p .
\end{aligned}
$$

For I_5 and I_7 we get

$$
\begin{aligned}
|I_5| &\leq p \int_{B_R} \left[\left(\frac{\alpha_1 c_2}{k} \right)^{1/p} |u_R| \, |\nabla \varphi_{x_0}| \right] \left[|H| \, \varphi_{x_0}^{p-1} \left(\frac{k}{\alpha_1 c_2} \right)^{1/p} \right] \\
&\leq \frac{\alpha_1 c_2}{k} \int_{B_R} |u_R|^p \, |\nabla \varphi_{x_0}|^p + (p-1) \left(\frac{k}{\alpha_1 c_2} \right)^{\frac{1}{p-1}} \|\varphi_0\|_\infty^{p-p'} \int_{B_R} \varphi_{x_0}^{p'} \, |H|^{p'} \\
&\leq \frac{\alpha_1 c_2}{k} \kappa_0^p \int_{B(x_0,1)} 1_{B_R} \, |u_R|^p + (p-1) \left(\frac{k}{\alpha_1 c_2} \right)^{\frac{1}{p-1}} \|\varphi_0\|_\infty^{p-p'} \int_{B_R} \varphi_{x_0}^{p'} \, |H|^{p'} ;
\end{aligned}
$$

$$
\begin{aligned}
|I_7| &\leq \int_{B_R} \left[\left(\frac{\alpha_1 c_2 p}{k} \right)^{1/p} \kappa_0 \, |u_R| \, \varphi_{x_0} \right] \left[\frac{1}{\kappa_0} \, |h| \, \varphi_{x_0}^{p-1} \left(\frac{\alpha_1 c_2 p}{k} \right)^{-\frac{1}{p}} \right] \\
&\leq \frac{\alpha_1 c_2}{k} \kappa_0^p \int_{B_R} \varphi_{x_0}^p \, |u_R|^p + \frac{1}{p' \kappa_0^{p'}} \left(\frac{\alpha_1 c_2 p}{k} \right)^{-\frac{1}{p-1}} \int_{B_R} \varphi_{x_0}^p \, |h|^{p'} \\
&\leq \frac{\alpha_1 c_2}{k} \kappa_0^p \delta \int_{B_R} \varphi_{x_0}^p \, |u_R|^p + \frac{1}{p' \kappa_0^{p'} \delta^{\frac{p'}{p}}} \left(\frac{\alpha_1 c_2 p}{k} \right)^{-\frac{1}{p-1}} \|\varphi_0\|_\infty^{p-p'} \int_{B_R} \varphi_{x_0}^{p'} \, |h|^{p'}
\end{aligned}
$$

where δ is given by (5.9) and where in the last terms in the last inequalities above in I_5 and in I_7, we have considered the fact that $p' \leq p$ since $p \geq 2$, so that $\varphi_{x_0}^p \leq \|\varphi_0\|_\infty^{p-p'} \varphi_{x_0}^{p'}$. As for I_6 we have

$$
|I_6| \leq \int_{B_R} \varphi_{x_0}^p \, |H| \, |r + \nabla u_R| + |r| \int_{B_R} \varphi_{x_0}^p \, |H|
$$

$$\begin{aligned}&\leq (p-1)c_2\left(\frac{k}{\alpha_1}\right)^{\frac{1}{p-1}}\int_{B_R}\varphi_{x_0}^p\,|r+\nabla u_R|^p+\frac{C_0^{-\frac{1}{p-1}}}{p'}\int_{B_R}\varphi_{x_0}^p\,|H|^{p'}\\&+\frac{1}{p}\int_{B_R}\varphi_{x_0}^p+\frac{1}{p'}\int_{B_R}\varphi_{x_0}^p\,|H|^{p'}\\&\leq (p-1)c_2\left(\frac{k}{\alpha_1}\right)^{\frac{1}{p-1}}\int_{B_R}\varphi_{x_0}^p\,|r+\nabla u_R|^p+\frac{C_0^{-\frac{1}{p-1}}+1}{p'}\,\|\varphi_0\|_\infty^{p-p'}\int_{B_R}\varphi_{x_0}^{p'}\,|H|^{p'}\\&+\frac{1}{p}\int_{B_R}\varphi_{x_0}^p\end{aligned}$$

where we have set $C_0=p(p-1)c_2\left(\frac{k}{\alpha_1}\right)^{\frac{1}{p-1}}$. Putting the above estimates together and taking the $\sup_{x_0\in\mathbb{R}^d}$ we reach at

$$\begin{aligned}&\left[\alpha_1-3(p-1)c_2\left(\frac{k}{\alpha_1}\right)^{\frac{1}{p-1}}\right]\|1_{B_R}(r+\nabla u_R)\|^p_{L^p_{uloc}(\mathbb{R}^d)}\\&+\left[1-4\frac{\alpha_1c_2}{k}\kappa_0^p\delta\right]\|1_{B_R}u_R\|^p_{L^p_{uloc}(\mathbb{R}^d)}\\&\leq\left((p-1)c_2\,\|1_{B_R}\|^p_{L^p_{uloc}(\mathbb{R}^d)}+c_2\,|r|+\frac{c_2}{p^p}\,|r|^p+\frac{1}{p}+\alpha_3\right)\|1_{B_R}\|^p_{L^p_{uloc}(\mathbb{R}^d)}\\&+\|\varphi_0\|_\infty^{p-p'}\left(\frac{C_0^{-\frac{1}{p-1}}+1}{p'}+(p-1)\left(\frac{k}{\alpha_1c_2}\right)^{\frac{1}{p-1}}\right)\|H\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)}\\&+\frac{1}{p'\kappa_0^{p'}\delta^{\frac{p'}{p}}}\left(\frac{\alpha_1c_2p}{k}\right)^{-\frac{1}{p-1}}\|\varphi_0\|_\infty^{p-p'}\,\|h\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)}\\&\leq C(r,p,c_2,\alpha_3)+C(\alpha_1,p,c_2,\alpha_3,d)(\|h\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)}+\|H\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)}).\end{aligned}$$

Then we choose k and κ_0 such that

$$\alpha_1-3(p-1)c_2\left(\frac{k}{\alpha_1}\right)^{\frac{1}{p-1}}=\frac{\alpha_1}{2}\text{ and }1-4\frac{\alpha_1c_2}{k}\kappa_0^p\delta=\frac{1}{2}$$

to obtain the estimate

$$\|1_{B_R}(r+\nabla u_R)\|^p_{L^p_{uloc}(\mathbb{R}^d)}+\|1_{B_R}u_R\|^p_{L^p_{uloc}(\mathbb{R}^d)}\leq C+C(\|h\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)}+\|H\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)})\tag{5.10}$$

where the constant C does not depend on R. It readily follows from the estimate (5.10) that the sequence $(u_R)_R$ is bounded in $W^{1,p}_{uloc}(\mathbb{R}^d)$, and thus in $W^{1,p}_{loc}(\mathbb{R}^d)$. Hence, there exist $u\in W^{1,p}_{loc}(\mathbb{R}^d)$ and a subsequence not relabeled such that

$$u_R\to u\text{ in }W^{1,p}_{loc}(\mathbb{R}^d)\text{-weak.}$$

It is straightforward to see that u is a solution of (5.4), and that from the lower semi-continuity, the next estimate holds true:

$$\|r+\nabla u\|^p_{L^p_{uloc}(\mathbb{R}^d)}+\|u\|^p_{L^p_{uloc}(\mathbb{R}^d)}\leq C+C(\|h\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)}+\|H\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)})\tag{5.11}$$

where $C=C(\alpha_1,p,c_2,\alpha_3,d,r)>0$. It follows that $u\in W^{1,p}_{uloc}(\mathbb{R}^d)$ and further

$$\|\nabla u\|^p_{L^p_{uloc}(\mathbb{R}^d)}+\|u\|^p_{L^p_{uloc}(\mathbb{R}^d)}\leq C+C(\|h\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)}+\|H\|^{p'}_{L^{p'}_{uloc}(\mathbb{R}^d)})\tag{5.12}$$

for a positive constant C depending on the same occurrences as in (5.11). This completes the proof of our result. □

The following result provides us with the solution of (5.5).

Lemma 5.1. *Consider $T \geq 1$ and let $r \in \mathbb{R}^d$ be fixed. Suppose $(h, H) \in B_A^{p'}(\mathbb{R}^d) \times B_A^{p'}(\mathbb{R}^d)^d$ ($p' = p/(p-1)$). Then the equation (5.5) possesses at least one solution $u_T \in B_A^{1,p}(\mathbb{R}^d)$ verifying*

$$\sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} \left(|\nabla u_T|^p + T^{-p}|u_T|^p\right) \leq C + C \sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} \left(T^{p'}|h|^{p'} + |H|^{p'}\right) \tag{5.13}$$

where $C = C(\alpha_1, p, c_2, \alpha_3, d, r) > 0$. Furthermore u_T is unique up to addition of a function $v \in B_A^{1,p}(\mathbb{R}^d)$ such that $M(|v|^p) = 0$.

Proof. In (5.5) we need to make the change of unknown function $v_T(y) = T^{-1}u_T(Ty)$. Then v_T is the solution of the equation

$$-\operatorname{div} \boldsymbol{a}_T(\cdot, r + \nabla v_T) + |v_T|^{p-2} v_T = h_T + \operatorname{div} H_T \text{ in } \mathbb{R}^d$$

where $\boldsymbol{a}_T(y,\xi) = \boldsymbol{a}(Ty,\xi)$, $h_T(y) = Th(Ty)$ and $H_T(y) = H(Ty)$. Noticing that $(h_T, H_T) \in L_{uloc}^{p'}(\mathbb{R}^d) \times L_{uloc}^{p'}(\mathbb{R}^d)^d$ since $B_A^{p'}(\mathbb{R}^d) \subset L_{uloc}^{p'}(\mathbb{R}^d)$, we infer from Proposition 5.1 that v_T (and thus u_T) exists in $W_{uloc}^{1,p}(\mathbb{R}^d)$ and verifies

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} \left(|\nabla v_T|^p + |v_T|^p\right) \leq C + C \sup_{x\in\mathbb{R}^d} \int_{B(x,1)} \left(|h_T|^{p'} + |H_T|^{p'}\right),$$

i.e.

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} T^{-p}\left(|\nabla u_T(Ty)|^p + |u_T(Ty)|^p\right) dy \leq C + C \sup_{x\in\mathbb{R}^d} \int_{B(x,1)} \left(T^{p'}|h(Ty)|^{p'} + |H(Ty)|^{p'}\right) dy.$$

Then making the change of variables $z = Ty$, we obtain

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d} T^{-d} \int_{B(Tx,T)} T^{-p}\left(|\nabla u_T(z)|^p + |u_T(z)|^p\right) dz \\ &\leq C + C \sup_{x\in\mathbb{R}^d} T^{-d} \int_{B(Tx,T)} \left(T^{p'}|h(z)|^{p'} + |H(z)|^{p'}\right) dz, \end{aligned}$$

or equivalently

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} T^{-p}\left(|\nabla u_T(z)|^p + |u_T(z)|^p\right) dz \\ &\leq C + C \sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} \left(T^{p'}|h(z)|^{p'} + |H(z)|^{p'}\right) dz. \end{aligned}$$

The proof of the fact that $u_T \in B_A^{1,p}(\mathbb{R}^d)$ can be seen in [4, proof of Lemma 2.1]. □

Lemma 5.2. *Consider the weak solution u_T of (5.5) given by Lemma 5.1. Then it holds*

$$\sup_{x\in\mathbb{R}^d} \fint_{B(x,R)} \left(|\nabla u_T|^p + T^{-p}|u_T|^p\right) dy \leq C + C \sup_{x\in\mathbb{R}^d} \fint_{B(x,R)} \left(|H|^{p'} + T^{p'}|h|^{p'}\right) dy \tag{5.14}$$

for any $R \geq T$, where $C = C(\alpha_1, p, c_2, \alpha_3, d, r) > 0$.

Proof. The proof uses Caccioppoli's inequality which we first state for (5.5). Let $\eta \in \mathcal{C}_0^\infty(B_{2R}(x))$ be a standard cut-off function: $\eta = 1$ in $B_R(x)$, $0 \le \eta \le 1$ and $|\nabla \eta| \le CR^{-1}$. We test (5.5) with $\eta^p u_T$ to get

$$\int_{B_{2R}(x)} \boldsymbol{a}(\cdot, r + \nabla u_T) \cdot \nabla(\eta^p u_T) + T^{-p} \int_{B_{2R}(x)} \eta^p |u_T|^p = \int_{B_{2R}(x)} (\eta^p u_T h - H \cdot \nabla(\eta^p u_T)) .$$

Making use of the properties of $\boldsymbol{a}$ we get

$$\begin{aligned}
&\alpha_1 \int_{B_{2R}(x)} \eta^p |r + \nabla u_T|^p + T^{-p} \int_{B_{2R}(x)} \eta^p |u_T|^p \\
&\le pc_2 \int_{B_{2R}(x)} \eta^{p-1} |u_T| |\nabla \eta| + pc_2 \int_{B_{2R}(x)} \eta^{p-1} |u_T| |\nabla \eta| |r + \nabla u_T|^{p-1} \\
&+ c_2 \int_{B_{2R}(x)} \eta^p |r| + c_2 \int_{B_{2R}(x)} \eta^p |r| |r + \nabla u_T|^{p-1} + p \int_{B_{2R}(x)} \eta^{p-1} |H| |u_T| |\nabla \eta| \\
&+ \int_{B_{2R}(x)} \eta^p |H| |r + \nabla u_T| + |r| \int_{B_{2R}(x)} \eta^p |H| + \int_{B_{2R}(x)} \eta^p |h| |u_T| + \alpha_3 \int_{B_{2R}(x)} \eta^p \\
&\le C \int_{B_{2R}(x)} |u_T|^p |\nabla \eta|^p + C \int_{B_{2R}(x)} \eta^p + \frac{\alpha_1}{4} \int_{B_{2R}(x)} \eta^p |r + \nabla u_T|^p + C \int_{B_{2R}(x)} |u_T|^p |\nabla \eta|^p \\
&+ (c_2 |r| + \alpha_3) \int_{B_{2R}(x)} \eta^p + \frac{\alpha_1}{4} \int_{B_{2R}(x)} \eta^p |r + \nabla u_T|^p + C |r|^p \int_{B_{2R}(x)} \eta^p \\
&+ C \int_{B_{2R}(x)} |u_T|^p |\nabla \eta|^p + C \int_{B_{2R}(x)} \eta^p |H|^{p'} + \frac{\alpha_1}{4} \int_{B_{2R}(x)} \eta^p |r + \nabla u_T|^p + C \int_{B_{2R}(x)} \eta^p |H|^{p'} \\
&+ C \int_{B_{2R}(x)} \eta^p |H|^{p'} + C |r|^p \int_{B_{2R}(x)} \eta^p + \frac{1}{p} T^{-p} \int_{B_{2R}(x)} \eta^p |u_T|^p + \frac{1}{p'} T^{p'} \int_{B_{2R}(x)} \eta^p |h|^{p'} .
\end{aligned}$$

After some simplifications and thanks to the properties of η together with the fact that

$$\int_{B_{2R}(x)} \eta^p |\nabla u_T|^p \le C \int_{B_{2R}(x)} \eta^p |r + \nabla u_T|^p + C |r|^p \int_{B_{2R}(x)} \eta^p,$$

we are led to

$$\begin{aligned}
\int_{B_R(x)} |\nabla u_T|^p + T^{-p} \int_{B_R(x)} |u_T|^p \le \frac{C}{R^p} \int_{B_{2R}(x)} |u_T|^p + C \int_{B_{2R}(x)} (T^{p'} |h|^{p'} + |H|^{p'}) \\
+ (C |r|^p + C |r| + C) |B_{2R}(x)|
\end{aligned}$$

(where $|B_{2R}(x)|$ denotes the Lebesgue measure of the ball $B_{2R}(x)$), or equivalently (dividing both members of the last inequality above by R^d),

$$\begin{aligned}
\fint_{B_R(x)} \left(|\nabla u_T|^p + T^{-p} |u_T|^p \right) \le \frac{C}{R^p} \fint_{B_{2R}(x)} |u_T|^p \\
+ C \fint_{B_{2R}(x)} (T^{p'} |h|^{p'} + |H|^{p'}) + C
\end{aligned} \tag{5.15}$$

where the constant C depends only on $\alpha_1, p, c_2, \alpha_3, d$ and r. Let us take the $\sup_{x \in \mathbb{R}^d}$ of both members of (5.15) and let us use the inequality

$$\sup_{x \in \mathbb{R}^d} \fint_{B_{2R}(x)} |V|^p \le C(d,p) \sup_{x \in \mathbb{R}^d} \fint_{B_R(x)} |V|^p ,$$

it emerges

$$\fint_{B_R(x)} \left(|\nabla u_T|^p + T^{-p} |u_T|^p\right) \leq C + \frac{C}{R^p} \sup_{x\in\mathbb{R}^d} \fint_{B_R(x)} |u_T|^p + C \sup_{x\in\mathbb{R}^d} \fint_{B_{2R}(x)} (T^{p'} |h|^{p'} + |H|^{p'}). \tag{5.16}$$

In (5.16) when we choose R so that $T^{-p} \geq 2CR^{-p}$, i.e. $R \geq T(2C)^{1/p}$, then the estimate (5.14) holds true. The case $T \leq R \leq T(2C)^{1/p}$ is obtained from the case $R = T$. □

Now, coming back to Eq (5.1), we obtain the following result.

Proposition 5.2. *Let $r \in \mathbb{R}^d$. Then there exists a function $u \in W^{1,p}_{loc}(\mathbb{R}^d)$ with $\nabla u \in B^p_A(\mathbb{R}^d)^d$ such that u is the solution of (5.1). Moreover ∇u is uniquely defined.*

Proof. Based on the estimate (5.14), we can see that the sequence $(\nabla u_T)_{T\geq 1}$ is bounded in $L^p_{loc}(\mathbb{R}^d)^d$, so that, proceeding exaclty as in the proof of [30, Theorem 1.3], we derive the existence of a function $u \in W^{1,p}_{loc}(\mathbb{R}^d)$ such that $\nabla u \in B^p_A(\mathbb{R}^d)^d$, $M(\nabla u) = 0$ and u solves (5.1). □

Remark 6. It is important to note that the framework leading to the existence of distributional corrector here is different from the one in [4, 30] in minimum two points of view.

(i) In [4], the function h and H are supposed to be L^∞-bounded. In our case we don't need to make such an assumption and moreover we first solve Eq. (5.5) in the space $W^{1,p}_{uloc}(\mathbb{R}^d)$, which was not the case in none of the previous references in the literature.

(ii) We rigorously prove, using the Caccioppoli's Inequality, the crucial estimate (5.14), which is sharp compared to its counterpart in [4, 30]. Indeed here it gives explicitly the dependence of the right-hand side in terms of the functions h and H.

6 Homogenization result

In what follows we suppose that A is an ergodic algebra with mean value on $\mathbb{R}^d$. We also assume that the distribution of the microstructures can be distributed anyhow in the medium in the following way:

(A6) $\boldsymbol{a}(\cdot, \lambda) \in (B^{p'}_A(\mathbb{R}^d))^d$ for all $\lambda \in \mathbb{R}^d$.

This means that the microstructures in the medium can be displayed anyhow in the deterministic fashion, provided that they are located in the way that there exists a mean value for their distribution function.

6.1 Passage to the limit

Let us set the space $Y = L^\infty(0, T; W^{1,p}_0(\Omega)) \cap H^1\left(0, T; L^2(\Omega)\right)$. Endowed with the norm $\|u\|_Y = \|u\|_{H^1(0,T;L^2(\Omega))} + \|u\|_{L^\infty(0,T;W^{1,p}_0(\Omega))}$, Y is a Banach Space. Owing to Lemma 3.2, the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in Y, so that the following preliminary result holds.

Proposition 6.1. *Let us consider $(u_\varepsilon)_{\varepsilon\in E}$ (where E is a fundamental sequence) be a bounded sequence in Y. Then there exist a subsequence E' of E and a couple $(u_0, u_1) \in Y \times L^p(Q, B^{1,p}_{\#A}(\mathbb{R}^d))$ such that as $E' \ni \varepsilon \to 0$,*

- $u_\varepsilon \to u_0$ *in* $L^\infty(0,T; W^{1,p}_0(\Omega))$*-weak* $*$
- $u'_\varepsilon \to u'_0$ *in* $L^2(Q)$*-weak*
- $u_\varepsilon \to u_0$ *in* $L^2(Q)$*-strong*
- $\nabla u_\varepsilon \to \nabla u_0 + \nabla_y u_1$ *in* $L^p(Q)^d$*-weak* Σ.

Proof. The proof is straightforward and is an easy consequence of both Theorem 4.2 and Lemma 3.2. □

Coming from the estimates in Lemma 3.2, and given the subsequence E' of Proposition 6.1, there exist a subsequence of E' not relabeled and functions $\mathbf{v} \in L^{p'}(Q; \mathcal{B}^{p'}_A(\mathbb{R}^d))^d$ and $w \in L^2(Q)$ such that

$$\boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon) \to \mathbf{v} \text{ in } L^{p'}(Q)^d\text{-weak } \Sigma \tag{6.1}$$

and

$$w_\varepsilon \to w \text{ in } L^2(Q)\text{-weak.} \tag{6.2}$$

We infer from Proposition 6.1 that $\mathbf{u} = (u_0, u_1)$ belongs to $\mathbb{F}^p_0$ where $\mathbb{F}^p_0 = V^p \times L^p(Q; B^{1,p}_{\#A}(\mathbb{R}^d))$, which is a Banach space with an obvious norm. It is an easy task in showing that the space $\mathcal{F}^\infty_0 = \mathcal{C}^\infty_0(Q) \times (\mathcal{C}^\infty_0(Q) \otimes A^\infty)$ is dense in $\mathbb{F}^p_0$.

For $\mathbf{v} = (v_0, v_1) \in \mathbb{F}^p_0$, we set $\mathbb{D}_i\mathbf{v} = \frac{\partial v_0}{\partial x_i} + \frac{\partial v_1}{\partial y_i}$ and $\mathbb{D}\mathbf{v} = (\mathbb{D}_i\mathbf{v})_{1\leq i\leq d} \equiv \nabla v_0 + \nabla_y v_1$. For $\Phi = (\psi_0, \psi_1) \in \mathcal{F}^\infty_0$, we define $\mathbb{D}\Phi$ accordingly.

Proposition 6.2. *The couple $\mathbf{u} = (u_0, u_1) \in \mathbb{F}^p_0$ and the function w determined above solve the following variational problem*

$$\left\{\begin{array}{l} -\int_Q (M(c)u_0 + w)\frac{\partial \psi_0}{\partial t} dxdt + \int_Q M\,(\mathbf{a}(\cdot, \mathbb{D}\mathbf{u}) \cdot \mathbb{D}\Phi) dxdt \\ = \int_Q g\psi_0 dxdt \text{ for all } \Phi = (\psi_0, \psi_1) \in \mathcal{F}^\infty_0. \end{array}\right. \tag{6.3}$$

Furthermore the function w has the representation $w = \mathcal{W}(u_0; \cdot)$ a.e. in Q.

Proof. To view this, we shall pass to the limit in the variational formulation of (1.1) provided that the assumptions (**A1**)-(**A6**) are fulfilled. For the sake of simplicity, we may omit throughout this section to precise that $E' \ni \varepsilon \to 0$ when dealing with a convergence result, although this will be kept in mind once for good. Bearing this in mind, we proceed as follows. Let $\Phi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon$ be defined by

$$\Phi_\varepsilon(x,t) = \psi_0(x,t) + \varepsilon\psi_1\left(x, t, \frac{x}{\varepsilon}\right) \ \ ((x,t) \in Q)$$

where $\psi_0 \in \mathcal{C}^\infty_0(Q)$ and $\psi_1 \in \mathcal{C}^\infty_0(Q) \otimes A^\infty$. Then $\Phi_\varepsilon \in \mathcal{C}^\infty_0(Q)$ and

$$\begin{aligned} &\Phi_\varepsilon \to \psi_0 \text{ in } L^p(0,T; W^{1,p}_0(\Omega))\text{-weak} \\ &\frac{\partial \Phi_\varepsilon}{\partial t} = \frac{\partial \psi_0}{\partial t} + \varepsilon\left(\frac{\partial \psi_1}{\partial t}\right)^\varepsilon \to \frac{\partial \psi_0}{\partial t} \text{ in } L^p(0,T; W^{1,p}(\Omega))\text{-weak} \\ &\nabla\Phi_\varepsilon = \nabla\psi_0 + \varepsilon(\nabla\psi_1)^\varepsilon + (\nabla_y\psi_1)^\varepsilon \to \nabla\psi_0 + \nabla_y\psi_1 \text{ in } L^p(Q)^d\text{-strong } \Sigma. \end{aligned} \tag{6.4}$$

Considering Φ_ε as a test function in the weak formulation of (1.1), we get

$$-\int_Q (c^\varepsilon u_\varepsilon + w_\varepsilon)\frac{\partial \Phi_\varepsilon}{\partial t} dxdt + \int_Q \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon)\cdot \nabla \Phi_\varepsilon dxdt = \int_Q g\Phi_\varepsilon dxdt. \tag{6.5}$$

Therefore utilizing Proposition 6.1, convergence results (6.1), (6.2), (6.4) and the monotonicity of $\boldsymbol{a}$, the passage to the limit in (6.5) is an easy exercise; see e.g., [19, 27]. We are led to (6.3) at once.

In order to conclude the proof of the proposition, we have to characterize the function w in terms of u_0.

We already noticed that the a priori estimates we found yield

$$u_\varepsilon \to u_0 \text{ in } L^p(0,T;W_0^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))\text{-weak}. \tag{6.6}$$

On the other hand, we can deduce that, it is possible to extract another subsequence from E', we have

$$u_\varepsilon \to u_0 \text{ uniformly in } [0,T] \text{ and a.e. in } \Omega.$$

Making use of the strong continuity of the operator $\mathcal{P}$, we obtain that

$$\mathcal{P}(u_\varepsilon;\cdot) \to \mathcal{P}(u_0;\cdot) \text{ uniformly in } [0,T] \text{ and a.e. in } \Omega.$$

Now, we define the functions

$$z_\varepsilon(x,t) = \mathcal{P}[u_\varepsilon(x,\cdot);x](t) \text{ and } z_0(x,t) = \mathcal{W}[u_0(x,\cdot);x](t).$$

It is an easy matter to see that $u_\varepsilon \to u_0$ in $L^2(\Omega;\mathcal{C}([0,T]))$-strong; in particular, $u_\varepsilon(x,\cdot) \to u_0(x,\cdot)$ in $\mathcal{C}([0,T])$, for a.e $x\in\Omega$. The fact that $w = \mathcal{P}(u_0;\cdot)$ can be showed arguing as in [25, Section IV.1] in particular we have to utilize some interpolation results and use the continuity of the hysteresis operator $\mathcal{P}$ uniformly in time, a.e. in space, which can be deduced from the local Lipschitz continuity property of $\mathcal{P}$. Hence, using the continuity of $\mathcal{P}$ supposed in (**A4**), we obtain $z_\varepsilon(x,\cdot) \to z(x,\cdot)$ in $\mathcal{C}([0,T])$, for a.e. $x\in\Omega$. In the following, in view of (1.3), we get

$$\sup_{0\le t\le T} |z_\varepsilon(x,t)| \le \kappa_0(x) + \gamma_0 \sup_{0\le t\le T} |u_\varepsilon(x,t)| \text{ for a.e. } x\in\Omega,$$

where the right-hand side converges strongly in $L^2(\Omega)$. Thus $z_\varepsilon \to z$ in $L^2(\Omega;\mathcal{C}([0,T]))$-strong. Knowing that w_ε is the linear interpolate of z_ε, an analogous argument shows that $w_\varepsilon - z_\varepsilon \to 0$ in $L^2(\Omega;\mathcal{C}([0,T]))$-strong.

In summary, as $w_\varepsilon(x,\cdot)$ is the time interpolate, we get $w_\varepsilon \to \mathcal{P}(u_0;\cdot)$ uniformly in $[0,T]$ and a.e. in Ω. Therefore, we obtain $w = \mathcal{P}(u_0;\cdot)$ a.e. in Q. The sequence $(\|w_\varepsilon(\cdot,t)\|_{\mathcal{C}([0,T])})_\varepsilon$ is uniformly integrable in Ω as the same holds for u_ε. Hence we have shown that $w_\varepsilon \to w = z$ in $L^2(\Omega;\mathcal{C}([0,T]))$-strong. □

6.2 Homogenized problem

In order to obtain the homogenized result, we have to deal with an equivalent expression of problem (6.3). As we can see, this problem is equivalent to the system (6.7)-(6.8) state below

$$\begin{cases} \int_Q (M(c)u_0 + w)\frac{\partial\psi_0}{dt} dxdt + \int_Q M\left(\boldsymbol{a}(\cdot, \nabla u_0 + \nabla_y u_1)\cdot\nabla_x\psi_0\right)dxdt \\ = \int_Q g\psi_0 dxdt \text{ for all } \psi_0 \in \mathcal{C}_0^\infty(Q) \\ w = \mathcal{P}(u_0;\cdot) \text{ a.e. in } Q \end{cases} \tag{6.7}$$

and

$$\int_Q M\left(\boldsymbol{a}\left(\cdot, \nabla u_0 + \nabla_y u_1\right) \cdot \nabla_y \psi_1\right) dxdt = 0 \quad \forall \psi_1 \in \mathcal{C}_0^\infty(Q) \otimes A^\infty. \tag{6.8}$$

Let us first deal with (6.8). We choose $\psi_1(x,t,y) = \varphi(x,t)\phi(y)$ with $\phi \in A^\infty$ and $\varphi \in \mathcal{C}_0^\infty(Q)$, (6.8) becomes

$$M\left(\boldsymbol{a}(\cdot, \mathbb{D}\mathbf{u}) \cdot \nabla_y \phi\right) = 0 \text{ for all } \phi \in A^\infty, \tag{6.9}$$

which is precisely the weak form in the duality arising from the mean value, of the following equation (in the usual sense of distributions in $\mathbb{R}^d$)

$$-\operatorname{div}_y \boldsymbol{a}(\cdot, \nabla u_0 + \nabla_y u_1) = 0 \text{ in } \mathbb{R}^d. \tag{6.10}$$

Then we fix $r \in \mathbb{R}^d$ and we hold the following corrector problem

$$\text{Find } \chi_r \in B^{1,p}_{\#A}(\mathbb{R}^d) \text{ such that } -\operatorname{div}_y \boldsymbol{a}(\cdot, r + \nabla_y \chi_r) = 0 \text{ in } \mathbb{R}^d. \tag{6.11}$$

Then appealing to Proposition 5.2, we derive the existence of a function $\chi_r \in B^{1,p}_{\#A}(\mathbb{R}^d)$ solution of (6.11) such that its gradient $\nabla\chi_r$ is uniquely defined. Now if we take $r = \nabla u_0$ in (6.11) and compare the resulting equation with (6.10) and next using the uniqueness of the gradient of the corresponding solution, we end up with $u_1 = \chi_{\nabla u_0}$ a.e. in Q, i.e. $u_1(x,t,y) = \chi_{\nabla u_0(x,t)}(y)$.

That said, let

$$\boldsymbol{a}^*(r) = M(\boldsymbol{a}(\cdot, r + \nabla_y \chi_r)) \text{ for } r \in \mathbb{R}^d. \tag{6.12}$$

The function $\boldsymbol{a}^*$ is well defined and is the so-called *homogenized coefficient*. We can verify that $\boldsymbol{a}^*$ satisfies properties similar to those of $\boldsymbol{a}$.

The next result provides us with the upscaled model of (1.1), which is *homogenized problem*, for which u_0 is the solution.

Proposition 6.3. *The function u_0 is the solution of the boundary value problem*

$$\begin{cases} \dfrac{\partial}{\partial t}(cu_0 + w) - \operatorname{div} \boldsymbol{a}^*(\nabla u_0) = g \text{ in } Q \\[2ex] w(x,t) = \mathcal{P}[u_0(x,\cdot); x](t) \text{ in } Q \\[2ex] u_0 = 0 \text{ on } \partial\Omega \times (0,T) \text{ and } u_0(x,0) = u^0(x) \text{ in } \Omega. \end{cases} \tag{6.13}$$

Proof. We just need to replace u_1 by $\chi_{\nabla u_0}$ in (6.7) and choose there $\psi_0(x,t) = \varphi(x,t)$, with $\varphi \in \mathcal{C}_0^\infty(Q)$. Then we readily get (6.13). □

The uniqueness of u_0 in (6.13) is ensured by the following result.

Proposition 6.4. *Consider u_0 and u_0^* be two solutions of (6.13) with the same initial condition u^0, then $u_0 = u_0^*$.*

Proof. Set $v_0 = u_0 - u_0^*$. Then appealing to [21, Theorem 5.1 and Corollary 5.1], it emerges that $v_0 = 0$. □

We are now able to state the main result of the work.

Theorem 6.1. *Let us suppose that assumptions* (**A1**)-(**A6**) *hold. For each* $\varepsilon > 0$ *let* u_ε *be the unique solution of* (1.1). *Then the sequence* $(u_\varepsilon)_{\varepsilon>0}$ *strongly converges in* $L^2(Q)$ *and weakly star in* $L^\infty(0,T;W_0^{1,p}(\Omega))$ *to the unique solution of the problem* (6.13).

Proof. Since the solution of (6.13) is unique, the conclusion of Theorem 6.1 follows from Propositions 6.1 and 6.3. □

7 Approximation of the homogenized coefficient

In this section, we use the same notations as in the preceding sections. The main goal here is to provide an approximate scheme for the homogenized coefficient $\boldsymbol{a}^*$ which is defined by (6.12). It is to be noted that the corrector equation is stated on the whole space $\mathbb{R}^d$ since our framework is the general non periodic deterministic setting, which, as in the periodic framework, cannot be reduced to a problem on bounded domain. It is worth recalling that the corrector problem reads as follows:

$$\chi_r \in B^{1,p}_{\#A}(\mathbb{R}^d) \text{ and } -\operatorname{div}_y \boldsymbol{a}(\cdot, r+\nabla_y\chi_r) = 0 \text{ in } \mathbb{R}^d. \tag{7.1}$$

Knowing that (7.1) is posed on $\mathbb{R}^d$ which is not bounded, it is not possible to expect a numerical computation of χ_r and thus we cannot compute the effective coefficient $\boldsymbol{a}^*$ that depends on χ_r, unless we find a suitable approximation of χ_r in bounded domains. That is the reason why taking the truncations of (7.1) in bounded domains is useful here as well as the study of the convergence of the sequence of the resulting approximate coefficients towards the homogenized coefficient. We consider here such a truncation on large balls $B_R = B(0,R)$ with Dirichlet boundary condition defined as follows:

$$-\nabla\cdot(\boldsymbol{a}(\cdot, r+\nabla\chi_{r,R})) = 0 \text{ in } B_R, \quad \chi_{r,R} = 0 \text{ on } \partial B_R. \tag{7.2}$$

We get the following result.

Proposition 7.1. *Problem* (7.2) *has a unique solution* $\chi_{r,R} \in W_0^{1,p}(B_R)$ *which verifies the estimate*

$$\left(\fint_{B_R} |\nabla\chi_{r,R}|^p\,dz\right)^{\frac{1}{p}} \le C\,|r| \text{ for any } R \ge 1. \tag{7.3}$$

The positive constant C *is independent of* R.

Proof. It is a piece of cake to see that (7.2) possesses a unique solution verifying (7.3). □

Let us consider $\chi_{r,R}$ as the solution of (7.2). Based on the above uniqueness argument, we can define the approximate effective coefficients (for every $r \in \mathbb{R}^d$) as follows:

$$\boldsymbol{a}^*_R(r) = \fint_{B_R} \boldsymbol{a}(y, r+\nabla\chi_{r,R}(y))dy. \tag{7.4}$$

Next, we are able to state the main result of this section.

Theorem 7.1. *It holds that* $\boldsymbol{a}^*_R(r) \to \boldsymbol{a}^*(r)$ *when* $R\to\infty$..

Proof. For $R \ge 1$, define the functions w_r^R and $\boldsymbol{a}_R$ on $B_1 =$ by

$$w_r^R(y) = R^{-1}\chi_{r,R}(Ry) \text{ and } \boldsymbol{a}_R(y,\lambda) = \boldsymbol{a}(Ry,\lambda) \text{ for } y \in B_1$$

Then they verify

$$-\operatorname{div} \boldsymbol{a}_R(y, r+\nabla_y w_r^R) = 0 \text{ in } B_1,\ w_r^R = 0 \text{ on } \partial B_1,$$

such that

$$w_r^R \in W_0^{1,p}(B_1) \text{ with } \left\|\nabla w_r^R\right\|_{L^p(B_1)} \leq C \tag{7.5}$$

where the constant C in (7.5) is independent of $R > 0$. In view of the uniform estimate in (7.5), we make use of the stationary version of the Σ -convergence (see [30]) to derive the existence of functions $w_{r,0} \in W_0^{1,p}(B_1)$ and $w_{r,1} \in L^p(B_1; B_{\#A}^{1,p}(\mathbb{R}^d))$ so that, up to a subsequence,

$$\begin{aligned} &w_r^R \to w_{r,0} \text{ in } W_0^{1,p}(B_1)\text{-weak}, \\ &\nabla w_r^R \to \nabla w_{r,0} + \nabla_y w_{r,1} \text{ in } L^p(B_1)\text{-weak } \Sigma. \end{aligned} \tag{7.6}$$

Moreover the couple $(w_{r,0}, w_{r,1})$ solves the equation

$$\int_{B_1} M\left(\boldsymbol{a}(\cdot, r+\nabla w_{r,0}+\nabla_y w_{r,1}) \cdot (\nabla\psi_0 + \nabla_y \psi_1)\right) dx = 0 \tag{7.7}$$

for all $\psi_0 \in \mathcal{C}_0^\infty(B_1)$ and $\psi_1 \in \mathcal{C}_0^\infty(B_1) \otimes A^\infty$. Equation (7.7) is equivalent to the following system:

$$\int_{B_1} M\left(\boldsymbol{a}(\cdot, r+\nabla w_{r,0}+\nabla_y w_{r,1})\right) \cdot \nabla\psi_0 dx = 0 \tag{7.8}$$

$$M\left(\boldsymbol{a}(\cdot, r+\nabla w_{r,0}+\nabla_y w_{r,1}) \cdot \nabla v\right) = 0 \quad \forall v \in A^\infty. \tag{7.9}$$

Consider the weak form of (7.9), which is

$$-\operatorname{div}_y \boldsymbol{a}(\cdot, r+\nabla w_{r,0}+\nabla_y w_{r,1}) = 0 \text{ in } \mathbb{R}^d. \tag{7.10}$$

Then let us fix $\xi \in \mathbb{R}^d$ and let us consider the equation

$$-\operatorname{div}_y \boldsymbol{a}(\cdot, r+\xi+\pi_r(\xi)) = 0 \text{ in } \mathbb{R}^d,\ \pi_r(\xi) \in B_{\#A}^{1,2}(\mathbb{R}^d).$$

It possesses a unique solution in the sense of Proposition 5.2 (that is, its gradient is unique). However it also possesses a unique solution $\chi_{r+\xi}$ in the sense of the same Proposition 5.2, such that $\pi_r(\xi) = \chi_{r+\xi}$. Therefore, if we take $\xi = \nabla w_{r,0}$, we get $w_{r,1} = \pi_r(\nabla w_{r,0}) = \chi_{r+\nabla w_{r,0}}$. Going back to (7.8) and replacing there $w_{r,1}$ by its value above, we end up with

$$\int_{B_1} M(\boldsymbol{a}(\cdot, r+\nabla_y \chi_{r+\nabla w_{r,0}} + \nabla w_{r,0})) \cdot \nabla\psi_0 dx = 0 \ \ \forall\psi_0 \in \mathcal{C}_0^\infty(B_1)$$

i.e.

$$\int_{B_1} M(\boldsymbol{a}(\cdot, r+\nabla w_{r,0} + \nabla_y \chi_{r+\nabla w_{r,0}})) \cdot \nabla\psi_0 dx = 0,$$

or

$$\int_{B_1} \boldsymbol{a}^*(r+\nabla w_{r,0}) \cdot \nabla\psi_0 dx = 0 \ \ \forall\psi_0 \in \mathcal{C}_0^\infty(B_1),$$

which is nothing else but

$$-\operatorname{div} \boldsymbol{a}^*(r+\nabla w_{r,0}) = 0 \text{ in } B_1,\ w_{r,0} \in W_0^{1,p}(B_1). \tag{7.11}$$

This shows that

$$\boldsymbol{a}_R(\cdot, r+\nabla w_r^R) \to \boldsymbol{a}^*(r+\nabla w_{r,0}) \text{ in } L^{p'}(B_1)\text{-weak} \tag{7.12}$$

for if $\phi \in L^p(B_1)^d$, taking into account (7.6), we get

$$\int_{B_1} \boldsymbol{a}_R(\cdot, r + \nabla w_r^R) \cdot \phi dx \to \int_{B_1} M\left(\boldsymbol{a}(\cdot, r + \nabla w_{r,0} + \nabla_y w_{r,1})\right) \cdot \phi dx.$$

Thus, making use of the equality $w_{r,1} = \chi_{r+\nabla w_{r,0}}$, we get $M\left(\boldsymbol{a}(\cdot, r + \nabla w_{r,0} + \nabla_y w_{r,1})\right) = \boldsymbol{a}^*\left(r + \nabla w_{r,0}\right)$. Next, in view of the properties of $\boldsymbol{a}^*$, it naturally follows that (7.11) possesses a unique solution which is actually 0, i.e. $w_{r,0} = 0$. Hence, taking the test function $\phi \equiv 1$ in (7.12), we are led to

$$\boldsymbol{a}_R^*(r) = \fint_{B_1} \boldsymbol{a}_R\left(y, r + \nabla \chi_{r,R}(y)\right) dy \to \fint_{B_1} \boldsymbol{a}^*\left(r + \nabla w_{r,0}\right) dy = \boldsymbol{a}^*(r).$$

□

Acknowledgments

The author acknowledges good working conditions at the University of Pretoria, where this paper has been finalized during his postdoctoral fellowship.

References

[1] L. Boccardo, T. Gallouët and F. Murat, Unicité de la solution de certaines équations elliptiques non linéaires, C.R. Acad. Sci. Paris **315** (1992) 1159–1164.

[2] H. Brézis, Opérateurs Maximaux Monotones et Semi-groups de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam, 1973.

[3] M. Brokate and J. Sprekels, Hysteresis and Phase Transitions, Appl. Math. Sci., Vol. **121**, Springer, New York, 1996.

[4] G. Cardone and J.L. Woukeng, Corrector problem in the deterministic homogenization of nonlinear elliptic equations. Appl. Anal. **98** (2019) 118–135.

[5] G. Cardone, A. Fouetio, S. Talla Lando and J. L. Woukeng. "Global dynamics of stochastic tidal equations." Nonlinear Analysis **225** (2022): 113137.

[6] J. Casado Diaz and I. Gayte, The two-scale convergence method applied to generalized Besicovitch spaces. Proc. R. Soc. Lond. A **458** (2002) 2925–2946.

[7] J. Franců, Homogenization of heat equation with hysteresis. Modelling 2001 (Pilsen). Math. Comput. Simulation **61** (2003) 591–597.

[8] J. Franců, Homogenization of diffusion equation with scalar hysteresis operator. Proceedings of Partial Differential Equations and Applications (Olomouc, 1999). Math. Bohem. **126** (2001) 363–377.

[9] J. Franců and P. Krejčí, Homogenization of scalar wave equations with hysteresis. Contin. Mech. Thermodyn. **11** (1999) 371–390.

[10] M. Hilpert, On uniqueness for evolution problems with hysteresis. In Mathematical Models for Phase Change Problems (J.F. Rodrigues, ed.). Birkhäuser, Basel (1989) 377–388.

[11] W. Jäger, A. Tambue and J.L. Woukeng, Approximation of homogenized coefficients in deterministic homogenization and convergence rates in the asymptotic almost periodic setting, arxiv preprint, arXiv: 1906.11501.

[12] J. Kopfová, A convergence result for spatially inhomogeneous Preisach operators. Z. Angew. Math. Phys. **58** (2007) 350–356.

[13] J. Kopfová, A homogenization result for a parabolic equation with Preisach hysteresis. ZAMM Z. Angew. Math. Mech. **87** (2007) 352–359.

[14] P. Krejčí (1996), Hysteresis, convexity and dissipation in hyperbolic equations, Tokyo: Gakkotosho.

[15] J. Leray and J.L. Lions, Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, Bull. Soc. Math. France **93** (1965) 97–107.

[16] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization. SIAM Journal on Mathematical Analysis, **20**(3):608-623, 1989.

[17] G. Nguetseng, Homogenization structures and applications I, Z. Anal. Anwen. **22** (2003) 73–107.

[18] G. Nguetseng, M. Sango and J.L. Woukeng, Reiterated ergodic algebras and applications, Commun. Math. Phys. **300** (2010) 835–876.

[19] G. Nguetseng and J.L.Woukeng, Deterministic homogenization of parabolic monotone operators with time dependent coefficients, Electron. J. Differ. Eq. 2004 (2004) 1–23.

[20] G. Nguetseng and J.L. Woukeng, Σ-convergence of nonlinear parabolic operators. Nonlin. Anal. TMA. **66** (2007), 968–1004.

[21] A. L. Pokam Kakeu and J.L. Woukeng, Well-Posedness and Long-Time Behaviour for a Nonlinear Parabolic Equation With Hysteresis, Commun. Math. Anal. **23** (2020) 38–62.

[22] A. L. Pokam Kakeu, Homogenization of Vlasov-Poisson equations with strong external magnetic field, revised in Partial Differential Equations and Applications PDEA.

[23] A. L. Pokam Kakeu and J. L. Woukeng, (2020), 'Homogenization of nonlinear parabolic equations with hysteresis', ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik p. e201900323.

[24] M. Sango, N. Svanstedt and J.L. Woukeng, Generalized Besicovitch spaces and application to deterministic homogenization, Nonlin. Anal. TMA **74** (2011) 351–379.

[25] A. Visintin, Differential Models of Hysteresis, Springer, Berlin, 1994.

[26] N. Wiener, 'Tauberian theorems', Annals of mathematics (1932), pp. 1–100.

[27] J.L. Woukeng, Deterministic homogenization of non-linear non-monotone degenerate elliptic operators, Adv. Math. **219** (2008) 1608–1631.

[28] J.L. Woukeng, Homogenization in algebra with mean value, Banach J. Math. Anal. **9** (2015) 142–182.

[29] J.L. Woukeng, Introverted algebras with mean value and applications, Nonlinear Anal. TMA **99** (2014) 190–215.

[30] J.L. Woukeng, Corrector problem and homogenization of nonlinear elliptic monotone PDE. In Shape optimization, homogenization and optimal control, 57–73, Internat. Ser. Numer. Math., **169**, Birkhäuser/Springer, Cham, 2018.

Achille Landri Pokam Kakeu University of Dschang
Current address: University of Pretoria

E-mail: pokakeu@yahoo.fr al.pokamkakeu@up.ac.za