TAMKANG JOURNAL OF MATHEMATICS
Volume vol, Number i, xxx-xxx, 20xx
DOI:10.5556/j.tkjm.55.2024.5276

# Unique continuation property for the Rosenau equation

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**Abstract**. In this work, using an appropriate Carleman-type estimate, we establish a unique continuation result for the Rosenau equation that models the dynamics of dense discrete systems with high order effects.

***Keywords.*** Rosenau equation, Carleman estimates, UCP, Treve's inequality

## 1 Introduction

To model the dynamics of dense discrete systems with high order effects, Philip Rosenau [14] derived the high order nonlinear partial differential equation,

$$u_{tt} + au_{xxxx} + bu_{xxxxtt} - \gamma u_{xx} = \big(f(u)\big)_{xx}, \tag{1.1}$$

where $a > 0$, $b > 0$, and $\gamma > 0$ are constants, $f(u) = -\beta|u|^p u$ with $\beta > 0$ and $p > 0$. The equation is called Rosenau equation. When $b = 0$ the Rosenau equation becomes the "good" Boussinesq equation which arises in the modeling of nonlinear strings.

S. Wang and G. Xu in [18] showed the well-posedness for the Cauchy problem associated to the model (1.1) in the Sobolev space $H^s(\mathbb{R})$, with $s > 1/2$, where $H^s(\mathbb{R})$ is the usual Sobolev space of order $s$ defined as the completion of the Schwartz class with respect to the norm

$$\|w\|_{H^s(\mathbb{R})} = \|\,(1+|\xi|)^s\,\widehat{w}(\xi)\|_{L^2_\xi},$$

where $\widehat{w}$ is the Fourier transform of $w$ in the space variable $x$ and $\xi$ is the variable in the frequency space related to the variable $x$. Specifically they proved the following result.

**Theorem 1.1.** *Assume that $s > 1/2$, $\varphi \in H^s(\mathbb{R})$, $\psi \in H^s(\mathbb{R})$ and $f \in C^N(\mathbb{R})$, where $N \geq \max\{1, s-2\}$ is an integer, then there exists a maximal time $T_0$ which depends only on $\varphi$ and $\psi$ such that for each $T < T_0$, the Cauchy problem*

$$\begin{cases} u_{tt} + au_{xxxx} + bu_{xxxxtt} - \gamma u_{xx} &= \big(f(u)\big)_{xx}, \quad x\in\mathbb{R}, t>0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) &= \psi(x), \quad x\in\mathbb{R}, \end{cases} \tag{1.2}$$

Received date: January 26, 2024; Published online: March 20, 2024.
2010 *Mathematics Subject Classification.* 58F15, 58F17, 53C35.
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*has a unique solution* $u \in C^1([0,T]\,;\, H^s(\mathbb{R}))$. *Moreover, if*

$$\sup_{t\in[0,T_0)} \left[\|u(\cdot,t)\|_{H^s} + \|u_t(\cdot,t)\|_{H^s}\right] < \infty,$$

*then* $T_0 = \infty$.

In the present work, we will prove a unique continuation result for the Rosenau equation (1.1) when $f(u) = -\beta u^{2k+1}$, $k \in \mathbb{N}$. More precisely, we show that if $u = u(x,t)$ is a solution of the model (1.1) in a suitable function space, for example

$$u \in L^2\left(-T,T; H^6_{loc}(\mathbb{R})\right), \quad u_t \in L^2\left(-T,T; H^2_{loc}(\mathbb{R})\right),$$

and $u$ vanishes on an open subset $\Omega$ of $\mathbb{R}\times[-T,T]$, then $u \equiv 0$ in the horizontal component of $\Omega$. We recall that the horizontal component $\Omega_1$ of an open subset $\Omega \subseteq \mathbb{R}\times\mathbb{R}$ is defined as the union of all segments $t = constant$ in $\mathbb{R}\times\mathbb{R}$ which contain a point of $\Omega$, this is,

$$\Omega_1 = \left\{(x,t) \in \mathbb{R}\times[-T,T] \;:\; \exists x_1 \in \mathbb{R},\; (x_1,t) \in \Omega\right\}.$$

The unique continuation property has been intensively studied for a long time due to the important role that plays in the applications (see V. Isakov [9] and J. L. Lions [12]). An important work on the subject was done by J. C. Saut and B. Scheurer in [15]. They showed a unique continuation result for a general class of dispersive equations including the well known KdV equation,

$$u_t + uu_x + u_{xxx} = 0,$$

and various generalizations. In a similar way, Y. Shang showed in [16] a unique continuation result for the symmetric regularized long wave equation,

$$u_{tt} - u_{xx} + \frac{1}{2}\left(u^2\right)_{xt} - u_{xxtt} = 0.$$

In the previous equations, a Carleman estimate is established to prove that if a solution $u$ vanishes on an open subset $\Omega$, then $u \equiv 0$ in the horizontal component of $\Omega$. By using the inverse scattering transform and some results from the Hardy function theory, B. Zhang in [19] established that that if $u$ is a solution of the KdV equation, then it cannot have compact support at two different moments unless it vanishes identically. In the paper [1], J. Bourgain introduced a different approach and prove that if a solution $u$ to the KdV equation has compact support in a nontrivial time interval $I = [t_1, t_2]$, then $u \equiv 0$. His argument is based on an analytic continuation of the Fourier transform via the Paley-Wiener Theorem and the dispersion relation of the linear part of the equation. It also applies to higher order dispersive nonlinear models, and to higher spatial dimensions; in particular, M. Panthee in [13] showed that if $u$ is a smooth solution of the Kadomtsev-Petviashvili (KP) equation,

$$u_t + u_{xxx} + uu_x + \partial_x^{-1} u_{yy} = 0,$$

such that, for some $B > 0$,

$$\operatorname{supp} u(t) \subset [-B,B]\times[-B,B] \quad \forall t \in [t_1,t_2],$$

then $u \equiv 0$.

More recently, C. Kenig, G. Ponce and L. Vega in [11] proposed a new method and proved that if a sufficiently smooth solution $u$ to a generalized KdV equation is supported in a half line at two different instants of time, then $u \equiv 0$. Moreover, L. Escauriaza, C. Kenig, G. Ponce and L. Vega in [6] established uniqueness properties of solutions of the $k$-generalized Korteweg- de Vries equation,

$$u_t + u^k u_x + u_{xxx} = 0, \quad k \in \mathbb{Z}^+. \tag{1.3}$$

They obtained sufficient conditions on the behavior of the difference $u_1 - u_2$ of two solutions $u_1$, $u_2$ of (1.3) at two different times $t_0 = 0$ and $t_1 = 1$ which guarantee that $u_1 \equiv u_2$. This kind of uniqueness results has been deduced under the assumption that the solutions coincide in a large sub-domain of $\mathbb{R}$ at two different times. In a similar fashion, E. Bustamante, P. Isaza and J. Mejía in [2] proved that if $u$ is a smooth solution of the Zakharov-Kuznetsov equation,

$$u_t + u_{xxx} + u_{xyy} + uu_x = 0,$$

such that, for some $B > 0$,

$$\operatorname{supp} u(t_2),\ \operatorname{supp} u(t_1) \subset [-B, B] \times [-B, B],$$

then $u \equiv 0$. Moreover, in [3] it was proved that if the difference of two sufficiently smooth solutions of the Zakharov-Kuznetsov equation decays as $e^{-a\left(x^2+y^2\right)^{3/4}}$ at two different times, for some $a > 0$ large enough, then both solutions coincide. More unique continuation results can be seen in [4], [5], [7], [8], [10].

Following from close the works of Saut-Scheurer [15], we base our analysis in finding an apppropiate Carleman-type estimate for the linear operator $\mathcal{L}$ associated to the equation (1.1). In order to do this we use a particular version of the well known Treves' inequality. For the operator $\mathcal{L}$ we also prove that if a solution vanishes in a ball in the $xt$ plane, which pass through the origen, then it also vanished in a neighborhood of the origen.

The paper is organized as follows. In Section 2, using a particular version of the Treves inequality, we establish a Carleman estimate for a differential operator $\mathcal{L}$ closely related to our problem. In Section 3, first we give some useful technical results. Later, we show the unique continuation result for the model (1.1).

## 2 Carleman estimates

In this section, using a particular version of the Treves' inequality, we establish a Carleman estimate for the differential operator $\mathcal{L}$ defined as

$$\mathcal{L} := \partial_t^2 + c_1\partial_x\partial_t + c_2\partial_x^5\partial_t + b\partial_x^4\partial_t^2 + a\partial_x^4 + c_3\partial_x^6 + f_1(x,t)\partial_x + f_2(x,t)\partial_x^2. \tag{2.1}$$

In what follows we are going to use the notation $D = (\partial_x, \partial_t)$. If $P = P(\xi_1, \xi_2)$ is a polynomial in two variables, has constant coefficients and degree $m$, then we consider the differential operator of order $m$ associated to $P$,

$$P(D) = P\left(\partial_x, \partial_t\right) = \sum_{|\alpha| \le m} a_\alpha D^\alpha,$$

where $D^\alpha = \partial_x^{\alpha_1}\partial_t^{\alpha_2}$ and $|\alpha| = \alpha_1 + \alpha_2$. By definition $P^{(\beta)}(\xi_1, \xi_2) = \partial_x^{\beta_1}\partial_t^{\beta_2}P(\xi_1, \xi_2)$ where $\beta$ is given by $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$.

**Theorem 2.1.** *(Treves' Inequality). Let $P(D) = P(\partial_x, \partial_t)$ be a differential operator of order $m$ with constant coefficients. Then for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, $\delta > 0$, $\tau > 0$, $\Psi \in C_0^\infty(\mathbb{R}^2)$ and $\psi(x,t) = (x-\delta)^2 + \delta^2 t^2$ we have that*

$$\frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_2}}{\alpha!}\int_{\mathbb{R}^2} |P^{(\alpha)}(D)\Psi|^2 e^{2\tau\psi}dxdt \le C(m,\alpha)\int_{\mathbb{R}^2} |P(D)\Psi|^2 e^{2\tau\psi}dxdt \tag{2.2}$$

*with*

$$|\alpha| = |\alpha_1| + |\alpha_2|, \quad \alpha! = \alpha_1!\alpha_2!,$$

*and*

$$C(m,\alpha) = \begin{cases} \sup_{|r+\alpha|\le m} \binom{r+\alpha}{\alpha}, & \text{if} \quad |\alpha| \le m, \\ 0, & \text{if} \quad |\alpha| > m. \end{cases}$$

*Proof.* See Corollary 1 in [16]. □

We present the Carleman estimate for the differential operator $\mathcal{L}$.

**Theorem 2.2.** *Let $\mathcal{L}$ the differential operator defined in (2.1), where $c_1, c_2, c_3$ are constants in $\mathbb{R}$ and $f_1, f_2 \in L^\infty_{loc}(\mathbb{R}^2)$. Let $\delta > 0$ and*

$$B_\delta := \{(x,t) \in \mathbb{R}^2 \,:\, x^2 + t^2 < \delta^2\}, \quad \psi(x,t) = (x-\delta)^2 + \delta^2 t^2.$$

*Then, there exists $C > 0$ such that for all $\Psi \in C_0^\infty(B_\delta)$ and $\tau > 0$ with*

$$\frac{\|f_1\|^2_{L^\infty(B_\delta)}}{\tau^6 c_3^2} \le \frac{1}{8}, \quad \frac{\|f_2\|^2_{L^\infty(B_\delta)}}{\tau^5\delta^4 b^2} \le \frac{1}{8},$$

*we have that*

$$\tau^6 c_3^2 \int_{B_\delta} |\Psi|^2 e^{2\tau\psi}dxdt + \tau^5\delta^4 b^2 \int_{B_\delta} |\partial_x\Psi|^2 e^{2\tau\psi}dxdt + \tau^4\delta^4 b^2 \int_{B_\delta} |\partial_x^2\Psi|^2 e^{2\tau\psi}dxdt \\ \le C\int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi}dxdt. \tag{2.3}$$

*Proof.* Let $\Psi \in C_0^\infty(B_\delta)$. Consider the polynomial

$$P(\xi_1,\xi_2) = \xi_2^2 + c_1\xi_1\xi_2 + c_2\xi_1^5\xi_2 + b\xi_1^4\xi_2^2 + a\xi_1^4 + c_3\xi_1^6.$$

and

$$P(D) = P(\partial_x, \partial_t) = \partial_t^2 + c_1\partial_x\partial_t + c_2\partial_x^5\partial_t + b\partial_x^4\partial_t^2 + a\partial_x^4 + c_3\partial_x^6$$

the differential operator associated to $P$. Then, simple calculations show that if $\alpha = (6,0)$ we have that

$$P^{(\alpha)}(\xi_1,\xi_2) = P^{(6,0)}(\xi_1,\xi_2) = 720c_3, \quad P^{(\alpha)}(D)\Psi = 720c_3\Psi,$$

$$C(6,\alpha) = \sup_{|r+\alpha|\le 6} \binom{r+\alpha}{\alpha} = 1.$$

Then, using Theorem 2.1 we see that

$$\tau^6 c_3^2 \int_{B_\delta} |\Psi|^2 e^{2\tau\psi}dxdt \le \frac{2^{12}\tau^6}{720}\int_{B_\delta} |720c_3\Psi|^2 e^{2\tau\psi}dxdt$$

$$
\begin{aligned}
&= \frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{\alpha_2}}{\alpha!}\int_{B_\delta}|P^{(\alpha)}(D)\Psi|^2e^{2\tau\psi}dxdt\\
&\le \int_{B_\delta}|P(D)\Psi|^2e^{2\tau\psi}dxdt. \qquad (2.4)
\end{aligned}
$$

Moreover,

$$
P^{(3,2)}(\xi_1,\xi_2)=48b\xi_1,\quad P^{(3,2)}(D)\Psi=48b\partial_x\Psi,\quad C(6,(3,2))=6.
$$

Then, using again the Theorem 2.1 we obtain that

$$
\begin{aligned}
\tau^5\delta^4b^2\int_{B_\delta}|\partial_x\Psi|^2e^{2\tau\psi}dxdt &\le \frac{2^{10}\tau^5\delta^4}{12}\int_{B_\delta}|P^{(3,2)}(D)\Psi|^2e^{2\tau\psi}dxdt\\
&\le 6\int_{B_\delta}|P(D)\Psi|^2e^{2\tau\psi}dxdt. \qquad (2.5)
\end{aligned}
$$

In a similar fashion

$$
P^{(2,2)}(\xi_1,\xi_2)=24b\xi_1^2,\quad P^{(2,2)}(D)\Psi=24b\partial_x^2\Psi,\quad C(6,(2,2))=6.
$$

Then, we have that

$$
\begin{aligned}
\tau^4\delta^4b^2\int_{B_\delta}|\partial_x^2\Psi|^2e^{2\tau\psi}dxdt &\le \frac{2^{8}\tau^4\delta^4}{4}\int_{B_\delta}|P^{(2,2)}(D)\Psi|^2e^{2\tau\psi}dxdt\\
&\le 6\int_{B_\delta}|P(D)\Psi|^2e^{2\tau\psi}dxdt. \qquad (2.6)
\end{aligned}
$$

From (2.4)-(2.6), there is $C>0$ such that

$$
\begin{aligned}
\tau^6c_3^2\int_{B_\delta}|\Psi|^2e^{2\tau\psi}dxdt+\tau^5\delta^4b^2\int_{B_\delta}|\partial_x\Psi|^2e^{2\tau\psi}dxdt+\tau^4\delta^4b^2\int_{B_\delta}|\partial_x^2\Psi|^2e^{2\tau\psi}dxdt&\\
\le C\int_{B_\delta}|P(D)\Psi|^2e^{2\tau\psi}dxdt.& \qquad (2.7)
\end{aligned}
$$

Now, we note that

$$
\mathcal{L}=\partial_t^2+c_1\partial_x\partial_t+c_2\partial_x^5\partial_t+b\partial_x^4\partial_t^2+a\partial_x^4+c_3\partial_x^6+f_1(x,t)\partial_x+f_2(x,t)\partial_x^2
$$

implies

$$
P(D)\Psi=\mathcal{L}\Psi-(f_1(x,t)\partial_x\Psi+f_2(x,t)\partial_x^2\Psi).
$$

Then, using the inequalities (2.5)-(2.6), we have that

$$
\begin{aligned}
&\int_{B_\delta}\left(|f_1(x,t)\partial_x\Psi|^2+|f_2(x,t)\partial_x^2\Psi|^2\right)e^{2\tau\psi}dxdt\\
&\qquad\le \|f_1\|^2_{L^\infty(B_\delta)}\int_{B_\delta}|\partial_x\Psi|^2e^{2\tau\psi}dxdt+\|f_2\|^2_{L^\infty(B_\delta)}\int_{B_\delta}|\partial_x^2\Psi|^2e^{2\tau\psi}dxdt\\
&\qquad\le A\int_{B_\delta}|P(D)\Psi|^2e^{2\tau\psi}dxdt\\
&\qquad\le 2A\int_{B_\delta}\left(|\mathcal{L}\Psi|^2+|f_1(x,t)\partial_x\Psi|^2+|f_2(x,t)\partial_x^2\Psi|^2\right)e^{2\tau\psi}dxdt, \qquad (2.8)
\end{aligned}
$$

where

$$A = \frac{\|f_1\|^2_{L^\infty(B_\delta)}}{\tau^6 c_3^2} + \frac{\|f_2\|^2_{L^\infty(B_\delta)}}{\tau^5\delta^4 b^2}.$$

Next, if we choose $\tau > 0$ large enough such that

$$\frac{\|f_1\|^2_{L^\infty(B_\delta)}}{\tau^6 c_3^2} \leq \frac{1}{8}, \quad \frac{\|f_2\|^2_{L^\infty(B_\delta)}}{\tau^5\delta^4 b^2} \leq \frac{1}{8}, \tag{2.9}$$

then from inequality (2.8) we have that

$$\int_{B_\delta} \left(|f_1(x,t)\partial_x\Psi|^2 + |f_2(x,t)\partial_x^2\Psi|^2\right) e^{2\tau\psi} dxdt$$
$$\leq \frac{1}{2}\int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt + \frac{1}{2}\int_{B_\delta} \left(|f_1(x,t)\partial_x\Psi|^2 + |f_2(x,t)\partial_x^2\Psi|^2\right) e^{2\tau\psi} dxdt$$

what implies

$$\int_{B_\delta} \left(|f_1(x,t)\partial_x\Psi|^2 + |f_2(x,t)\partial_x^2\Psi|^2\right) e^{2\tau\psi} dxdt \leq \int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt.$$

Thus,

$$\int_{B_\delta} |P(D)\Psi|^2 e^{2\tau\psi} dxdt \leq 2\int_{B_\delta} \left(|\mathcal{L}\Psi|^2 + |f_1(x,t)\partial_x\Psi|^2 + |f_2(x,t)\partial_x^2\Psi|^2\right) e^{2\tau\psi} dxdt$$
$$\leq 4\int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt.$$

Hence, from previous inequality and (2.7) we obtain the estimate (2.3).

□

**Remark 1.** The estimate (2.3) is invariant under changes of signs of $\mathcal{L}$.

**Corollary 2.3.** *Let $T > 0$. Assume that in addition to the hypotheses of the Theorem 2.2 we have that*

$$u \in L^2(-T,T; H^6_{loc}(\mathbb{R})), \quad u_t \in L^2(0,T; H^2_{loc}(\mathbb{R})),$$

*and the support of $u$ is compact contained in $B_\delta$. Then, the inequality (2.3) holds if we replace $\Psi$ by $u$. Indeed,*

$$\tau^6 c_3^2 \int_{B_\delta} |u|^2 e^{2\tau\psi} dxdt + \tau^5\delta^4 b^2 \int_{B_\delta} |\partial_x u|^2 e^{2\tau\psi} dxdt + \tau^4\delta^4 b^2 \int_{B_\delta} |\partial_x^2 u|^2 e^{2\tau\psi} dxdt$$
$$\leq C\int_{B_\delta} |\mathcal{L}u|^2 e^{2\tau\psi} dxdt. \tag{2.10}$$

*Proof.* Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a regularizing sequence (in two variables) and consider $u_\epsilon = \rho_\epsilon * u$ where $*$ denotes the usual convolution. Then we have that $u_\epsilon \in C_0^\infty(B_\delta)$ and the inequality (2.3) folds for $u_\epsilon$, that is

$$\tau^6 c_3^2 \int_{B_\delta} |\rho_\epsilon * u|^2 e^{2\tau\psi} dxdt + \tau^5\delta^4 b^2 \int_{B_\delta} |\partial_x(\rho_\epsilon * u)|^2 e^{2\tau\psi} dxdt$$

$$+\tau^4\delta^4b^2\int_{B_\delta}|\partial_x^2(\rho_\epsilon*u)|^2e^{2\tau\psi}dxdt\le C\int_{B_\delta}|\mathcal{L}u_\epsilon|^2e^{2\tau\psi}dxdt. \tag{2.11}$$

Now, for $n=0,1,2$ we have that

$$\begin{aligned}\|\partial_x^n(\rho_\epsilon*u)e^{\tau\psi}-\partial_x^nu\,e^{\tau\psi}\|_{L^2(B_\delta)}&=\|(\rho_\epsilon*\partial_x^nu)e^{\tau\psi}-\partial_x^nu\,e^{\tau\psi}\|_{L^2(B_\delta)}\\&\le C\|\partial_x^n(\rho_\epsilon*u)-\partial_x^nu\|_{L^2(B_\delta)}\to0,\end{aligned}$$

where $C$ is a positive constant depending only on $\tau$ and $\delta$. Similarly we have that

$$\int_{B_\delta}\Big(|\mathcal{L}u_\epsilon|^2\,e^{\tau\psi}-|\mathcal{L}u|^2\,e^{\tau\psi}\Big)dxdt\to0,\quad\text{as}\quad\epsilon\to0^+,$$

which allows us to pass to the limit in (2.11) to conclude the proof of Corollary 2.3.

□

# 3 Unique continuation

In this section we will prove the unique continuation result for the Rosenau equation (1.1). Before we do the proof, we establish the following results.

**Lemma 3.1.** *Let $T>0$ and $f_1,f_2\in L^\infty_{loc}(\mathbb{R}\times(-T,T))$. Let $u$ with*

$$u\in L^2(-T,T;H^6_{loc}(\mathbb{R})),\quad u_t\in L^2(-T,T\,;\,H^2_{loc}(\mathbb{R}))$$

*be a solution of $\mathcal{L}u=0$ in $\mathbb{R}\times(-T,T)$ where $\mathcal{L}$ is the differential operator defined in (2.1). Let*

$$\widetilde{u}=\begin{cases}u & if\quad t\ge0\\0 & if\quad t<0.\end{cases}$$

*Suppose that $\widetilde{u}\equiv0$ in the region $\{(x,t)\,:\,x<t\}$ intercepted with a neighborhood of $(0,0)$. Then there exists a neighborhood $\mathcal{O}_1$ of $(0,0)$ (in the plane $xt$) such that $\widetilde{u}\equiv0$ in $\mathcal{O}_1$.*

*Proof.* By hypotheses there is $0<\delta<1$ such that $\widetilde{u}\equiv0$ in $R_\delta=R_1\cup R_2$, where

$$R_1=\{(x,t)\,:\,x<t\}\cap B_\delta,\quad R_2=\{(x,t)\,:\,t<0\}\cap B_\delta,\quad B_\delta=\{(x,t)\,:\,x^2+t^2<\delta^2\}.$$

Next, consider $\chi\in C_0^\infty(B_\delta)$ such that $\chi=1$ in a neighborhood $\mathcal{O}$ of $(0,0)$ and define

$$\Psi:=\chi\widetilde{u}.$$

Then we have that

$$\Psi\in L^2(-T,T;H^6_{loc}(\mathbb{R})),\quad\Psi_t\in L^2(-T,T;H^2_{loc}(\mathbb{R})),$$

and

$$supp\,\Psi\subset B_\delta.$$

By using the definition of $\chi$, we note that $\mathcal{L}\Psi=0$ in $\mathcal{O}$. Thus, using the Corollary 2.3, we have for $\psi(x,t)=(x-\delta)^2+\delta^2t^2$ and $\tau>0$ large enough that

$$\tau^6c_3^2\int_{B_\delta}|\Psi|^2e^{2\tau\psi}dxdt+\tau^5\delta^4b^2\int_{B_\delta}|\partial_x\Psi|^2e^{2\tau\psi}dxdt+\tau^4\delta^4b^2\int_{B_\delta}|\partial_x^2\Psi|^2e^{2\tau\psi}dxdt$$

$$\leq C \int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt = C \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt. \tag{3.1}$$

Now, using again the definition of $\chi$ and the fact that $\widetilde{u} \equiv 0$ in $R_\delta$, we see that

$$supp\, \Psi \subset D, \quad supp\, \mathcal{L}\Psi \subset D \cap (B_\delta \setminus \mathcal{O}), \quad D = \{(x,t) : 0 \leq t \leq x < \delta < 1\}.$$

It follows that if $(x,t) \neq (0,0)$ and $(x,t) \in D$ then

$$\psi(x,t) = (x-\delta)^2 + \delta^2 t^2 \leq (t-\delta)^2 + \delta^2 t^2 = t^2(1+\delta^2) - 2t\delta + \delta^2 < \delta^2.$$

Thus, there exists $0 < \epsilon < \delta^2$ such that

$$\psi(x,t) \leq \delta^2 - \epsilon, \quad (x,t) \in D \cap (B_\delta \setminus \mathcal{O}).$$

Moreover, since $\psi(0,0) = \delta^2$, we can choose $\mathcal{O}_1 \subset \mathcal{O}$ a neighborhood of $(0,0)$ such that

$$\psi(x,t) > \delta^2 - \epsilon, \quad (x,t) \in \mathcal{O}_1.$$

From the above construction and the inequality (3.1) we have that there exists $C_1 > 0$ such that

$$\begin{aligned} \tau^6 e^{2\tau(\delta^2-\epsilon)} \int_{\mathcal{O}_1} |\Psi|^2 dxdt &\leq \tau^6 \int_{\mathcal{O}_1} |\Psi|^2 e^{2\tau\psi} dxdt \\ &\leq \tau^6 \int_{B_\delta} |\Psi|^2 e^{2\tau\psi} dxdt \\ &\leq C_1 \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdt \\ &\leq C_1 e^{2\tau(\delta^2-\epsilon)} \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 dxdt. \end{aligned}$$

Therefore

$$\int_{\mathcal{O}_1} |\Psi|^2 dxdt \leq \frac{C_1}{\tau^6} \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 dxdt.$$

Then, passing to the limit as $\tau \to +\infty$, we have that $\Psi \equiv 0$ in $\mathcal{O}_1$. Since $\widetilde{u} = \Psi$ in $\mathcal{O}$ and $\mathcal{O}_1 \subset \mathcal{O}$, we see that $\widetilde{u} = 0$ in $\mathcal{O}_1$.

□

Similarly we can also show the following result.

**Lemma 3.2.** *Let $T > 0$ and $f_1, f_2 \in L^\infty_{loc}(\mathbb{R} \times (-T,T))$. Let $u$ with*

$$u \in L^2(-T,T; H^6_{loc}(\mathbb{R})), \quad u_t \in L^2(-T,T; H^2_{loc}(\mathbb{R}))$$

*be a solution of $\mathcal{L}u = 0$ in $\mathbb{R} \times (-T,T)$ where $\mathcal{L}$ is the differential operator defined in (2.1). Let*

$$\widetilde{u} = \begin{cases} 0 & if \quad t \geq 0 \\ u & if \quad t < 0. \end{cases}$$

*Suppose that $\widetilde{u} \equiv 0$ in the region $\{(x,t) : x < -t\}$ intercepted with a neighborhood of $(0,0)$. Then there exists a neighborhood $\mathcal{O}_2$ of $(0,0)$ (in the plane $xt$) such that $\widetilde{u} \equiv 0$ in $\mathcal{O}_2$.*

**Corollary 3.1.** *Let $T>0$ and $F_1, F_2 \in L^\infty_{loc}(\mathbb{R}\times(-T,T))$. Let $u$ with*

$$u \in L^2(-T,T;H^6_{loc}(\mathbb{R})), \quad u_t \in L^2(-T,T;H^2_{loc}(\mathbb{R}))$$

*be a solution in $\mathbb{R}\times(-T,T)$ of the equation*

$$u_{tt} + au_{xxxx} + bu_{xxxxtt} + F_1(x,t)u_x + F_2(x,t)u_{xx} = 0.$$

*Let $\gamma$ be a circumference passing through the origin $(0,0)$. Suppose that $u\equiv 0$ in the interior of the circle (with boundary $\gamma$) in a neighborhood of $(0,0)$. Then, there exists a neighborhood of $(0,0)$ where $u\equiv 0$.*

*Proof.* Let us assume that the circumference (a piece of it) $\gamma$ is given by $x=g(t)$ with $g''(t)<0$ in a neighborhood of $(0,0)$. By using the hypotheses, we have that $u\equiv 0$ in the region $\{(x,t) : x< g(t)\}$ intercepted with a neighborhood of $(0,0)$. Then, we can see that there exists $\omega\in\mathbb{R}\setminus\{0,1\}$ such that $u\equiv 0$ in a neighborhood of $(0,0)$ in the region $\{(x,t) : x<h(t)\}$ where

$$h(t) = \begin{cases} \omega t & \text{if} \quad t\geq 0 \\ -\frac{1}{\omega}t & \text{if} \quad t<0. \end{cases}$$

Now, we consider the following change of variables $(x,t)\to(X,T)$ with

$$X = x - h(t) + |t|$$
$$T = t.$$

Notice that in the new variables, if $T\geq 0$ then the function $u=u(X,T)$ is a solution of

$$\partial_T^2 u + c_1\partial_X\partial_T u + c_2\partial_X^5\partial_T u + b\partial_X^4\partial_T^2 + a\partial_X^4 u + c_3\partial_X^6 u + f_1(X,T)\partial_X u + f_2(X,T)\partial_X^2 u = 0$$

with

$$c_1 = 2(1-\omega),\ c_2 = bc_1,\ c_3 = b(1-\omega)^2,\ f_1 = F_1,\ f_2 = (1-\omega)^2 + F_2.$$

Then, $u\equiv 0$ in the region $\{(X,T) : X<T,\ T\geq 0\}$ intercepted with a neighborhood of $(0,0)$ and $u$ satisfies

$$\mathcal{L}u = 0 \quad \text{if} \quad T\geq 0,$$

where

$$\mathcal{L} = \partial_T^2 + c_1\partial_X\partial_T + c_2\partial_X^5\partial_T + b\partial_X^4\partial_T^2 + a\partial_X^4 + c_3\partial_X^6 + f_1(X,T)\partial_X + f_2(X,T)\partial_X^2.$$

So, using the Lemma 3.1 with the previous differential operator $\mathcal{L}$, we obtain that there exists a neighborhood $\mathcal{O}_1$ of $(0,0)$ in the plane $XT$ where $u\equiv 0$.

In a similar fashion, $u\equiv 0$ in the region $\{(X,T) : X<-T, T<0\}$ intercepted with a neighborhood of $(0,0)$ and $u$ satisfies

$$\mathcal{L}u = 0 \quad \text{if} \quad T<0,$$

where

$$c_1 = 2\left(\frac{1}{\omega}-1\right),\ c_2 = bc_1,\ c_3 = b\left(\frac{1}{\omega}-1\right)^2,$$

and

$$f_1 = F_1, \quad f_2 = \left(\frac{1}{\omega}-1\right)^2 + F_2.$$

Then, from Lemma 3.2 we have that there exists a neighborhood $\mathcal{O}_2$ of $(0,0,)$ in the plane $XT$ where $u\equiv 0$. Thus, returning to the original variables $(x,t)$ we have the result.

□

Now we have the main result on the unique continuation property for the equation (1.1).

**Theorem 3.2.** *Let $T > 0$ and $u$ with*

$$u \in L^2(-T, T; H^6_{loc}(\mathbb{R})), \quad u_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}))$$

*be a solution in $\mathbb{R} \times (-T, T)$ of the Rosenau equation (1.1). If $u \equiv 0$ in an open subset $\Omega \subset \mathbb{R} \times (-T, T)$, then $u \equiv 0$ in the horizontal component of $\Omega$.*

*Proof.* By defining the functions

$$F_1(x,t) = 2k(2k+1)\beta u^{2k-1} u_x, \quad F_2(x,t) = -\gamma + (2k+1)\beta u^{2k}, \quad \beta, \gamma > 0, \quad k \in \mathbb{N},$$

the Rosenau equation (1.1) takes the form

$$u_{tt} + a u_{xxxx} + b u_{xxxxtt} + F_1(x,t) u_x + F_2(x,t) u_{xx} = 0, \tag{3.2}$$

with $F_1, F_2 \in L^\infty_{loc}(\mathbb{R} \times (-T, T))$. Then, we will show the result for model (3.2).

Denote by $\Omega_1$ the horizontal component of $\Omega$ and let

$$\Lambda = \{(x,t) \in \Omega_1 : u \equiv 0 \quad \text{in a neighborhood of} \quad (x,t)\}.$$

Let $Q \in \Omega_1$ arbitrary. Choose $P \in \Lambda$ and let $\Gamma$ be a continuous curve contained in $\Omega_1$ joining $P$ to $Q$, parametrized by a continuous function $f : [0,1] \to \Omega_1$ with $f(0) = P$ and $f(1) = Q$. Since $P \in \Lambda$, there exists $r > 0$ such that

$$u \equiv 0 \quad \text{in} \quad B_r(P). \tag{3.3}$$

Taking $0 < r_0 < \min\{r, dist(\Gamma, \partial\Omega_1)\}$, where $\partial\Omega_1$ denotes the boundary of $\Omega_1$, we have that

$$B_{r_0}(P) \subset \Lambda.$$

Now, if $r_1 < \frac{r_0}{4}$ we see that

$$B_{2r_1}(f(s)) \subset \Omega_1, \quad \text{for all } s \in [0,1]; \tag{3.4}$$

in fact, if $w \in B_{2r_1}(f(s))$ and $w \notin \Omega_1$ then

$$\|w - f(s)\| < 2r_1 < r_0 < dist(\Gamma, \partial\Omega_1) \leq \|w - f(s)\|,$$

which is a contradiction.

Next, let

$$\Lambda_1 = \{(x,t) \in \Lambda : u \equiv 0 \quad \text{in} \quad B_{r_1}(x,t) \cap \Omega_1\}$$

and

$$S = \{0 \leq \ell \leq 1 : f(s) \in \Lambda_1 \quad \text{whenever} \quad 0 \leq s \leq \ell\}, \quad \ell_0 = \sup S.$$

We will prove that $f(\ell_0) \in \Lambda_1$. If $w \in B_{r_1}(f(\ell_0))$ and $r_2 = \|w - f(\ell_0)\|$ then there exists $0 < \delta < \ell_0$ such that $\|f(\ell_0) - f(\ell_0 - \delta)\| < r_1 - r_2$. Therefore

$$\|w - f(\ell_0 - \delta)\| \leq \|w - f(\ell_0)\| + \|f(\ell_0) - f(\ell_0 - \delta)\| < r_1,$$

and so $w \in B_{r_1}(f(\ell_0 - \delta))$. Now, from the definition of $\ell_0$ there exists $\ell_\delta \in S$ such that $\ell_0 - \delta < \ell_\delta \leq \ell_0$, what implies $f(\ell_0 - \delta) \in \Lambda_1$. Then, using (3.4) we see that

$$u \equiv 0 \quad \text{in} \quad B_{r_1}(f(\ell_0 - \delta)) \cap \Omega_1 = B_{r_1}(f(\ell_0 - \delta)). \tag{3.5}$$

Consequently we obtain that $u(w) = 0$ and then

$$u \equiv 0 \quad \text{in} \quad B_{r_1}(f(\ell_0)). \tag{3.6}$$

Hence, we have showed $f(\ell_0) \in \Lambda_1$.

If $\ell_0 = 1$ then from previous analysis we have that $Q = f(1) \in \Lambda_1 \subset \Lambda$. Thus, since $Q$ was arbitrarily chosen we obtain that $u \equiv 0$ in $\Omega_1$, which proves Theorem 3.2. Then to finish the proof of Theorem 3.2 remains to prove that $\ell_0 = 1$. In fact, let us suppose that $\ell_0 < 1$ and let

$$G = \{Y \in \Omega_1 \, : \, \|Y - f(\ell_0)\| = r_1\}.$$

For $w = (x_1, t_1) \in G$ fixed, we consider the change of variable $(x, t) \to (X, T)$ where

$$X = x - x_1,$$
$$T = t - t_1.$$

Notice that $(0, 0) \in G^* = \{Y = (X, T) \, : \, \|Y - (f(\ell_0) - w)\| = r_1\}$. Moreover, from (3.6) we see that

$$u(X, T) = 0, \quad (X, T) \in B_{r_1}(f(\ell_0) - w).$$

So that, by using Corollary 3.1, there exists $r_w^* > 0$ such that

$$u(X, T) = 0, \quad (X, T) \in B_{r_w^*}(0, 0).$$

Returning to the original variables we have that for each $w \in G$ there exists $r_w^* > 0$ such that

$$u \equiv 0 \quad \text{in} \quad B_{r_w^*}(w).$$

Then, using (3.6) and the compactness of $G$, we have that there is $\epsilon_1 > 0$ such that

$$u \equiv 0 \quad \text{in} \quad B_{r_1+\epsilon_1}(f(\ell_0)). \tag{3.7}$$

Now, we note that there exists $0 < \delta_1 < 1 - \ell_0$ such that if $w \in B_{r_1}(f(\ell_0 + \delta_1))$ then

$$\|w - f(\ell_0)\| \leq \|w - f(\ell_0 + \delta_1)\| + \|f(\ell_0 + \delta_1) - f(\ell_0)\| < r_1 + \epsilon_1.$$

Thus, $w \in B_{r_1+\epsilon_1}(f(\ell_0))$ and so $B_{r_1}(f(\ell_0 + \delta_1)) \subset B_{r_1+\epsilon_1}(f(\ell_0))$. Therefore, using (3.7) we have that $u \equiv 0$ in $B_{r_1}(f(\ell_0 + \delta_1))$. Consequently $f(\ell_0 + \delta_1) \in \Lambda_1$, which contradicts the definition of $\ell_0$. So, $\ell_0 = 1$ and the proof of Theorem 3.2 is complete.

□

## Acknowledgments

R. Córdoba was supported by University of Nariño (Colombia) and Anyi D. Corredor was supported by University of Cauca (Colombia).

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