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Addendum to: Solving an inverse problem for the Sturm-Liouville operator with singular potential by Yurko's method (Tamkang J. Math. 52 (2021), no. 1, 125-154)

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Abstract. This addendum outlines a simpler proof of Theorem 2.1 from [N.P. Bondarenko, Tamkang J. Math. 52(1), 125-154 (2021)].

Keywords. transformation operator, singular potentials, integral equations

The proof of Theorem 2.1 can be simplified. Specifically, the mappings $I_{\mathscr{K}}(\mathscr{K}, \mathscr{N}, \mathscr{C})$, $I_{\mathscr{N}}(\mathscr{K}, \mathscr{N}, \mathscr{C})$, and $I_{\mathscr{C}}(\mathscr{K}, \mathscr{C})$, which are defined on pages 129–130, can be represented as follows:

$$
\begin{aligned}
I_{\mathscr{K}}(\mathscr{K}, \mathscr{N}, \mathscr{C}) = & \frac{1}{2}\int_{x-t}^{x} \big(\mathscr{K}(s, t-x+s) + \mathscr{N}(s, t-x+s)\big)\sigma(s)\,ds \\
& + \frac{1}{2}\int_{\frac{x-t}{2}}^{x-t} \big(\mathscr{K}(s, x-s-t) - \mathscr{N}(s, x-s-t)\big)\sigma(s)\,ds \\
& + \frac{1}{2}\int_{\frac{x+t}{2}}^{x} \big(\mathscr{K}(s, x-s+t) - \mathscr{N}(s, x-s+t)\big)\sigma(s)\,ds \\
& - \frac{1}{2}\int_{t}^{x} d\xi \Big(\int_{x-\xi}^{x} \mathscr{K}(s, \xi-x+s)\sigma^2(s)\,ds \\
& + \int_{\frac{x-\xi}{2}}^{x-\xi} \mathscr{K}(s, x-s-\xi)\sigma^2(s)\,ds \\
& - \int_{\frac{x+\xi}{2}}^{x} \mathscr{K}(s, x-s+\xi)\sigma^2(s)\,ds\Big) - \int_0^{x-t} \mathscr{C}(s)\sigma(s)\,ds, \qquad (0.1)
\end{aligned}
$$

$$
\begin{aligned}
I_{\mathscr{N}}(\mathscr{K}, \mathscr{N}, \mathscr{C}) = & -\frac{1}{2}\int_{x-t}^{x} \big(\mathscr{K}(s, t-x+s) + \mathscr{N}(s, t-x+s)\big)\sigma(s)\,ds \\
& - \frac{1}{2}\int_{\frac{x-t}{2}}^{x-t} \big(\mathscr{K}(s, x-s-t) - \mathscr{N}(s, x-s-t)\big)\sigma(s)\,ds \\
& + \frac{1}{2}\int_{\frac{x+t}{2}}^{x} \big(\mathscr{K}(s, x-s+t) - \mathscr{N}(s, x-s+t)\big)\sigma(s)\,ds \\
& + \frac{1}{2}\int_{t}^{x} d\xi \Big(\int_{x-\xi}^{x} \mathscr{K}(s, \xi-x+s)\sigma^2(s)\,ds
\end{aligned}
$$

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$$+\int_{\frac{x-\xi}{2}}^{x-\xi}\mathscr{K}(s,x-s-\xi)\sigma^2(s)\,ds$$
$$+\int_{\frac{x+\xi}{2}}^{x}\mathscr{K}(s,x-s+\xi)\sigma^2(s)\,ds\Big)+\int_0^{x-t}\mathscr{C}(s)\sigma(s)\,ds, \quad (0.2)$$
$$I_{\mathscr{C}}(\mathscr{K},\mathscr{C})=-\frac{1}{2}\int_0^x d\xi\Big(\int_{x-\xi}^{x}\mathscr{K}(s,\xi-x+s)\sigma^2(s)\,ds$$
$$+\int_{\frac{x-\xi}{2}}^{x-\xi}\mathscr{K}(s,x-s-\xi)\sigma^2(s)\,ds$$
$$+\int_{\frac{x+\xi}{2}}^{x}\mathscr{K}(s,x-s+\xi)\sigma^2(s)\,ds\Big)-\int_0^{x}\mathscr{C}(s)\sigma(s)\,ds. \quad (0.3)$$

Using these representations, we get that, for $n \ge 1$, the functions $\mathscr{K}_n(x,t)$, $\mathscr{N}_n(x,t)$, and $\mathscr{C}_n(x)$ are continuous for $0 \le t \le x \le \pi$ and $0 \le x \le \pi$, respectively. Moreover, they fulfill the estimates

$$|\mathscr{K}_n(x,t)|,\ |\mathscr{N}_n(x,t)|,\ |\mathscr{C}_n(x)| \le a^n Q^n(x)\sqrt{\tfrac{x^{n-1}}{(n-1)!}}, \quad n \ge 1, \quad (0.4)$$

with some constant a depending on $\|\sigma\|_{L_2(0,\pi)}$. This immediately implies that the series

$$\sum_{n=1}^{\infty}\mathscr{K}_n(x,t), \quad \sum_{n=1}^{\infty}\mathscr{N}_n(x,t) \quad (\text{without } n=0), \quad \sum_{n=0}^{\infty}\mathscr{C}_n(x)$$

converge absolutely and uniformly to continuous functions. Adding the terms $\mathscr{K}_0(x,t)$ and $\mathscr{N}_0(x,t)$, which belong to $L_2(\mathbb{D})$, one can conclude the proof.

Note that the integrals $\int_{x-\xi}^{x}\mathscr{K}(s,\xi-x+s)\sigma^2(s)\,ds$, $\int_{\frac{x-\xi}{2}}^{x-\xi}\mathscr{K}(s,x-s-\xi)\sigma^2(s)\,ds$, and $\int_{\frac{x+\xi}{2}}^{x}\mathscr{K}(s,x-s+\xi)\sigma^2(s)\,ds$ in the formulas (0.1), (0.2), (0.3) are understood as L_1-functions of ξ for each fixed x. One can change the order of integration to obtain inner integrals that converge absolutely for any fixed x, t, s:

$$I_{\mathscr{K}}(\mathscr{K},\mathscr{N},\mathscr{C})=\frac{1}{2}\int_{x-t}^{x}\big(\mathscr{K}(s,t-x+s)+\mathscr{N}(s,t-x+s)\big)\sigma(s)\,ds$$
$$+\frac{1}{2}\int_{\frac{x-t}{2}}^{x-t}\big(\mathscr{K}(s,x-s-t)-\mathscr{N}(s,x-s-t)\big)\sigma(s)\,ds$$
$$+\frac{1}{2}\int_{\frac{x+t}{2}}^{x}\big(\mathscr{K}(s,x-s+t)-\mathscr{N}(s,x-s+t)\big)\sigma(s)\,ds$$
$$-\frac{1}{2}\Big(\int_0^x\sigma^2(s)\,ds\int_0^{\min\{s,x-t\}}\mathscr{K}(s,s-\xi)\,d\xi$$
$$+\int_0^{x-t}\sigma^2(s)\,ds\int_s^{\min\{2s,x-t\}}\mathscr{K}(s,\xi-s)\,d\xi$$
$$-\int_{\frac{x+t}{2}}^{x}\sigma^2(s)\,ds\int_t^{2s-x}\mathscr{K}(s,x+\xi-s)\,d\xi\Big)-\int_0^{x-t}\mathscr{C}(s)\sigma(s)\,ds,$$
$$I_{\mathscr{N}}(\mathscr{K},\mathscr{N},\mathscr{C})=-\frac{1}{2}\int_{x-t}^{x}\big(\mathscr{K}(s,t-x+s)+\mathscr{N}(s,t-x+s)\big)\sigma(s)\,ds$$
$$-\frac{1}{2}\int_{\frac{x-t}{2}}^{x-t}\big(\mathscr{K}(s,x-s-t)-\mathscr{N}(s,x-s-t)\big)\sigma(s)\,ds$$

$$
\begin{aligned}
&+ \frac{1}{2} \int_{\frac{x+t}{2}}^{x} \big(\mathscr{K}(s, x-s+t) - \mathscr{N}(s, x-s+t)\big)\sigma(s)\, ds \\
&+ \frac{1}{2}\bigg(\int_0^x \sigma^2(s)\, ds \int_0^{\min\{s, x-t\}} \mathscr{K}(s, s-\xi)\, d\xi \\
&+ \int_0^{x-t} \sigma^2(s)\, ds \int_s^{\min\{2s, x-t\}} \mathscr{K}(s, \xi - s)\, d\xi \\
&+ \int_{\frac{x+t}{2}}^{x} \sigma^2(s)\, ds \int_t^{2s-x} \mathscr{K}(s, x+\xi-s)\, d\xi\bigg) + \int_0^{x-t} \mathscr{C}(s)\sigma(s)\, ds, \\
I_{\mathscr{C}}(\mathscr{K}, \mathscr{C}) = &-\frac{1}{2}\int_0^x \sigma^2(t)\, dt \bigg(\int_{x-t}^{x} \mathscr{K}(t, \xi - x + t)\, d\xi + \int_{x-2t}^{x-t} \mathscr{K}(t, x - \xi - t)\, d\xi\bigg) \\
&- \int_0^x \mathscr{C}(s)\sigma(s)\, ds.
\end{aligned}
$$

Usage of these relations leads to the same estimates (0.4).

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