TAMKANG JOURNAL OF MATHEMATICS
Volume vol, Number i, xxx-xxx, 20xx
DOI:10.5556/j.tkjm.56.2025.5386

Second-order noncanonical mixed type difference equations of unstable type: new oscillation criteria

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Abstract. This paper is concerned with the oscillatory properties of the second-order noncanonical difference equation with a deviating argument of the form

$$\Delta(a_n \Delta y_n) = q_n y_{\sigma(n)}.$$

The authors first transform the noncanonical equation into canonical form so that the discrete Kneser theorem can be applied to classify the nonoscillatory solutions into two types. Some new monotonic properties of the nonoscillatory solutions are then obtained, and they are used to eliminate certain type of nonoscillatory solutions. This leads to the development of new oscillation criteria for the equation. The results obtained are new and complement those currently existing in the literature. Examples to illustrate the importance of the main results are also presented.

Keywords. Second-order difference equation, noncanonical form, delay and advanced arguments, oscillation

1 Introduction

Consider the second-order noncanonical difference equation with a deviating argument of the form

$$\Delta(a_n \Delta y_n) = q_n y_{\sigma(n)}, \ n \in \mathbb{N}(n_0), \tag{E}$$

where $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$ and n_0 is a nonnegative integer. We shall assume that

(H_1) $\{a_n\}$ and $\{q_n\}$ are sequences of positive real numbers;

(H_2) $\{\sigma(n)\}$ is an increasing sequence of integers such that $\sigma(n) \to \infty$ as $n \to \infty$.

By a *solution* of equation (E), we mean a real sequence $\{y_n\}$ defined and satisfying equation (E) for $n \in \mathbb{N}(n_0)$ and with $\sup\{|y_s| : s \geq n\} > 0$ for $n \in \mathbb{N}(n_0)$. Such a solution $\{y_n\}$ is called *oscillatory* if for any $n_1 \in \mathbb{N}(n_0)$, there are integers n_2, $n_3 \geq n_1$ such that $y_{n_2} y_{n_3} \leq 0$, and

Received date: September 10, 2024; Published online: October 30, 2024.
2010 *Mathematics Subject Classification*. 39A21.
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is called *nonoscillatory* otherwise. The equation is said to be oscillatory if all its solutions are oscillatory.

Equation (E) is in noncanonical form since we are assuming that

$$A_n = \sum_{s=n}^{\infty} \frac{1}{a_s} \quad \text{with} \quad A_{n_0} < \infty. \tag{1.1}$$

The determination of oscillation criteria for difference equations of different forms and orders continues to be an area of great interest in recent years; see, for example, the monographs [1, 2, 3], the papers [4, 6, 10, 11, 12, 13, 14], and the references cited therein. In dynamical models, deviation and oscillation scenarios are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, for example, [7, 9]. The determination of oscillation criteria for difference equations of different forms has also been an active area of research; see [5] and the references contained within.

While it is known that the equation

$$\Delta^2 y_{n-1} = q_n y_n$$

possesses positive decreasing and positive increasing solutions, the situation is quite different for the equation with deviating arguments

$$\Delta^2 y_n = q_n y_{\sigma(n)}. \tag{1.2}$$

For example, Lalli and Zhang [8] showed that if $\sigma(n) = n - k$, where k is a positive integer, the condition

$$\limsup_{n\to\infty} \sum_{s=n-k}^{n-1} (s + 1 - k) q_s > 1 \tag{1.3}$$

eliminates positive decreasing solutions. However, (1.2) does not possess positive increasing solution if $\sigma(n) = n + m$, where m is a positive integer, and

$$\limsup_{n\to\infty} \sum_{s=n}^{n+m-1} (n + m - s) q_s > 1. \tag{1.4}$$

A review of the literature reveals that there are very few results ensuring that (1.2) is oscillatory, (see, for example [6, 8, 10, 12, 11, 14, 13]), but this type of equation has been well studied in the literature in the case where $\{q_n\}$ is negative; see for example, the monographs [1, 2, 3] and their numerous references. The aim of this paper is to obtain corresponding results for the second-order noncanonical functional difference equation (E) that are new and complement existing ones in the literature.

Our paper is organized as follows. First, in Section 2, we transform equation (E) into a canonical type equation so that we can directly use the discrete Kneser theorem [2] to obtain the structure of the nonoscillatory solutions. This is an essential step in obtaining our oscillation criteria. Second, we obtain new monotonic properties of the nonoscillatory solutions of the transformed equation (equation (E_c) below). Using these properties, we are then able to eliminate the existence of positive decreasing solutions of (E_c) if it has a delay argument, and eliminate the positive increasing solutions of (E_c) in case it has an advanced argument. In Section 3, we combine the results to get oscillation of all solutions of equation (E) if it involves both delayed and advanced arguments. Examples are presented in Section 4 to show the importance and novelty of our main results. It should be noted that the research in this paper is partially inspired by the recent results in a very nice paper by Baculíková and Džurina [4] on differential equations.

2 Preliminary Results

For the sake of convenience, we define

$$b_n = a_n A_n A_{n+1},\ Q_n = A_{n+1} q_n A_{\sigma(n)},\ B_n = \sum_{s=n_0}^{n-1} \frac{1}{b_s},$$

and

$$z_n = \frac{y_n}{A_n}.$$

We begin with the following theorem.

Theorem 2.1. *The noncanonical operator $Dy_n = \Delta(a_n \Delta y_n)$ can be written in the equivalent canonical form as*

$$Dy_n = \frac{1}{A_{n+1}} \Delta \left(a_n A_n A_{n+1} \Delta \left(\frac{y_n}{A_n} \right) \right). \tag{2.1}$$

Proof. A direct computation shows that

$$\Delta \left(a_n A_n A_{n+1} \Delta \left(\frac{y_n}{A_n} \right) \right) = \Delta(A_n a_n \Delta y_n + y_n) = A_{n+1} \Delta(a_n \Delta y_n).$$

Furthermore,

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n A_n A_{n+1}} = \sum_{n=n_0}^{\infty} \Delta \left(\frac{1}{A_n} \right) = \lim_{n\to\infty} \frac{1}{A_n} - \frac{1}{A_{n_0}} = \infty.$$

Hence, the right hand side of (2.1) is in canonical form, and this completes the proof of the theorem. □

From the above theorem, the following observations and results are immediate. The noncanonical equation (E) can be rewritten in equivalent canonical form as

$$\Delta \left(a_n A_n A_{n+1} \Delta \left(\frac{y_n}{A_n} \right) \right) = A_{n+1} q_n y_{\sigma(n)}.$$

Theorem 2.2. *The noncanonical difference equation (E) possesses a solution $\{y_n\}$ if and only if the canonical equation*

$$\Delta(b_n \Delta z_n) = Q_n z_{\sigma(n)} \tag{E_c}$$

has the solution $\{z_n\} = \left\{ \frac{y_n}{A_n} \right\}$.

Corollary 2.3. *The noncanonical difference equation (E) has an eventually positive solution if and only if the canonical equation (E_c) has an eventually positive solution.*

Corollary 2.3 simplifies the investigation of the noncanonical equation (E) since for (E_c), we use the discrete Kneser's theorem (see [6, Theorem 1.8.11]) that any eventually positive solution of (E_c) satisfies one of the following conditions:

$$S_0 : b_n \Delta z_n < 0,\ \Delta(b_n \Delta z_n) > 0,$$

$$S_2 : b_n\Delta z_n > 0,\ \Delta(b_n\Delta z_n) > 0$$

for $n \geq n_1 \in \mathbb{N}(n_0)$.

If we let S denote the set of all positive solutions of (E_c), then it has the decomposition

$$S = S_0 \cup S_2.$$

Next, we have some useful auxiliary results regarding the monotonic properties of nonoscillatory solutions of (E_c).

Lemma 2.1. *Let $\sigma(n) = n - k$, where k is a positive integer, and assume that $\{z_n\}$ is a positive solution of (E_c) belongs to S_0. If there exists a constant $\delta \in (0, 1)$ such that*

$$B_n\left(\sum_{s=n}^{n+k} Q_s\right) \geq \delta,\ for\ n \geq n_0, \tag{2.2}$$

then $\{B_n z_n\}$ is decreasing.

Proof. Assume that $\{z_n\}$ is a positive solution of (E_c) belonging to the class S_0. Since $b_n\Delta z_n < 0$, z_n is decreasing. Therefore, a summation of (E_c) from $n-k$ to n gives

$$-b_{n-k}\Delta z_{n-k} \geq z_{n-k}\left(\sum_{s=n-k}^{n} Q_s\right),$$

that is,

$$-B_n b_n \Delta z_n \geq z_{n+1}B_n\left(\sum_{s=n}^{n+k} Q_s\right) \geq \delta z_{n+1}$$

by (2.2). This implies

$$\Delta(B_n^\delta z_n) = z_{n+1}\Delta B_n^\delta + B_n^\delta \Delta z_n \leq \frac{B_n^{\delta-1}}{b_n}\left(\delta z_{n+1} + B_n b_n \Delta z_n\right) \leq 0,$$

from which we conclude that the sequence $\{B_n^\delta z_n\}$ is decreasing. This completes the proof. ☐

Theorem 2.4. *Let $\sigma(n) = n - k$, where k is a positive integer, and assume that (2.2) holds. If*

$$\limsup_{n\to\infty} B_{n-k}^\delta \sum_{s=n-k}^{n} \frac{1}{b_s}\left(\sum_{t=s}^{n} \frac{Q_t}{B_{t-k}^\delta}\right) > 1, \tag{2.3}$$

then $S_0 = \emptyset$.

Proof. Assume, to the contrary, that (E_c) has an eventually positive solution $\{z_n\}$ belonging to the class S_0. Summing (E_c) from j to n and applying the monotonicity of $\{B_n^\delta z_n\}$ gives

$$-b_j\Delta z_j \geq B_{n-k}^\delta z_{n-k}\sum_{s=j}^{n} \frac{Q_s}{B_{s-k}^\delta}.$$

Summing again form j to n, we obtain

$$z_j \geq B_{n-k}^\delta z_{n-k}\sum_{s=j}^{n} \frac{1}{b_s}\left(\sum_{t=s}^{n} \frac{Q_t}{B_{t-k}^\delta}\right).$$

Taking $j = n - k$ gives

$$z_{n-k} \geq B^{\delta}_{n-k} z_{n-k} \sum_{s=n-k}^{n} \frac{1}{b_s} \left(\sum_{t=s}^{n} \frac{Q_t}{B^{\delta}_{t-k}} \right),$$

which contradicts (2.3), and so S_0 is empty. This proves the theorem. □

Next, we investigate the monotonic properties of possible positive increasing solutions of (E_c).

Lemma 2.2. *Let $\sigma(n) = n + l$ with l a positive integer, and assume that $\{z_n\}$ is a positive solution of (E_c) that belongs to the class S_2. If there exists a constant $\alpha \in (0, 1)$ such that*

$$B_n \left(\sum_{s=n-l}^{n-1} Q_s \right) \geq \alpha, \text{ for } n \geq n_0, \tag{2.4}$$

then $\left\{ \frac{z_n}{B^{\alpha}_n} \right\}$ is increasing.

Proof. Assume that $\{z_n\}$ is a positive solution of (E_c) belonging to S_2. Then $b_n \Delta z_n > 0$, and z_n is increasing. Summing (E_c) from n to $n + l - 1$ and then using the monotonicity of $\{B_n z_n\}$ from Lemma 2.1, we obtain

$$b_{n+l} \Delta z_{n+l} \geq z_{n+l} \left(\sum_{s=n}^{n+l-1} Q_s \right)$$

and hence in view of (2.4)

$$B_n b_n \Delta z_n \geq z_n B_n \left[\sum_{s=n-l}^{n-1} Q_s \right] \geq \alpha z_n.$$

Therefore,

$$\Delta \left(\frac{z_n}{B^{\alpha}_n} \right) = \frac{B^{\alpha}_n \Delta z_n - z_n \Delta B^{\alpha}_n}{B^{\alpha}_n B^{\alpha}_{n+1}} \geq \frac{B_n b_n z_n - \alpha z_n}{b_n B_n B^{\alpha}_{n+1}} \geq 0.$$

We see that $\left\{ \frac{z_n}{B^{\alpha}_n} \right\}$ is increasing and this finishes the proof. □

Theorem 2.5. *Let $\sigma(n) = n + l$, l be a positive integer, and (2.4) hold. If*

$$\limsup_{n\to\infty} \frac{1}{B^{\alpha}_{n+l}} \sum_{s=n}^{n+l-1} \frac{1}{b_s} \left(\sum_{t=n}^{s} Q_t B^{\alpha}_{t+l} \right) > 1, \tag{2.5}$$

then $S_2 = \emptyset$.

Proof. Assume, to the contrary, that (E_c) has a positive solution $\{z_n\}$ belonging to S_2. Summing (E_c) from n to $j - 1$ and then using the fact that $\left\{ \frac{z_n}{B^{\alpha}_n} \right\}$ is increasing gives

$$b_j \Delta z_j \geq \frac{z_{n+l}}{B^{\alpha}_{n+l}} \sum_{s=n}^{j-1} Q_s B^{\alpha}_{s+l}.$$

Summing again from n to $j-1$,

$$z_j \geq \frac{z_{n+l}}{B_{n+l}^{\alpha}} \sum_{s=n}^{j-1} \frac{1}{b_s} \left(\sum_{t=n}^{s-1} Q_t B_{t+l}^{\alpha} \right).$$

Letting $j = n+l$,

$$z_{n+l} \geq \frac{z_{n+l}}{B_{n+l}^{\alpha}} \sum_{s=n}^{n+l-1} \frac{1}{b_s} \left(\sum_{t=n}^{s-1} Q_t B_{t+l}^{\alpha} \right),$$

which clearly contradicts condition (2.5). Therefore, the class S_2 is empty, and this proves the theorem. □

Next, we present a new monotonic property for the first differences of nonoscillatory solutions of (E_c). This will lead to another result similar to those given in Theorems 2.4 and 2.5.

Lemma 2.3. *Let $\sigma(n) = n-k$, where k is a positive integer and let $\{z_n\}$ be a positive solution of (E_c) that belongs to (S_0). If there exists a constant $\gamma \in (0,1)$ such that*

$$Q_n[B_{n+1} - B_{n-k}]B_n b_n \geq \gamma, \text{ for } n \geq n_0, \tag{2.6}$$

then $\{-B_n^{\gamma} b_n \Delta z_n\}$ is decreasing.

Proof. Assume that $\{z_n\}$ is a positive solution of (E_c) in S_0. Since $-b_n\Delta z_n$ is positive and decreasing,

$$z_{n-k} \geq \sum_{s=n-k}^{n} -\Delta z_s \geq -\Delta z_n b_n \sum_{s=n-k}^{n} \frac{1}{b_s} \geq -\Delta z_{n+1} b_{n+1}(B_{n+1} - B_{n-k}). \tag{2.7}$$

Using (2.7) in (E_c), we obtain

$$\Delta(b_n \Delta z_n) \geq Q_n(-\Delta z_{n+1} b_{n+1})(B_{n+1} - B_{n-k}),$$

which in view of (2.6) implies

$$\Delta(b_n \Delta z_n) B_n b_n \geq \gamma(-\Delta z_{n+1} b_{n+1}). \tag{2.8}$$

Applying the discrete Mean-Value theorem, we have

$$\Delta(-B_n^{\gamma} b_n \Delta z_n) \leq \frac{B_n^{\gamma-1}}{b_n} \left[-b_{n+1}\Delta z_{n+1} - \Delta(b_n \Delta z_n) B_n b_n \right] \leq 0$$

by (2.8). Hence, $\{-B_n^{\gamma} b_n \Delta z_n\}$ is decreasing and this proves the lemma. □

As indicated above, here is another result ensuring that the class S_2 is empty.

Theorem 2.6. *Let $\sigma(n) = n-k$ with k a positive integer and let (2.6) hold. If*

$$\limsup_{n\to\infty} B_{n-k}^{\gamma} \sum_{s=n-k}^{n} Q_s \left(B_{n-k+1}^{1-\gamma} - B_{s-k}^{1-\gamma} \right) > 1 - \gamma, \tag{2.9}$$

then the class S_0 is empty.

Proof. Let $\{z_n\}$ be a positive solution of (E_c) that belongs to (S_0). By Lemma 2.3, $-B_n^\gamma b_n \Delta z_n$ is positive and decreasing, so

$$\begin{aligned} z_{s-k} &\geq \sum_{t=s-k}^{n-k} -\Delta z_t \frac{B_t^\gamma b_t}{B_t^\gamma b_t} \\ &\geq -\Delta z_{n-k} B_{n-k}^\gamma b_{n-k} \sum_{t=s-k}^{n-k} \frac{1}{B_t^\gamma b_t} \\ &\geq -\Delta z_{n-k} B_{n-k}^\gamma b_{n-k} \sum_{t=s-k}^{n-k} \frac{1}{B_{t+1}^\gamma b_t} \\ &\geq -\Delta z_{n-k} B_{n-k}^\gamma b_{n-k} \frac{(B_{n-k+1}^{1-\gamma} - B_{s-k}^{1-\gamma})}{1-\gamma}. \end{aligned} \tag{2.10}$$

Summing (E_c) from $n-k$ to n and then using (2.10), we see that

$$-b_{n-k}\Delta z_{n-k} \geq \sum_{s=n-k}^{n} Q_s z_{s-k} \geq -\Delta z_{n-k} \frac{b_{n-k}B_{n-k}^\gamma}{1-\gamma} \sum_{s=n-k}^{n} Q_s \left(B_{n-k+1}^{1-\gamma} - B_{s-k}^{1-\gamma}\right),$$

which contradicts (2.9) and proves that S_0 is empty. □

Lemma 2.4. *Let $\sigma(n) = n + l$, l be a positive integer, and let $\{z_n\}$ be a positive solution of (E_c) belonging to S_2. If there exists a constant $d \in (0, 1)$ such that*

$$Q_n(B_{n+1-l} - B_{n+1})B_n b_n \geq d, \text{ for } n \geq n_0, \tag{2.11}$$

then $\left\{\dfrac{b_n \Delta z_n}{B_n^d}\right\}$ is an increasing sequence.

Proof. Let $\{z_n\}$ be a positive solution of (E_c) in (S_2). Since $b_n\Delta z_n$ is positive and increasing, it is not difficult to see that

$$z_{n+l} \geq \sum_{s=n}^{n+l-1} \Delta z_s \geq b_n \Delta z_n \sum_{s=n}^{n+l-1} \frac{1}{b_s} = b_n\Delta z_n(B_{n+l} - B_n). \tag{2.12}$$

Substituting (2.12) into (E_c) gives

$$\Delta(b_n\Delta z_n) \geq Q_n b_n \Delta z_n (B_{n+l} - B_n). \tag{2.13}$$

In view of (2.11), we have

$$\Delta(b_n\Delta z_n)B_n b_n \quad \geq \quad d b_n \Delta z_n. \tag{2.14}$$

Now

$$\Delta\left(\frac{b_n\Delta z_n}{B_n^d}\right) = \frac{B_n^d \Delta(b_n\Delta z_n) - b_n\Delta z_n \Delta(B_n^d)}{B_n^d B_{n+1}^d}.$$

By the discrete Mean Value Theorem, we have

$$\Delta\left(B_n^d\right) \leq d B_n^{d-1} \frac{1}{b_n}$$

and using this we see that

$$\Delta\left(\frac{b_n\Delta z_n}{B_n^d}\right) \geq \frac{1}{b_nB_n^{1+d}}[B_nb_n\Delta(b_n\Delta z_n) - db_n\Delta z_n] \geq 0$$

by (2.14). Hence, we conclude that $\left\{\frac{b_n\Delta z_n}{B_n^d}\right\}$ is increasing. This completes the proof. □

Our last result of this type gives conditions under which $S_2 = \emptyset$.

Theorem 2.7. *Let $\sigma(n) = n + l$, l be a positive integer, and condition (2.11) hold. If*

$$\limsup_{n\to\infty} \frac{1}{B_{n+l}^d}\sum_{s=n}^{n+l-1} Q_s\left(B_{s+l}^{1+d} - B_{n+l}^{1+d}\right) > 1 + d, \tag{2.15}$$

then the class S_2 is empty.

Proof. Let $\{z_n\}$ be a positive solution of (E_c) belonging to (S_2). Since $\left\{\frac{b_n\Delta z_n}{B_n^d}\right\}$ is increasing, we have

$$z_{s+l} \geq \sum_{t=n+l}^{s+l-1} \Delta z_t \frac{B_t^d b_t}{B_t^d b_t} \geq \frac{\Delta z_{n+l}b_{n+l}}{B_{n+l}^d}\sum_{t=n+l}^{s+l-1}\frac{B_t^d}{b_t} = \frac{b_{n+l}\Delta z_{n+l}}{B_{n+l}^d}\frac{(B_{s+l}^{1+d} - B_{n+l}^{1+d})}{1+d}.$$

Summing (E_c) from n to $n + l - 1$ and using the above estimate, we obtain

$$b_{n+l}\Delta z_{n+l} \geq \sum_{s=n}^{n+l-1} Q_s z_{s+l} \geq \frac{b_{n+l}\Delta z_{n+l}}{(1+d)B_{n+l}^d}\sum_{s=n}^{n+l-1} Q_s\left(B_{s+l}^{d+1} - B_{n+l}^{d+1}\right),$$

which contradicts (2.15) and proves that S_2 is empty. □

3 Oscillation Theorems

In view of the results in Section 2, we might expect that all solutions will be oscillatory for equations that contain both a delay and an advanced argument. In the following theorems we show that this can in fact happen. We consider the equation

$$\Delta(a_n\Delta y_n) = q_ny_{n-k} + p_ny_{n+l},\ n \in \mathbb{N}(n_0), \tag{3.1}$$

where $\{a_n\}$ and $\{q_n\}$ satisfy condition (H_1) and

(H_3) $\{p_n\}$ is a positive real sequence and l and k are positive integers.

Using Theorem 2.1, we can transform the noncanonical equation (3.1) into a canonical type equation

$$\Delta(b_n\Delta z_n) = Q_nz_{n-k} + Q_n^*z_{n+l},\ n \geq \mathbb{N}(n_0) \tag{3.2}$$

where $\{b_n\}$, $\{Q_n\}$, and $\{z_n\}$ are as defined earlier and

$$Q_n^* = A_{n+1}p_nA_{n+l}.$$

Theorem 3.1. *Let conditions* (2.2) *and* (2.3) *hold and assume there exists a constant* $\beta \in (0,1)$ *such that*

$$B_n \left(\sum_{s=n-l}^{n-1} Q_s^* \right) \geq \beta, \ \textit{for } n \in \mathbb{N}(n_0). \tag{3.3}$$

If

$$\limsup_{n\to\infty} \frac{1}{B_{n+l}^{\beta}} \sum_{s=n}^{n+l-1} \frac{1}{b_s} \left(\sum_{t=n}^{s} Q_t^* B_{t+1}^{\beta} \right) > 1, \tag{3.4}$$

then equation (3.1) *is oscillatory.*

Proof. Let $\{y_n\}$ be an eventually positive solution of (3.1). Then by Theorem 2.2, $\{z_n\}$ is a positive solution of (3.2) such that $z_n \in S_0$ or S_2 for all $n \in \mathbb{N}(n_1)$.

Suppose z_n is in the class S_0. It is not difficult to see that (3.2) implies

$$\Delta(b_n \Delta z_n) \geq Q_n z_{n-k}.$$

As in the proof of Theorem 2.4, it can be shown that (2.3) guarantees that $S_0 = \emptyset$.

Now assume that $z_n \in S_2$. From (3.2) we see that

$$\Delta(b_n \Delta z_n) \geq Q_n^* z_{n+l}.$$

Then, as we did in the proof of Theorem 2.5, we see that $S_2 = \emptyset$. This shows that $\{z_n\}$ must be oscillatory, and by the transformation $\{y_n\} = \{A_n z_n\}$, it is not difficult to see that $\{y_n\}$ is also oscillatory. This completes the proof of the theorem. □

Theorem 3.2. *Let conditions* (2.6) *and* (2.9) *hold and assume that there is a constant* $d_0 \in (0,1)$ *such that*

$$B_n Q_n^* (B_{n+l} - B_n) b_n \geq d_0, \ \textit{for } n \geq n_0.$$

If

$$\limsup_{n\to\infty} B_{n+l}^{-d_0} \sum_{s=n}^{n+l-1} Q_s^* \left(B_{s+l}^{1+d_0} - B_{n+l}^{1+d_0} \right) > 1 + d_0,$$

then equation (3.1) *is oscillatory.*

Since the proof is similar to that of Theorem 3.1, we omit the details.

4 Examples

In this section we present some examples to illustrate the applicability of our main results.

Example 1. Consider the second-order noncanonical delay difference equation

$$\Delta\left(n(n+1)\Delta y_n\right) = a(n+1)(n-2) y_{n-2}, \ n \geq 3, \tag{4.1}$$

where $a > 0$ is a constant. Here $a_n = n(n+1)$, $q_n = a(n+1)(n-2)$, and $\sigma(n) = n-2$. Simple calculation shows that

$$A_n = \frac{1}{n}, \ b_n = 1, \ B_n \simeq n, \ Q_n = a,$$

and the transformed equation is

$$\Delta^2 z_n = az_{n-2},\ n \geq 3,$$

which is clearly in canonical form. Choosing $\delta = \frac{1}{2}$, and $a = 0.17$, we see that conditions (2.2) and (2.3) hold. Therefore, by Theorem 2.4 the class S_0 is empty. In other words every bounded solution of (4.1) is oscillatory if $a = 0.17$.

Example 2. Consider the second-order noncanonical advanced difference equation

$$\Delta\left(n(n+1)\Delta y_n\right) = a(n+2)(n+1)y_{n+2},\ n \geq 1, \tag{4.2}$$

where $a > 0$ is a constant. Here we have $a_n = n(n+1)$, $q_n = a(n+2)(n+1)$, and $\sigma(n) = n+2$. Some simple computations show that

$$A_n = \frac{1}{n},\ b_n = 1,\ B_n \simeq n,\ Q_n = a,$$

and the transformed equation is

$$\Delta^2 z_n = az_{n+2},\ n \geq 1,$$

which is clearly a canonical type equation. Choosing $\delta = \frac{1}{2}$ and $a = 0.34$, we see that conditions (2.4) and (2.5) hold. Therefore, by Theorem 2.5, the class S_2 is empty. In other words, every unbounded solution of (4.2) is oscillatory if $a = 0.34$.

Example 3. Consider the second-order mixed type difference equation

$$\Delta\left(n(n+1)\Delta y_n\right) = a(n+1)(n-2)y_{n-2} + d(n+1)(n+2)y_{n+2},\ n \geq 3, \tag{4.3}$$

where $a > 0$ and $d > 0$ are constants. The transformed equation is

$$\Delta^2 z_n = az_{n-2} + dz_{n+2},\ n \geq 3,$$

which is in canonical form. Choosing $\delta = \beta = \frac{1}{2}$, $a = 0.17$, and $d = 0.34$, all conditions of Theorem 3.1 hold. Thus, equation (4.3) is oscillatory if $a = 0.17$ and $d = 0.34$.

5 Conclusions

In this paper, by using a canonical transform method, we converted equation (E) into canonical form. Then, we derived some new monotonic properties of nonoscillatory solutions of the transformed equation. Next, we then used these properties and the summation averaging method to eliminate certain types of nonoscillatory solutions. In this way we obtained oscillation results for mixed type difference equations. The results established in this paper are new and complement existing results in the literature. Moreover, no currently known results apply to equations (4.1)–(4.3) since these equations are in noncanonical form.

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