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Existence theory for a fractional q-integral equations

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Abstract. The paper focuses on establishing sufficient conditions for the existence of the solutions for a functional equation involving q-fractional integrals, particularly in Banach spaces. In this method, the technique of measures of noncompactness and Petryshyn's fixed point theorem Banach space is used. We provide some examples of equations, which confirm that our result is applicable to a wide class of integral equations.

Keywords. Measure of noncompactness, existence of solution, q-integral equation, Petryshyn's fixed point theorem

1 Introduction

The existence of solutions for integral equations ($\mathbb{IE}$s) is a major challenge in mathematical research, leading to extensive investigation and development of various methodologies. Two notable frameworks for addressing solution existence are Schauder's theorem and Darbo's Fixed Point Theorem (F.P.T). A key concept in these approaches is the measure of noncompactness (M.N.C), which plays a foundational role in proving the existence results. The M.N.C is a highly significant and practical concept in non-linear analysis. It originated from Kuratowski's seminal paper [22], providing powerful tools for solving a wide range of $\mathbb{IE}$s and fractional differential equations ($\mathbb{FDE}$s).

In 1955, Darbo's fundamental F.P.T in [9], introduced the concept of M.N.C, which has since been widely employed by researchers to analyze the solvability of various types of $\mathbb{IE}$s and $\mathbb{DE}$s [1]. In fact, researchers have successfully generalized this theorem to address specific problem types they encounter. In [6], via a measure of non-compactness concept, Banas obtained an existence result nonlinear functional $\mathbb{IE}$s (for more details, see [7, 29, 31]). Caballero *et al.* in [8] studied the solvability of the functional $\mathbb{IE}$ as follows,

$$x(t) = f\left(t, \int_0^t v(t,s,x(s))\,\mathrm{d}s,\, x(\alpha(t))\right) \cdot g\left(t, \int_0^a x(t)u(t,s,x(s))\,\mathrm{d}s, x(\beta(t))\right), \tag{1.1}$$

for $x \in C(I_a)$, $I_a := [0,a]$. They showed under the some conditions the functional $\mathbb{IE}$s (1.1) has at least one solution $x \in C(I_a)$. In [10], authors with the help of generalized Darbo's F.P.T

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established the existence of solution of a $\mathbb{IE}$ with generalized fractional integral of two variables. Metwalli *et al.* in [24], studied the existence and the uniqueness of a.e. monotonic solutions of following problem,

$$\begin{aligned} &x(t) = h(t) + m(t) \cdot g(t, x(t-\tau)) + \int k(t,s) f(s, x(s-\tau))\,\mathrm{d}s, \\ &x(t) = \phi(t), \qquad t \in [-\tau, o). \end{aligned} \tag{1.2}$$

Mishra *et al.* in [25] obtained some results on the existence of solutions for a nonlinear Erdelyi-Kober fractional quadratic $\mathbb{IE}$ with deviating arguments. The solutions of system of functional $\mathbb{IE}$s in the setting of M.N.C investigated [26]. In [33], authors investigated existence of solutions in Banach algebra the nonlinear functional $\mathbb{IE}$ comprising of Hadamard fractional operators under certain relevant assumptions in conjunction with F.P. theory. Using the techniques of Darbo's F.P.T associated with M.N.C, authors established the existence results for functional stochastic $\mathbb{IE}$s in Banach Algebra [11]. Srivastava *et al.* [34] established the existence of the solution of a functional $\mathbb{IE}$ of two variables, which is of the form of the product of two operators in the Banach algebra $C(I_a \times I_a)$.

The existence of solution of q-integral equations (q-$\mathbb{IE}$s) have been analyzed by several researchers [2, 4, 5, 23, 30, 32]. In [17], the authors discussed the q-$\mathbb{IE}$,

$$x(t) = F\left(t, x(a(t)), \frac{x(b(t))}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s, x(s))\,\mathrm{d}_q s\right), \qquad t \in I_1, \tag{1.3}$$

where $\alpha > 1$, $q \in (0,1)$. With a generalized version of Darbo's theorem, they have established a sufficient conditions for the existence of at least one solution to (1.3). In [3], the author examined quadratic $\mathbb{IE}$ as follows,

$$y(t) = y(t) + \frac{(Ay)(t)}{\Gamma_q(\beta)} \int_0^t k(t,s)(t-qs)^{(\beta-1)} (By)(s)\,\mathrm{d}_q s, \qquad t \in I_1, \tag{1.4}$$

where $\beta > 0$, $q \in (0,1)$ and $A, B : C(I_1) \to C(I_1)$. Recently, Kazemi and Ezzati [19] utilized Petryshyn's F.P.T for the existence of solutions of nonlinear functional $\mathbb{IE}$s. They showed that Petryshyn's theorem is much more useful than Darbo's theorem. In fact, it dose not require to confirm the used operator of maps a closed convex subset onto itself [20].

Motivated and stimulated by the [19, 20], we introduced and analyze the existence result for the following a class of functional q-$\mathbb{IE}$ with fractional order,

$$\varkappa(t) = \Psi\left(t, \varkappa(a(t)), \pi(t, \varkappa(c(t))), \frac{1}{\Gamma_q(\varsigma)} \int_0^{r(t)} (r(t)-qs)^{(\varsigma-1)} p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\right), \tag{1.5}$$

where $0 < q < 1$, $\beta > 0$ and $\varsigma > 1$.

The paper is structured as follows. In Section 2, we collect some definitions, lemmas and theorems, which are essential to prove our main results. In Section 3, we introduce a new functional q-$\mathbb{IE}$ for existence of a solution a Banach space. Then, we establish and prove a existence theorem by utilized Petryshyn's F.P.T . In Section 4, we also give some examples to support our main theorem. Finally, in Section 5, concludes the paper.

2 Auxiliary facts and notations

In this section,we review some definitions and theorems, by stating some auxiliary facts and notations. Let

- X: Banach space,
- N_ε: A ball of radius ε,
- $\bar{N}_\varepsilon$: Sphere in X with radius ε,
- $C(I_a)$: All real functions continuous on I_a,
- $B(X)$: Bounded set of X,
- Fix(T): the set of fixed points of T in$X$.

In 1910, Jackson introduced the concept of quantum calculus in [16]. There are several applications of q-calculus in physical problems such as molecular problems [13], elementary particle physics, and chemical physics [12, 15].Basic definitions and properties of quantum calculus can be found in the book [18]. In follow, we recall some basic facts on quantum calculus and present additional properties that will be used later.

For $q \in (0,1)$, we define $[\varkappa]_q = \frac{1-q^{\varkappa}}{1-q}, \varkappa \in R$. The q-analogue of the power function $(1-\vartheta)^k$ with $k \in \{0,1,2,\ldots\}$ is,

$$(1-\vartheta)^{(0)} = 1, \qquad (1-\vartheta)^{(k)} = \prod_{i=0}^{k-1}(1-\vartheta q^i), \quad k \in N, \vartheta \in \mathbb{R}. \tag{2.1}$$

Also, The q-integral of a function f defined on the interval I_a is given by,

$$\int_0^\nu f(\zeta)\, \mathrm{d}_q\zeta = \nu(1-q)\sum_{i=0}^{\infty} f(\nu q^i)q^i, \qquad \nu \in I_a. \tag{2.2}$$

For any $m, n > 0$, $B_q(m,n) = \int_0^1 \zeta^{(m-1)}(1-q\zeta)^{(n-1)}\, \mathrm{d}_q\zeta$ is called the q-beta function.

Lemma 2.1 ([28]). *Let $m, n > 0$, and $0 < q < 1$. Then we have*

$$\int_0^\nu (\nu - q\zeta)^{(m-1)}\zeta^{(n)}\, \mathrm{d}_q\zeta = \nu^{m+n}B_q(m, n+1). \tag{2.3}$$

Definition 1 ([22]). Let $\mathrm{T} \subset B(X)$, then

$$\phi(\mathrm{T}) = \inf\{\varepsilon : \textit{there exist a finite number of sets of diameter} \leq \varepsilon \textit{ that can cover } \mathrm{T}\}, \tag{2.4}$$

is said to be the Kuratowski M.N.C.

Definition 2 ([14]). Let $\mathrm{T} \subset B(X)$, then

$$\zeta(\mathrm{T}) = \inf\Big\{\varepsilon > 0 : \mathrm{T} \text{ has a finite } \varepsilon-\text{net in } X\Big\}, \tag{2.5}$$

is said to be the Hausdorff M.N.C.

Definition 3 ([14]). For $\mathrm{T} \subset B(X)$, the M.N.C ϕ and ζ fulfill $\zeta(\mathrm{T}) \leq \phi(\mathrm{T}) \leq 2\zeta(\mathrm{T})$.

For $h \in C(I_a)$ and $\varepsilon \geq 0$ denoted by $\delta_\varepsilon(h)$, the modulus of continuity of the function h, i.e.,

$$\delta_\varepsilon(h) = \sup \Big\{ |h(t) - h(\hat{t})| \,:\, |t - \hat{t}| \leq \varepsilon \Big\}. \tag{2.6}$$

The uniformly continuous h on I_a implies that $\delta_\varepsilon(h) \to 0$ as $\varepsilon \to 0$.

Theorem 2.1 ([21]). *For* $\mathrm{T} \subset B(X)$, *the* M.N.C *in* $C(I_a)$ *is denoted by,*

$$\mathrm{T} = \limsup_{\varepsilon \to 0} \delta_\varepsilon t, \;\; t \in \mathrm{T}. \tag{2.7}$$

Theorem 2.2 ([27]). *Let* $\mathfrak{F}, \mathfrak{E} \subset X$ *then*

1. $\zeta(\mathfrak{F} \bigcup \mathfrak{E} = \max \big\{ \zeta(\mathfrak{F}), \zeta(\mathfrak{E}) \big\}$;
2. $\zeta(\mathfrak{F} + \mathfrak{E}) \leq \zeta(\mathfrak{F}) + \zeta(\mathfrak{E})$;
3. $\zeta(\varsigma\mathfrak{F}) = |\varsigma|\zeta(\mathfrak{F})$, *where* $\varsigma\mathfrak{F} = \big\{ \varsigma m : \; m \in \mathfrak{F} \big\}$;
4. $\zeta(\mathfrak{F}) \leq \zeta(\mathfrak{E})$, *for* $\mathfrak{F} \subset \mathfrak{E}$;
5. $\zeta(\bar{co}\mathfrak{F}) = \zeta(\mathfrak{F})$;

Recall that $\digamma \in C(X)$ is called a κ-set contraction if for $K \subset B(X)$, $\digamma$ is bounded and $\phi(\digamma K) \leq \kappa \phi(K)$ for each $0 < \kappa < 1$. Moreover, If $\phi(\digamma K) \leq \phi(K)$, $\digamma$ is called condensing map.

Let us recall that Petryshyn's F.P.T,

Theorem 2.3 ([27]). *Let* $\digamma : N_\varepsilon \to X$ *be a condensing mapping and satisfying the following boundary condition,*

$$\text{if } \digamma(x) = \kappa x, \text{ for some } x \in \bar{N}_\varepsilon, \text{ with } \kappa \leq 1. \tag{2.8}$$

Then $\mathrm{Fix}_{N_\varepsilon}(\digamma) \neq \varnothing$.

In order to prove the main results, we first establish the following lemma.

Lemma 2.2. *The following inequality is hold*

$$\begin{aligned} \Big| \int_0^{r(t)} & (r(t) - qs)^{(\varsigma-1)} p(t, s, \varkappa(b(s)))\, \mathrm{d}_q s \\ & - \int_0^{r(\hat{t})} (r(\hat{t}) - qs)^{(\varsigma-1)} p(\hat{t}, s, \varkappa(b(s)))\, \mathrm{d}_q s \Big| \\ & \leq \hat{\delta}_\tau(u) D^\varsigma B_q(\varsigma, 1) + 2L B_q(\varsigma, 1) \delta_\tau(w)^\varsigma, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} & \delta_\tau(r) = \sup \Big\{ |r(t) - r(\hat{t})| \,:\, t, \hat{t} \in I_a, \, |t - \hat{t}| \leq \tau \Big\}, \\ & L = \sup \Big\{ |p(t, s, z)| \,:\, t \in I_a, \, s \in [0, D], \, z \in [-\varepsilon, \varepsilon] \Big\}, \\ & \hat{\delta}_\tau(u) = \sup \Big\{ |p(t, x, y) - p(\hat{t}, x, y)| \,:\, |t - \hat{t}| \leq \tau, \, x \in [0, D], \, y \in [-\varepsilon, \varepsilon] \Big\}. \end{aligned} \tag{2.10}$$

Proof. We have

$$
\begin{aligned}
&\bigg| \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(t, s, \varkappa(b(s)))\, \mathrm{d}_q s \\
&\qquad - \int_0^{r(\hat{t})} (r(\hat{t}) - qs)^{(\varsigma-1)} p(\hat{t}, s, \varkappa(b(s)))\, \mathrm{d}_q s \bigg| \\
&\qquad \le \bigg| \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(t, s, \varkappa(b(s)))\, \mathrm{d}_q s \\
&\qquad - \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(\hat{t}, s, \varkappa(b(s)))\, \mathrm{d}_q s \bigg| \\
&\qquad + \bigg| \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(\hat{t}, s, \varkappa(b(s)))\, \mathrm{d}_q s \\
&\qquad - \int_0^{r(\hat{t})} (r(t) - qs)^{(\varsigma-1)} p(\hat{t}, s, \varkappa(b(s)))\, \mathrm{d}_q s \bigg| \\
&\qquad + \bigg| \int_0^{r(\hat{t})} (r(t) - qs)^{(\varsigma-1)} p(\hat{t}, s, \varkappa(b(s)))\, \mathrm{d}_q s \\
&\qquad - \int_0^{r(\hat{t})} (r(\hat{t}) - qs)^{(\varsigma-1)} p(\hat{t}, s, \varkappa(b(s)))\, \mathrm{d}_q s \bigg| \\
&\qquad \le \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} \Big| p(t, s, \varkappa(b(s))) - p(\hat{t}, s, \varkappa(b(s)))\, \mathrm{d}_q s \Big| \\
&\qquad + \int_{r(\hat{t})}^{r(t)} (r(t) - qs)^{(\varsigma-1)} \Big| p(\hat{t}, s, \varkappa(b(s))) \Big|\, \mathrm{d}_q s \\
&\qquad + \int_0^{r(\hat{t})} [(r(\hat{t}) - qs)^{(\varsigma-1)} - (r(t) - qs)^{(\varsigma-1)}] \Big| p(\hat{t}, s, \varkappa(b(s))) \Big|\, \mathrm{d}_q s \\
&\qquad \le \hat{\delta}_\tau(u) \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)}\, \mathrm{d}_q s \\
&\qquad + L \int_0^{r(\hat{t})} (r(\hat{t}) - qs)^{(\varsigma-1)} - (r(t) - qs)^{(\varsigma-1)}\, \mathrm{d}_q s \\
&\qquad + \int_{r(\hat{t})}^{r(t)} (r(t) - qs)^{(\varsigma-1)}\, \mathrm{d}_q s \\
&\qquad = \hat{\delta}_\tau(u) r(t)^\varsigma B_q(\varsigma, 1) + L B_q(\varsigma, 1) \Big[r(\hat{t})^\varsigma - r(t)^\varsigma + 2(r(t) - r(\hat{t}))^\varsigma \Big] \\
&\qquad \le \hat{\delta}_\tau(u) r(t)^\varsigma B_q(\varsigma, 1) + 2L B_q(\varsigma, 1)(r(t) - r(\hat{t}))^\varsigma \\
&\qquad \le \hat{\delta}_\tau(u) D^\varsigma B_q(\varsigma, +1) + 2L B_q(\varsigma, 1) \delta(r, \tau)^\varsigma.
\end{aligned} \tag{2.11}
$$

□

3 A Existence Theorem

In this section, we will study the existence of q-$\mathbb{IE}$ (1.5) under the following assumptions, S1) $\varkappa \in C(I_a)$, $\Psi \in C(I_a \times \mathbb{R}^{3})$, $\pi \in C(I_a \times \mathbb{R})$, $p \in C(I_a \times [0, D] \times \mathbb{R})$, $c : I_a \to I_a$, $b : \mathbb{R}^+ \to I_a$, $r : I_a \to \mathbb{R}^+$ are continuous and for each $t \in I_a$ and $r(t) \le D$;

S2) There exist nonnegative constant k_1, k_2, k_3 and k_4 with $k_1 + k_2k_4 \le 1$ s.t.

$$|\Psi(t, t_1, t_2, t_3) - \Psi(t, \hat{t_1}, \hat{t_2}, \hat{t_3})| \le k_1|t_1 - \hat{t_1}| + k_2|t_2 - \hat{t_2}| + k_3|t_3 - \hat{t_3}|, \tag{3.1}$$

and $|\pi(x, y) - \pi(x, \hat{y})| \le k_4|y - \hat{y}|$;

S3) For $\varepsilon \ge 0$, the operator Ψ satisfies the following condition,

$$\sup \Big\{ |\Psi(t, x, y, z)| : t \in I_a,\ x, y \in [-\varepsilon, \varepsilon], \\ -\frac{D^\varsigma}{\Gamma_q(\varsigma)}\ L\ B_q(\varsigma, 1) \le z \le \frac{D^\varsigma}{\Gamma_q(\varsigma)}\ L\ B_q(\varsigma, 1) \Big\} \le \varepsilon, \tag{3.2}$$

where $L = \sup \Big\{ |p(t, s, z)| : t \in I_a,\ s \in [0, D], z \in [-\varepsilon, \varepsilon] \Big\}$.

Theorem 3.1. *Under the assumption* (S1)-(S3)*, the functional q-IIE* (1.5) *has at least one solution in $C(I_a)$.*

Proof. Let

$$\begin{cases} F : N_\varepsilon \to X, \\ (F\varkappa)(t) = \Psi\Big(t, \varkappa(a(t)), \pi(t, \varkappa(c(t))), \\ \qquad \dfrac{1}{\Gamma_q(\varsigma)} \displaystyle\int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(t, s, \varkappa(b(s)))\, \mathrm{d}_q s\Big). \end{cases} \tag{3.3}$$

We divided the proof into several steps.

Step 1. The operator F is continuous on the ball N_ε. Consider $\tau > 0$ and any $\varkappa, \hat{\varkappa} \in N_\varepsilon$ s.t. $\|\varkappa - \hat{\varkappa}\| < \tau$. We have,

$$\begin{aligned} |(F\varkappa)(t) - (F\hat{\varkappa})(t)| = \Big| &\Psi\Big(t, \varkappa(a(t)), \pi(t, \varkappa(c(t))), \\ &\frac{1}{\Gamma_q(\varsigma)} \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(t, s, \varkappa(b(s)))\, \mathrm{d}_q s\Big) \\ &- \Psi\Big(t, \hat{\varkappa}(a(t)), \pi(t, \hat{\varkappa}(c(t))), \\ &\frac{1}{\Gamma_q(\varsigma)} \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(t, s, \hat{\varkappa}(b(s)))\, \mathrm{d}_q s\Big) \Big| \\ \le \Big| &\Psi\Big(t, \varkappa(a(t)), \pi(t, \varkappa(c(t))), \\ &\frac{1}{\Gamma_q(\varsigma)} \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(t, s, \varkappa(b(s)))\, \mathrm{d}_q s\Big) \\ &- \Psi\Big(t, \hat{\varkappa}(a(t)), \pi(t, \varkappa(c(t))), \\ &\frac{1}{\Gamma_q(\varsigma)} \int_0^{r(t)} (r(t) - qs)^{(\varsigma-1)} p(t, s, \varkappa(b(s)))\, \mathrm{d}_q s\Big) \Big| \end{aligned}$$

$$
\begin{aligned}
&+\Bigg|\Psi\bigg(t,\hat{\varkappa}(a(t)),\pi(t,\varkappa(c(t))),\\
&\frac{1}{\Gamma_q(\varsigma)}\int_0^{r(t)}(r(t)-qs)^{(\varsigma-1)}p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\\
&-\Psi\bigg(t,\hat{\varkappa}(a(t)),\pi(t,\hat{\varkappa}(c(t))),\\
&\frac{1}{\Gamma_q(\varsigma)}\int_0^{r(t)}(r(t)-qs)^{(\varsigma-1)}p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\Bigg|\\
&+\Bigg|\Psi\bigg(t,\hat{\varkappa}(a(t)),\pi(t,\hat{\varkappa}(c(t))),\\
&\frac{1}{\Gamma_q(\varsigma)}\int_0^{r(t)}(r(t)-qs)^{(\varsigma-1)}p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\\
&-\Psi\bigg(t,\hat{\varkappa}(a(t)),\pi(t,\hat{\varkappa}(c(t))),\\
&\frac{1}{\Gamma_q(\varsigma)}\int_0^{r(t)}(r(t)-qs)^{(\varsigma-1)}p(t,s,\hat{\varkappa}(b(s)))\,\mathrm{d}_q s\bigg)\Bigg|\\
&\le k_1\Big|\varkappa(a(t))-\hat{\varkappa}(a(t))\Big|+k_2\Big|g(t,\varkappa(c(t)))-g(t,\hat{\varkappa}(c(t)))\Big|\\
&+\frac{k_3}{\Gamma_q(\varsigma)}\int_0^{r(t)}(r(t)-qs)^{(\varsigma-1)}\Big|p(t,s,\varkappa(b(s)))-p(t,s,\hat{\varkappa}(b(s)))\Big|\,\mathrm{d}_q s\\
&\le k_1\Big|\varkappa(a(t))-\hat{\varkappa}(a(t))\Big|+k_2k_4\Big|\varkappa(c(t))-\hat{\varkappa}(c(t))\Big|\\
&+\frac{k_3}{\Gamma_q(\varsigma)}\delta_\tau(u)r(t)^{(\varsigma)}B_q(\varsigma,1)\\
&\le (k_1+k_2k_4)\|\varkappa-\hat{\varkappa}\|+\frac{k_3}{\Gamma_q(\varsigma)}\delta_\tau(u)D^{\varsigma}B_q(\varsigma,1),
\end{aligned}\tag{3.4}
$$

where

$$
\begin{aligned}
\delta_\tau(u)=\sup\Big\{&|p(t,x,y)-p(t,x,\hat{y})|\,:\,t\in I_a,\ x\in[0,D],\\
&y,\hat{y}\in[-\varepsilon,\varepsilon],\ |y-\hat{y}|\le\varepsilon\Big\}.
\end{aligned}\tag{3.5}
$$

The uniformly continuity of $p(\cdot,\cdot,\cdot)$ on the subset $I_a\times[0,D]\times\mathbb{R}$ implies that $\delta_\tau(u)\to 0$ as $\tau\to 0$. This shows that the operator $\digamma$ is continuous on N_ε.

Step 2. The densifying condition operator $\digamma$. Let E_τ be a bounded subset in X and $t,\hat{t}\in I_a$ such that $t\le\hat{t}$ and $t-\hat{t}\le\tau$ for $\tau>0$, Lemma 2.2 implies that,

$$
\begin{aligned}
|(\digamma\varkappa)(t)-(\digamma\varkappa)(\hat{t})|=&\Bigg|\Psi\bigg(t,\varkappa(a(t)),\pi(t,\varkappa(c(t))),\\
&\frac{1}{\Gamma_q(\varsigma)}\int_0^{r(t)}(r(t)-qs)^{(\varsigma-1)}p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\\
&-\Psi\bigg(\hat{t},f(\hat{t},\varkappa(a(\hat{t}))),\pi(t,\varkappa(c(\hat{t}))),\\
&\frac{1}{\Gamma_q(\varsigma)}\int_0^{r(\hat{t})}(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Bigg|\Psi\bigg(t, \varkappa(a(t)), \pi(t, \varkappa(c(t))),\\
&\quad \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(t)}(r(t)-qs)^{(\varsigma-1)}p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\\
&\quad -\Psi\bigg(t, \varkappa(a(t)), \pi(t, \varkappa(c(t))),\\
&\quad \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(\hat{t})}(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\Bigg|\\
&\quad +\Bigg|\Psi\bigg(t, \varkappa(a(t)), \pi(t, \varkappa(c(t))),\\
&\quad \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(\hat{t})}(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\\
&\quad -\Psi\bigg(t, \varkappa(a(t)), \pi(\hat{t}, \varkappa(c(\hat{t}))),\\
&\quad \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(\hat{t})}(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\Bigg|\\
&\quad +\Bigg|\Psi\bigg(t, \varkappa(a(\hat{t}))), \pi(\hat{t}, \varkappa(c(\hat{t}))),\\
&\quad \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(\hat{t})}(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\Bigg|\\
&\quad -\Psi\bigg(t, \varkappa(a(\hat{t}))), \pi(\hat{t}, \varkappa(c(\hat{t}))),\\
&\quad \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(\hat{t})}(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\Bigg|\\
&\quad +\Bigg|\Psi\bigg(t, f(\hat{t}, \varkappa(a(\hat{t}))), \pi(\hat{t}, \varkappa(c(\hat{t}))),\\
&\quad \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(\hat{t})}(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\\
&\quad -\Psi\bigg(\hat{t}, f(\hat{t}, \varkappa(a(\hat{t}))), \pi(\hat{t}, \varkappa(c(\hat{t}))),\\
&\quad \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(\hat{t})}(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\bigg)\Bigg|\\
&\leq \frac{k_3}{\Gamma_q(\varsigma)}\Bigg|\int_0^{r(t)}(r(t)-qs)^{(\varsigma-1)}p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\\
&\quad -\int_0^{r(\hat{t})}s(r(\hat{t})-qs)^{(\varsigma-1)}p(\hat{t},s,\varkappa(b(s)))\,\mathrm{d}_q s\Bigg|\\
&\quad +k_2\Big|g(t,\varkappa(c(t))-\pi(t,\varkappa(c(\hat{t}))\Big| + k_2\Big|g(t,\varkappa(c(t))-\pi(\hat{t},\varkappa(c(\hat{t})))\Big|\\
&\quad +k_1\Big|\varkappa(a(t))-\varkappa(a(\hat{t}))\Big| + k_1\Big|\varkappa(a(t))-f(\hat{t},\varkappa(a(\hat{t})))\Big| + \delta(\Psi,\tau)
\end{aligned}
$$

$$\le \frac{k_3\hat{\delta}_\tau(u)}{\Gamma_q(\varsigma)} D^\varsigma B_q(\varsigma,1) + \frac{2k_3 L}{\Gamma_q(\varsigma)} B_q(\varsigma,1)\delta_\tau(w)^\varsigma + k_1\Big|\varkappa(a(t)) - \varkappa(a(\hat{t})\Big|$$
$$+ k_2k_4\Big|\varkappa(c(t)) - \varkappa(c(\hat{t}))\Big| k_1\delta_\tau(\pi) + k_2\delta_\tau(f) + \delta_\tau(\Psi), \tag{3.6}$$

where

$$\delta_\tau(\Psi) = \sup\Big\{|\Psi(t,x,y,z) - \Psi(\hat{t},x,y,z)| \,:\, |t-\hat{t}| \le \tau,$$
$$x,y \in [-\varepsilon,\varepsilon],\, z \in [-DL, DL]\Big\},$$
$$\delta_\tau(r) = \sup\Big\{|r(t) - r(\hat{t})| \,:\, t,\hat{t} \in I_a, |t-\hat{t}| \le \tau\Big\},$$
$$\delta_\tau(\pi) = \sup\Big\{|\pi(t,x) - \pi(\hat{t},x)| \,:\, |t-\hat{t}| \le \tau,\, x \in [-\varepsilon,\varepsilon]\Big\},$$
$$\hat{\delta}_\tau(u) = \sup\Big\{|p(t,x,y) - p(\hat{t},x,y)| \,:\, |t-\hat{t}| \le \tau,\, x \in [0,D],\, y \in [-\varepsilon,\varepsilon]\Big\}. \tag{3.7}$$

So

$$\delta(F\varkappa,\tau) \le \frac{k_3\hat{\delta}_\tau(u)}{\Gamma_q(\varsigma)} D^\varsigma B_q(\varsigma,1) + \frac{2k_3 L}{\Gamma_q(\varsigma)} B_q(\varsigma,1)\delta_\tau(r)^\varsigma$$
$$+ k_1\delta(\varkappa,\delta_\tau(a(t))) + k_2k_4\delta(\varkappa,\delta_\tau(c(t))). \tag{3.8}$$

Now, by taking limit of (3.8) equation as $\tau \to 0$ would give easily following inequality

$$\delta(F\varkappa,\tau) \le (k_1 + k_2k_4)\delta(\varkappa,\tau), \tag{3.9}$$

which this provide $\zeta(FE_\tau) \le (k_1 + k_2k_4)\zeta(E_\tau)$. Therefore, Ψ is condensing map.

Step 3. The boundary condition. Suppose that $z \in \bar{N}_\varepsilon$. If $\|Fz\| = k\|z\| = k\varepsilon$, the condition (S3) implies that

$$\Big|(Fz)(t)\Big| = \Bigg|\Psi\Bigg(t, \varkappa(a(t)), \pi(t,\varkappa(c(t))),$$
$$\frac{1}{\Gamma_q(\varsigma)}\int_0^{r(t)} (r(t)-qs)^{(\varsigma-1)} p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\Bigg)\Bigg| \le \varepsilon, \qquad \forall t \in I_a. \tag{3.10}$$

Hence $\|Fz\| \le \varepsilon$, i.e., $k \le 1$. This complete the proof. □

Corollary 3.2. *Assume that the following conditions are holds,*

(P1) $\varkappa \in C(I_a \times I_a \times \mathbb{R})$, $g \in C(I_a \times \mathbb{R})$, $r \in C(I_a, \mathbb{R}^+)$, $r(t) \le D$, *and there exist positive constant* l_i, $i = 1,\dots,5$ *s.t.*

$$|\Psi(t,x,y) - \Psi(t,\bar{x},\bar{y})| \le l_1|x-\bar{x}| + l_2|y-\bar{y}|, \tag{3.11}$$

$|\Psi(t,0,0)| \le l_3$, $|\pi(t,x) - \pi(t,\bar{x})| \le l_4|x-\bar{x}|$, *and* $|\pi(t,0)| \le l_5$*;*

(P2) $p(t,s,x) \in C(I_a \times I_a \times \mathbb{R})$ *and* $\exists c_1, c_2 > 0$ *s.t.* $|p(t,s,x)| \le c_1 + c_2|x|$, $\forall t,s \in I_a$, $x \in \mathbb{R}$*;*

(P3) $l_1 + \frac{1}{\Gamma_q(\varsigma)} l_2(l_4 + l_5) D^\varsigma B_q(\varsigma,1)(c_1+c_2) < 1$.

Then

$$\varkappa(t) = \Psi\Bigg(t, \varkappa(a(t)), \frac{\pi(t,\varkappa(c(t)))}{\Gamma_q(\varsigma)}\int_0^{r(t)} (t-qs)^{(\varsigma-1)} p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\Bigg), \tag{3.12}$$

has at least one solution in $C(I_a)$.

Proof. One can easily check, $(P1)$ implied that $(S2)$ in Theorem 3.1. Now, we prove $(S3)$ holds. Let $\varepsilon > 0, \|\varkappa\| \leq \varepsilon$. We obtain,

$$\begin{aligned}
|\varkappa(t)| = &\Bigg|\Psi\Bigg(t, \varkappa(a(t)),\\
&\frac{\pi(t,\varkappa(c(t)))}{\Gamma_q(\varsigma)} \int_0^{r(t)} (t-qs)^{(\varsigma-1)} p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\Bigg)\Bigg|\\
\leq &\Bigg|\Psi\Bigg(t, \varkappa(a(t)),\\
&\frac{\pi(t,\varkappa(c(t)))}{\Gamma_q(\varsigma)} \int_0^{r(t)} (t-qs)^{(\varsigma-1)} p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\Bigg)\\
&-\Psi(t,0,0)\Bigg| + \big|\Psi(t,0,0)\big|\\
\leq &\, l_1\|\varkappa(a(t))\|\\
&+ l_2\Bigg|\frac{\pi(t,\varkappa(c(t)))}{\Gamma_q(\varsigma)} \int_0^{r(t)} (t-qs)^{(\varsigma-1)} p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\Bigg| + l_3\\
\leq &\, l_1\|\varkappa(a(t))\| + l_2\Big(|\pi(t,\varkappa(c(t))) - \pi(t,0)|\\
&+ |\pi(t,0)|\Big)\frac{1}{\Gamma_q(\varsigma)}\Bigg|\int_0^{r(t)} (t-qs)^{(\varsigma-1)} p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\Bigg|\\
\leq &\, l_1\|\varkappa(a(t))\| + l_2\Big(l_4\|\varkappa(c(t))\|\\
&+ l_5\Big)\frac{1}{\Gamma_q(\varsigma)} r(t)^{\varsigma} B_q(\varsigma,1)\Big(c_1 + c_2\|\varkappa(c(t))\|\Big).
\end{aligned} \tag{3.13}$$

Hence, ε in (S3) is real number that satisfies

$$l_1\varepsilon + \frac{1}{\Gamma_q(\varsigma)} l_2(l_4\varepsilon + l_5) D^{\varsigma} B_q(\varsigma,1)(c_1 + c_2\varepsilon) \leq \varepsilon. \tag{3.14}$$

Now, we define the continuous function $\Sigma : [0,1] \to \mathbb{R}$ as follows,

$$\Sigma(\varepsilon) = l_1\varepsilon + \frac{1}{\Gamma_q(\varsigma)} l_2(l_4\varepsilon + l_5) D^{\varsigma+} B_q(\varsigma,1)(c_1 + c_2\varepsilon) - \varepsilon. \tag{3.15}$$

The property $(P3)$ implies that $\Sigma(0)\Sigma(1) < 0$, then $\exists\varepsilon \in (0,1)$ s.t. $\Sigma(\varepsilon) = 0$. □

Remark 1. Jleli *et al.* [17] introduced a functional q-$\mathbb{IE}$ (1.3) that include the assumptions (A1)-(A9). They proved the functional q-$\mathbb{IE}$ has at least one solution in $C([0,1])$. By utilization of suggestion technique in Theorem 3.1 and Corollary 3.2, one can achieve the desired result with the least conditions compared to Jleli article.

Corollary 3.3. *Let*

1) $\varkappa \in C(I_a, \mathbb{R})$, $\varphi \in C(I_a \times \mathbb{R}^2)$, $f, \pi \in C(I_a \times \mathbb{R})$, $p \in C(I_a \times [0,D] \times \mathbb{R})$, *and* $c : I_a \to I_a$, $b : \mathbb{R}^+ \to I_a$, $r : I_a \to \mathbb{R}^+$ *are continuous and for each* $t \in I_a$, $r(t) \leq D$;
2) *There exist nonnegative constant* k_1, k_2, k_3 *and* k_4 *with* $k_1k_3 + k_4 \leq 1$ *s.t.,*

$$|\varphi(t,t_1,t_2) - \Psi(t,\hat{t_1},\hat{t_2})| \leq k_1|t_1 - \hat{t_1}| + k_2|t_2 - \hat{t_2}|, \tag{3.16}$$

$|\pi(x,y) - \pi(x,\hat{y})| \leq k_3|y - \hat{y}|$, *and* $|f(s,y) - f(s,\bar{y})| \leq k_4|y - \bar{y}|$;

3) *For $\varepsilon \geq 0$, the operator Ψ satisfies the following condition,*

$$\sup\Bigg\{|\varphi(t,y,z)| : t \in I_a,\ y \in [-\varepsilon,\varepsilon], \\ -\frac{D^{\varsigma}}{\Gamma_q(\varsigma)}\ L\ B_q(\varsigma,1) \leq z \leq \frac{D^{\varsigma}}{\Gamma_q(\varsigma)}\ L\ B_q(\varsigma,1)\Bigg\} \leq \varepsilon, \tag{3.17}$$

where $L = \sup\{|p(t,s,z)| : t \in I_a, s \in [0,D],\ z \in [-\varepsilon,\varepsilon]\}$.

Then

$$\varkappa(t) = f(t,\varkappa(a(t))) + \varphi\Bigg(t, \pi(t,\varkappa(c(t))), \\ \frac{1}{\Gamma_q(\varsigma)}\int_0^{r(t)} (r(t)-qs)^{(\varsigma-1)} p(t,s,\varkappa(b(s)))\,\mathrm{d}_q s\Bigg), \tag{3.18}$$

has at least one solution in $C(I_b)$.

4 Examples

In this section, based on the explained approach, we incline to present some examples by using Maple software to grantee Theorem 3.1.

Example 1. Consider the following nonlinear q-$\mathbb{IE}$,

$$\varkappa(t) = \frac{1}{2}e^{-t^2}\sin\frac{\varkappa(t)}{2} + \frac{e^{-\sqrt{t}}t^2}{4+4t^2}\ln(1+|\varkappa(t)|) \\ + \frac{1}{\Gamma_{1/2}\left(\frac{3}{2}\right)}\int_0^{\sqrt{t}}\left(\sqrt{t}-\frac{1}{2}s\right)^{1/2}\frac{\varkappa(s)}{2+s^2}\,\mathrm{d}_q s, \qquad t \in I_1. \tag{4.1}$$

The assumptions (S1) and (S2) of Theorem 3.1 are satisfied. Now, we check that (S3) also holds. Suppose that $\|\varkappa(t)\| \leq \varepsilon$, then,

$$|\varkappa(t)| = \Bigg|\frac{1}{2}e^{-t^2}\sin\frac{\varkappa(t)}{2} + \frac{e^{-\sqrt{t}}t^2}{4+4t^2}\ln(1+|\varkappa(t)|) \\ + \frac{1}{\Gamma_{1/2}\left(\frac{3}{2}\right)}\int_0^{\sqrt{t}}\left(\sqrt{t}-\frac{1}{2}s\right)^{1/2}\frac{\varkappa(s)}{2+s^2}\,\mathrm{d}_q s\Bigg| \\ \leq \frac{1}{2} + \frac{1}{4}\varepsilon + \frac{1}{2\Gamma_{1/2}\left(\frac{3}{2}\right)}B_{1/2}\left(\frac{3}{2},1\right)\ \varepsilon \leq \varepsilon. \tag{4.2}$$

Hence, (S3) holds if, $\varepsilon \geq 1.51496$. This implies that the equation has at least one solution in $C(I_1)$.

Example 2. Consider the following nonlinear q-$\mathbb{IE}$,

$$\varkappa(t) = \frac{1}{4}\left(te^{-t} + t^3\varkappa(t)\right)$$

$$+\frac{1}{\Gamma_{1/4}\left(\frac{5}{2}\right)}\int_0^{t^2}\left(t^2-\frac{1}{4}s\right)^{3/2}\left(\frac{1}{2}\varkappa(s^2)+se^{-2t}\frac{\sin(t)}{2+|\cos(t)|}\right)\mathrm{d}_q s. \tag{4.3}$$

One can see that, the assumptions (S1) and (S2) of Theorem 3.1 are satisfied. Now, we check that (S3) also holds. Suppose that $\|\varkappa(t)\| \leq \varepsilon$, then,

$$\begin{aligned}|\varkappa(t)| &= \left|\frac{1}{4}\left(te^{-t}+t^3\varkappa(t)\right)\right. \\ &\quad \left.+\frac{1}{\Gamma_{1/4}\left(\frac{5}{2}\right)}\int_0^{t^2}\left(t^2-\frac{1}{4}s\right)^{3/2}\left(\frac{1}{2}\varkappa(s^2)+se^{-2t}\frac{\sin(t)}{2+|\cos(t)|}\right)\mathrm{d}_q s\right| \\ &\leq \frac{1}{4}(\varepsilon+1)+\frac{1}{\Gamma_{1/4}\left(\frac{5}{2}\right)}B_{1/4}\left(\frac{5}{2},1\right)\left(\frac{1}{2}\varepsilon+\frac{1}{2}\right)\leq\varepsilon. \end{aligned} \tag{4.4}$$

Thus, (S3) holds if, $\varepsilon \geq 1.50045$. This implies that the equation has at least one solution in $C(I_1)$.

Example 3. Consider the following nonlinear q-$\mathbb{IE}$,

$$\begin{aligned}\varkappa(t) &= \frac{e^{-2t}}{3}+\frac{\ln(1+|\varkappa(\sqrt{t})|)}{4+t^2}+\frac{|\varkappa(\sqrt{t})|)}{1+|\varkappa(\sqrt{t})|)} \\ &\quad +\frac{1}{\Gamma_{1/4}\left(\frac{5}{2}\right)}\int_0^{t^2}\left(t^2-\frac{1}{4}s\right)^{3/2}\sqrt{\frac{1+\varkappa(\sqrt{t})}{1+st}}\,\mathrm{d}_q s, \qquad t\in[0,1]. \end{aligned} \tag{4.5}$$

The assumptions (S1) and (S2) of Theorem 3.1 are satisfied. Now, we check that (S3) also holds. Suppose that $\|\varkappa(t)\| \leq \varepsilon$, then,

$$\begin{aligned}|\varkappa(t)| &= \left|\frac{e^{-2t}}{3}+\frac{\ln(1+|\varkappa(\sqrt{t})|}{4+t^2}+\frac{|\varkappa(\sqrt{t})|)}{1+|\varkappa(\sqrt{t})|}\right. \\ &\quad \left.+\frac{1}{\Gamma_{1/4}\left(\frac{5}{2}\right)}\int_0^{t^2}\left(t^2-\frac{1}{4}s\right)^{3/2}\sqrt{\frac{1+\varkappa(\sqrt{t})}{1+st}}\,\mathrm{d}_q s\right| \\ &\leq \frac{1}{3}+\frac{1}{4}\varepsilon+\frac{1}{3}+\frac{1}{\Gamma_{1/4}\left(\frac{5}{2}\right)}B_{1/4}\left(\frac{5}{2},1\right)\sqrt{1+\varepsilon}\leq\varepsilon. \end{aligned} \tag{4.6}$$

Hence, (S3) holds if, $\varepsilon \geq 2.67961$. This implies that the equation has at least one solution in $C(I_1)$.

Remark 2. Since there is no constants c_1 and c_2 satisfying the inequalities (Sublinear condition), $|p(t,s,x)| \leq c_1 + c_2|x|$, for all $t, s \in I_a$ and $x \in \mathbb{R}$, the results in [17] and [3] are inapplicable to the q-$\mathbb{IE}$ 4.5.

5 Conclusion

Because many problems of q-$\mathbb{IE}$s are unsolvable, it is important to ensure that a solution exists in these equations. Therefore, many researchers have published papers in this field and have explained their methods with the results obtained. Accordingly in this paper, the authors presented a new method based on the technique of M.N.C and Petryshyn's F.P.T. The advantages of the proposed method compared to other similar methods are that it has fewer conditions, and there is no need to verify the involved operator maps a closed convex subset onto itself.

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