TAMKANG JOURNAL OF MATHEMATICS
Volume 56, Number 4, 369-381, 2025
DOI:10.5556/j.tkjm.56.2025.5428

# Antimagic graph constructions with triangle and three-path unions

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**Abstract**. Let $G = (V, E)$ be a graph with $p$ edges and let $f$ be a bijective function from $E(G)$ to $\{1, 2, \ldots, p\}$. For any vertex $v$, let $\phi_f(v)$ denote the sum of $f(e)$ at all edges $e$ incident to $v$. If $\phi_f(v) \neq \phi_f(u)$ holds for any two distinct vertices $u$ and $v$, then $f$ is called an antimagic labeling of $G$. A graph $G$ is considered antimagic if it admits such a labeling. In this study, we investigate the antimagic properties of graph unions, particularly focusing on structures composed of multiple triangles and 3-paths. We employ Skolem sequences and extended Skolem sequences to construct antimagic labelling for these graph unions. Specifically, we demonstrate that for any integer $n \geq 9$, the graph formed by the disjoint union of $m$ copies of the triangle $C_3$ and $n$ copies of the path $P_3$ is antimagic for $m \geq \lceil \frac{n}{3} \rceil$.

***Keywords.*** Antimagic labelling, skolem sequences, extended Skolem sequences

## 1 Introduction

Graphs are a fundamental structure in mathematics and computer science, often used to model and analyze various systems and networks. One intriguing aspect of graph theory is the study of graph labellings, where elements of a graph, such as vertices or edges, are assigned labels subject to certain conditions. Among these, antimagic labellings have garnered significant interest due to their unique properties and applications.

The graphs considered in this paper are not necessarily connected, unless otherwise indicated. Let $G = (V, E)$ be a graph with $p = |E(G)|$ edges and $q = |V(G)|$ vertices. An antimagic labelling of a graph $G$ with $p$ edges and $q$ vertices is a one-to-one correspondence $f$ between the $E(G)$ to the label set $\{1, 2, \ldots, p\}$ such that $\phi_f(u) \neq \phi_f(v)$, for any two distinct vertices of $u, v \in V(G)$, where $\phi_f(v)$ is defined as the sum of the labels of the edges that are incident to a vertex $v$ in $G$. A graph is antimagic if it admits an antimagic labelling.

Antimagic labelling, a concept introduced by Hartsfield and Ringel [9], involves assigning distinct integers to the edges of a graph in such a way that the sums of these integers at each vertex are unique. This labelling is termed "antimagic" because it ensures that no two vertices share the same sum of incident edge labels. In their foundational work, Hartsfield and Ringel [9] demonstrated that several basic graph types, including paths, cycles, and complete graphs $K_n$ for

Received date: July 29, 2024; Published online: April 16, 2025.
2010 *Mathematics Subject Classification*. 05C30, 05C78, 05B07.
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$n \geq 3$, exhibit antimagic properties. They conjectured that this property holds for all connected graphs, except for the simple case of $K_2$. While this conjecture remains unproven in its entirety, significant strides have been made towards its verification. Alon and his colleagues in 2004 [1] provided evidence that the conjecture is valid for graphs with high edge density, reinforcing the belief that most connected graphs are indeed antimagic.

In 1990, Hartsfield and Ringel [9] also suggested that, except for $K_2$, every tree can be labelled in an antimagic manner. This conjecture has seen partial validation, particularly for trees with specific structures. For instance, Kaplan, Lev, and Roditty (2009) [10] showed that trees with no more than one vertex of degree 2 are antimagic. Furthermore, the research by Deng and Li (2019) [8] on caterpillar trees—trees where all vertices are within one edge of a central path—demonstrated that caterpillars with a maximum degree of 3 possess the antimagic property.

In the recent years, numerous researchers are currently concentrating on antimagic labelling for a wide range of graph structures ([8], [10], [13]). The significance of antimagic labellings lies not only in theoretical interest but also in potential applications in network theory, where distinct sums at vertices can represent unique signatures or frequencies in communication networks [3]. In the context of finite groups, the associated graphs often exhibit natural antimagic labellings [11]. These types of graphs have important connections to automata theory, and understanding their properties can yield valuable insights.

Our approach builds on this line of research by applying Skolem sequences and extended Skolem sequences, which have been effectively used in combinatorial design and graph labelling problems. Skolem sequences were first introduced by Skolem [12] and have since been utilized in various contexts to solve problems involving difference sets and graph embeddings. Extended Skolem sequences [4], a generalization of the original sequences, further expand the possibilities for constructing labellings in more complex graph structures. In this paper, we shall focus on the graphs for unions of graphs with many triangles and three-paths. In particular, we shall give constructions of the antimagic labelling for unions of graphs with many triangles and three-paths using the skolem and extended skolem sequences.

## 2 Some constructions of anti-magic labellings

In [6], every $k$ regular graph has been shown to be antimagic for $k \geq 2$. As a specific case, the triangle $C_3$ is proven to be antimagic, as shown in the Figure 1.

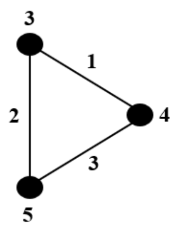


Figure 1: Antimagic labelling $C_3$.

Let $G$ be a graph composed of disjoint triangles $m$, denoted as $G = mC_3$ where $m \geq 1$. For each triangle $j$, let the edges be labelled with $t_{j_1}, t_{j_2}, t_{j_3}$ for $j = 1, 2, \ldots, m$. The labels of the vertices in this triangle are determined by the sums of the edges incident to each vertex.

Specifically, the vertex labels $\phi(v_{j_1}), \phi(v_{j_2}), \phi(v_{j_2})$ for the triangle $j$ are given by the set

$$\{t_{j_1} + t_{j_2}, t_{j_2} + t_{j_3}, t_{j_3} + t_{j_1}\}$$

as shown in Figure 2.

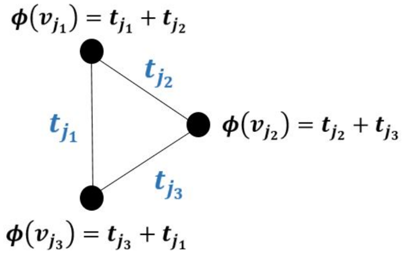


Figure 2: The labels of the vertices and edges of the $j$th triangle where $j = 1, 2, \ldots, m$.

In the following, we denote the edge labels of the $j$-th triangle by $(t_{j_1}, t_{j_2}, t_{j_3})$, $j = 1, 2, \ldots, m$. We show that there exists a unique way to assign the edge labels such that each vertex has a distinct sum for $mC_3$, $m \geq 1$.

**Proposition 2.1.** *Let $G = mC_3$ where $m \geq 1$. Then $G$ is antimagic.*

*Proof.* $G$ consists of $3m$ edges and the edge labels are $\{1, 2, \ldots, 3m\}$. Let

$$(t_{j_1}, t_{j_2}, t_{j_3}) = (3j - 2, 3j - 1, 3j)$$

for $j = 1, 2, \ldots, m$. Note that the sum of the edge labels incident to the vertices $v_{j_1}, v_{j_2}, v_{j_3}$ is $\{\phi(v_{j_1}), \phi(v_{j_2}), \phi(v_{j_3})\} = \{6j - 3, 6j - 2, 6j - 1\}$, for $j = 1, 2, \ldots, m$. Since $|V(G)| = 3m$, each vertex in $G$ receives a unique sum, thereby confirming that $G$ is antimagic. □

Let $G = tP_3$ where $t \geq 1$. We denote the edge labels of the $k$-th path $P_3$ by $(p_{k_1}, p_{k_2})$. Let $\{\phi(v_{k_1}), \phi(v_{k_2}), \phi(v_{k_3})\} = \{p_{k_1}, p_{k_2}, p_{k_1} + p_{k_2}\}$ be the vertex labels of the $k$-th path $P_3$, for $k = 1, 2, \ldots, t$ as illustarted in Figure 3:

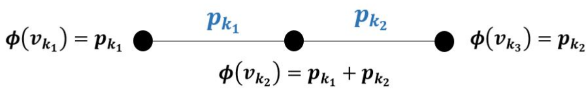


Figure 3: The labels of the vertices and edges of the $k$th $P_3$ where $k = 1, 2, \ldots, t$.

It is clear that $P_3$ is antimagic as illustrated in the Figure 4. However, as demonstrated in [5], the graph consisting of two disjoint path $2P_3$ is not antimagic.

**Proposition 2.2.** *Let $G = tP_3$ where $t \geq 2$ is a positive integer. Then $G$ is not antimagic.*

*Proof.* $G$ consists of $2t$ edges and the edge labels are $\{1, 2, \ldots, 2t\}$. Let $p_{k_1}, p_{k_2} \in \{1, 2, \ldots, 2t\}$, for $k = 1, 2, \ldots, t$. If $G = tP_3$ is antimagic, then $p_{k_1} + p_{k_2} > 2t$ for all $k$. Consider labelling a 3-path with $(1, 2t)$. The resulting sum is $2t + 1$ which is greater than $2t$. Now, if we try to label a

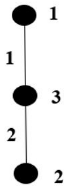


Figure 4: Antimagic labelling $P_3$.

second 3-path with any of the pairs from $(2,3),(2,4),\dots,(2,2t-1)$, we find that these pairs fail to provide the distinct sums required for an antimagic labelling. Hence, under these conditions, $G = tP_3$ cannot be considered antimagic. □

Next, we shall focus on the $G = C_3 \cup P_3$. By calculating all the combinations, we notice that there are two antimagic labellings for $C_3 \cup P_3$ as given in the Figure 5.

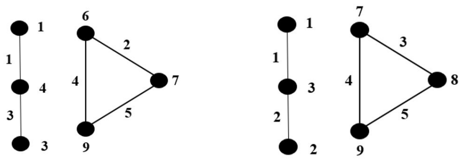

Figure 5: All antimagic labellings of $G = C_3 \cup P_3$.

Let $G = mC_3 \cup tP_3$ for positive integers $m, t \geq 0$. Let $f$ be the antimagic labelling of $G$. By definition of $f$, $f : E(G) \to \{1, 2, \dots, 3m + 2t\}$ is bijective such that $\phi(u) \neq \phi(v)$ for any two distinct vertices $u, v \in G$. It is clear that $1 \leq \phi(u) \leq 6m + 4t - 1$ for all vertices $u \in V(G)$.

**Proposition 2.3.** *If $G = mC_3 \cup P_3$ where $m$ is an integer and $m \geq 1$, then $G$ is antimagic.*

*Proof.* We first label the only 3-path by $(1, 2)$ and label the $j$-th triangle by

$$(t_{j_1}, t_{j_2}, t_{j_3}) = (j + 2, m + j + 2, 2m + j + 2)$$

for $j = 1, 2, \dots, m$. Hence,

$$\phi(V) = \{1, 2, 3\} \cup \{m + 2j + 4, 2m + 2j + 4, 3m + 2j + 4 | j = 1, 2, \dots, m\}$$

and $|\phi(V)| = 3m + 3$. Thus, $G$ is antimagic. □

Figure 6 showed an example of antimagic labelling of graph $G = 5C_3 \cup P_3$, using the constructions given in Proposition 2.3.

**Proposition 2.4.** *Let $m$ and $t$ be positive integers. If $G = mC_3 \cup tP_3$ is antimagic, then $G' = G \cup C_3$ is also antimagic.*

*Proof.* If $G$ is antimagic, then is a one-to-one correspondence $f$ between the $E(G)$ to the label set $\{1, 2, \dots, 3m + 2t\}$ such that $\phi_f(u) \neq \phi_f(v)$, for any two distinct vertices of $u, v \in V(G)$. We

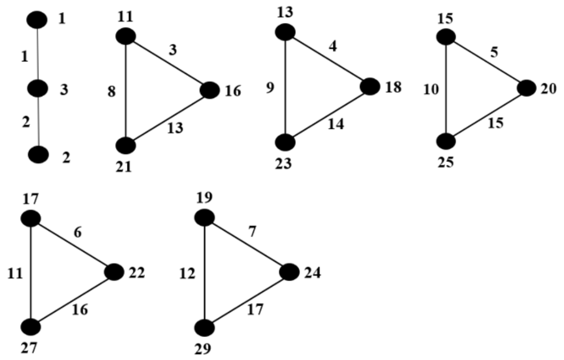


Figure 6: An antimagic labelling of $G = 5C_3 \cup P_3$.

label the $(m+1)$-th triangle by $(3m+2t+1, 3m+2t+2, 3m+2t+3)$ and the sum of the edge labels incident to the $(m+1)$-th triangle yields

$$\{\phi(v_{(m+1)_1}), \phi(v_{(m+1)_2}), \phi(v_{(m+1)_3})\} = \{6m+4t+3, 6m+4t+4, 6m+4t+5\}$$

. Since each vertex in $G'$ receive a unique sum, the graph $G'$ is antimagic. $\square$

Consider a graph $G = mC_3 \cup tP_3$ where $m, t \geq 1$, consisting of $m$ disjoint cycles $C_3$ and $t$ disjoint paths $P_3$. If $G$ is antimagic, adding an additional path $P_3$ to form $G' = G \cup P_3$ does not necessarily preserve the antimagic property. For instance, in Figure 7, the graph $G = C_3 \cup P_3$ is antimagic. However, when we add another $P_3$ to $G$ and label its edges with 6 and 7, it is evident that one of the vertices in this new $P_3$ must be labelled 7. This creates a repetition of vertex labels in the graph, thereby violating the antimagic property. Consequently, $G' = G \cup P_3$ is not antimagic in this case.

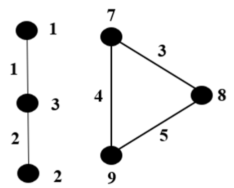


Figure 7: An antimagic labelling of $G = C_3 \cup P_3$.

In Figure 8, we start with an antimagic graph $G = C_3 \cup 2P_2$. When a new path $P_3$ is added on the left to form $G' = C_3 \cup 3P_2$, it becomes evident that $G'$ remains antimagic.

## 3 Skolem sequences and antimagic labelling

A *Skolem sequence* of order $n$ [12] is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions for every $k \in \{1, 2, \ldots, n\}$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$ and if $s_i = s_j = k$ with $i < j$, then $j - i = k$. Skolem sequences are also written as collections of ordered pairs $\{(a_i, b_i) : 1 \leq i \leq n, b_i - a_i = i\}$ with $\bigcup_{i=1}^{n} \{a_i, b_i\} = \{1, 2, \ldots, 2n\}$. Note that $(a_i, b_i)$ can be written in the triples $(i, a_i + n, b_i + n)$ for all $i = 1, \ldots, n$.

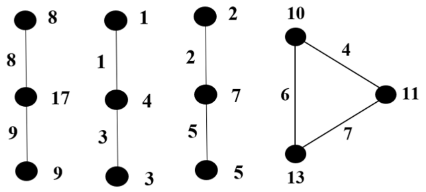


Figure 8: An antimagic labelling of $G = C_3 \cup 3P_3$.

Consider a skolem sequence of order 4, given by $S = (4, 2, 3, 2, 4, 3, 1, 1)$. This sequence can be represented as a set of tuples: $\{(1, 11, 12), (2, 6, 8), (3, 7, 10), (4, 5, 9)\}$. Accordingly, the labels of the four 3-paths are $(1, 11), (2, 6), (3, 7), (4, 5)$ as shown in the Figure 9. Note that the largest

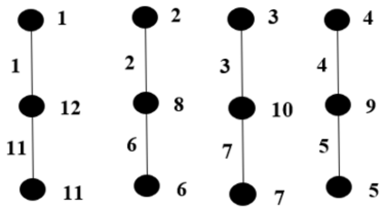


Figure 9: Labels of the edges of four $P_3$.

edge label among the 3-paths is 11. Since the integers $\{1, 2, 3, 4, 5, 6, 7, 11\}$ are used to label the edges of three-paths, the remaining integers $\{8, 9, 10\}$ are available to label the edges of triangle $C_3$. Consequently, the combined graph $G = C_3 \cup 4P_3$ is antimagic as illustrated in Figure 10.

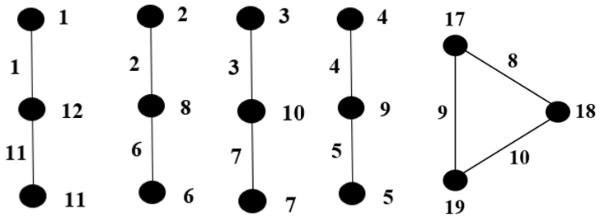


Figure 10: Antimagic labelling of $C_3 \cup 4P_3$.

In the following theorem, we will detail the Skolem sequence for $n = 4s$ and $n = 4s+1$ given in [7]. Subsequently, we will show how to use this strategy to create an antimagic labelling for graphs composed of unions of triangles and three-paths.

**Theorem 3.1.** *A Skolem sequence of order n exists if and only if $n \equiv 0, 1$ (mod 4). When $n = 1$, take* $(1, 1)$. *When $n = 4$, take* $(1, 1, 3, 4, 2, 3, 2, 4)$. *When $n = 5$, take* $(2, 4, 2, 3, 5, 4, 3, 1, 1, 5)$.

*When $n > 5$, use the construction*

$$n = 4s: \quad \begin{cases} (4s+r-1, 8s-r+1), & r = 1,2,\ldots,2s; \\ (r, 4s-r-1), & r = 1,2,\ldots,s-2; \\ (s+r+1, 3s-r), & r = 1,2,\ldots,s-2; \\ (s-1,3s),(s,s+1),(2s,4s-1), & (2s+1,6s). \end{cases}$$

$$n = 4s+1: \quad \begin{cases} (4s+r+1, 8s-r+3), & r = 1,2,\ldots,2s; \\ (r, 4s-r+1), & r = 1,2,\ldots,s; \\ (s+r+2, 3s-r+1), & r = 1,2,\ldots,s-2; \\ (s+1,s+2),(2s+1,6s+2), & (2s+2,4s+1). \end{cases}$$

Following this, we will illustrate how to use this Skolem sequences to construct an antimagic labelling.

**Theorem 3.2.** *Given that $n = 4s$ where $s \geq 3$ is a positive integer, there exists a graph $G = sC_3 \cup 4sP_3$ which is antimagic.*

*Proof.* In the case where $n = 4s$, the ordered pairs in Theorem 3.1 can be expressed as triples $(i, a_i + n, b_i + n)$, for $i = 1, 2 \ldots, n$ as detailed below:

$$\begin{cases} (4s-2r+2, 8s+r-1, 12s-r+1), & r = 1,2,\ldots,2s; \\ (4s-2r-1, 4s+r, 8s-r-1), & r = 1,2,\ldots,s-2; \\ (2s-2r-1, 5s+r+1, 7s-r), & r = 1,2,\ldots,s-2; \\ (2s+1,5s-1,7s),(1,5s,5s+1),(2s-1,6s,8s-1), & \\ (4s-1,6s+1,10s). & \end{cases}$$

To label the edges of the $i$-th 3-path, we use the pairs $(i, a_i + n)$, where $i = 1, 2 \ldots, n$. It's important to note that the highest value of $a_i + n$ in this context is $10s - 1$. We then identify the set $B$, defined as $B = \{b_i + n | b_i + n \leq 2n\}$ and the elements within $B$ can be listed as follows:

$$\begin{aligned} B &= \{5s+1, 7s, 8s-1\} \cup \{7s-r, 8s-r-1 | r = 1,2,\ldots,s-2\} \\ &= \{5s+1\} \cup \{6s+2,\ldots,8s-1\}. \end{aligned}$$

Observe that $|B| = 2s - 1$. By adding the elements $11s - 1$ and $11s$ to $B$, we let $B' = B \cup \{11s - 1, 11s\}$. If $G$ is to be antimagic, there must exists a bijection function $f$ mapping $E(G)$ to the set $\{1, 2, \ldots, 11s\}$. We will label the triangles in $G$ based on the following cases:

(i) For $s = 3k$ where $k \geq 1$, label the edges of all $3k$ triangles as follows:

$$\begin{cases} (6s+3j-1, 6s+3j, 6s+3j+1), & j = 1,2,\ldots,2k-1; \\ (10s+3j-1, 10s+3j, 10s+3j+1), & j = 1,2,\ldots,k-1; \\ (5s+1, 11s-1, 11s),(8s-1, 10s, 10s+1); & \end{cases}$$

(ii) When $s = 3k + 1$ where $k \geq 1$, assign labels to the edges of each of the $3k + 1$ triangles as described below:

$$\begin{cases} (6s+3j-1, 6s+3j, 6s+3j+1), & j = 1,2,\ldots,2k; \\ (10s+3j-3, 10s+3j-2, 10s+3j-1), & j = 1,2,\ldots,k; \\ (5s+1, 11s-1, 11s); & \end{cases}$$

(iii) For $s = 3k + 2$ where $k \geq 1$, proceed to label the edges of all $3k + 2$ triangles as follows:

$$\begin{cases} (6s+3j-1, 6s+3j, 6s+3j+1), & j = 1, 2, \ldots, 2k; \\ (10s+3j-2, 10s+3j-1, 10s+3j), & j = 1, 2, \ldots, k; \\ (5s+1, 11s-1, 11s), (8s-2, 8s-1, 10s). & \end{cases}$$

Since the labels for all vertices are distinct, this confirms that $G = sC_3 \cup 4sP_3$ is antimagic. $\square$

For the case when $s = 1$ and $s = 2$, the existence of the antimagic labelling for $G = sC_3 \cup 4sP_3$ is given in the Table 1:

Table 1: Antimagic labelling for $G = sC_3 \cup 4sP_3$, $n = 4s$.

| $s$ | $n$ | Labellings of $C_3$ | Labellings of $P_3$ |
|---|---|---|---|
| 1 | 4 | (6,10,11) | (1,5),(4,8),(2,9),(3,7) |
| 2 | 8 | (11,21,22),(14,15,20) | (1,10),(2,19),(3,12),(4,18),<br>(5,9),(6,17),(7,13),(8,16) |

An extended Skolem sequence of order $n$ is a sequence $ES = (s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ integers satisfying the conditions for every $k \in \{1, 2, \ldots, n\}$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$ with $i < j$, then $j - i = k$ and there is exactly one $s_i \in ES$ such that $s_i = 0$. An extended Skolem sequence of order $n$ exists for all $n$ [2].

For the case $n = 4s + 1$, $n = 4s + 2$ and $n = 4s - 1$, we shall use the extended skolem sequence to construct the antimagic labelling. Let's look at an explicit construction for the extended Skolem sequence [7]:

$$n = 4s+1, n > 5: \quad \begin{cases} (r, 4s-r+2), & r = 1, 2, \ldots, 2s; \\ (5s+r, 7s-r+3), & r = 1, 2, \ldots, s; \\ (4s+r+2, 8s-r+3), & r = 1, 2, \ldots, s-2; \\ (2s+1, 6s+2), (6s+1, 8s+4), (7s+3, 7s+4). & \end{cases}$$

$$n = 4s+2, n > 2: \quad \begin{cases} (r, 4s-r+3), & r = 1, 2, \ldots, 2s; \\ (4s+r+4, 8s-r+4), & r = 1, 2, \ldots, s-1; \\ (5s+r+3, 7s-r+3), & r = 1, 2, \ldots, s-2; \\ (2s+1, 6s+3), (2s+2, 6s+2), (4s+4, 6s+4), & \\ (7s+3, 7s+4), (8s+4, 8s+6). & \end{cases}$$

$$n = 4s-1, n > 3: \quad \begin{cases} (r, 4s-r), & r = 1, 2, \ldots, 2s-1; \\ (4s+r+1, 8s-r), & r = 1, 2, \ldots, s-2; \\ (5s+r, 7s-r-1), & r = 1, 2, \ldots, s-2; \\ (2s, 6s-1), (5s, 7s+1), (4s+1, 6s), (7s-1, 7s). & \end{cases}$$

In the upcoming theorems, we will demonstrate the use of extended Skolem sequences of order $n$ to construct antimagic labellings for the graph $G$ when $n = 4s + 1$, $n = 4s + 2$ and $n = 4s - 1$.

**Theorem 3.3.** *For any positive integer $n$ such that $n = 4s + 1$ and $s \geq 3$, there exists an antimagic graph $G = (\frac{n+3}{4})C_3 \cup nP_3$.*

*Proof.* When $n = 4s + 1$, the ordered pairs found in the extended Skolem sequences can be organized into sets of three, as demonstrated by the following triples:

$$\begin{cases} (4s-2r+2, 4s+r+1, 8s-r+3), & r = 1, 2, \ldots, 2s; \\ (2s-2r+3, 9s+r+1, 11s-r+4), & r = 1, 2, \ldots, s; \\ (4s-2r+1, 8s+r+3, 12s-r+4), & r = 1, 2, \ldots, s-2; \\ (4s+1, 6s+2, 10s+3), (2s+3, 10s+2, 12s+5), (1, 11s+4, 11s+5). \end{cases}$$

We label the edges of the $i$-th 3-path with the pairs $(i, a_i+n)$ for $i = 1, 2 \ldots, n$. The maximum value of $a_i + n$ is $11s + 4$. Following this, we define the set $B$ as $B = \{b_i + n | b_i + n \leq 2n\}$ and enumerate its elements as shown below:

$$\begin{aligned} B &= \{10s+3\} \cup \{8s-r+3 | r = 1, \ldots, 2s\} \cup \{11s-r+4 | r = 1, \ldots, s\} \\ &= \{6s+3, 6s+4, \ldots, 8s+2\} \cup \{10s+3, 10s+4, \ldots, 11s+3\}. \end{aligned}$$

In the extended Skolem sequence, we have $s_{n+1} = 0$. Thus we define $B' = B \cup \{8s+3\}$. Since $|B| = 3s+1$, it follows that $|B'| = 3s+2$. Recognizing that each triangle comprises three edges, we further expand our set to $B'' = B' \cup \{11s+5\}$. Consequently, the size of $B''$ is $|B''| = 3(s+1)$. For $G$ to be considered antimagic, there must exist a bijective function $f$ from the edges of $G$, denoted by $E(G)$, to the set $\{1, 2, \ldots, 11s+5\}$. Using the elements in $B''$, we labels the edges of all $(s+1)$ triangles as follows:

(i) When $s = 3k$ where $k \geq 1$, label the edges of each of the $(3k+1)$ triangles in the following manner:

$$\begin{cases} (6s+3j, 6s+3j+1, 6s+3j+2), & j = 1, 2, \ldots, 2k; \\ (10s+3j+2, 10s+3j+3, 10s+3j+4), & j = 1, 2, \ldots, k-1; \\ (8s+3, 10s+3, 10s+4), (11s+2, 11s+3, 11s+5). \end{cases}$$

(ii) When $s = 3k+1$, $k \geq 1$, apply the following labels to the edges of all $(3k+2)$ triangles:

$$\begin{cases} (6s+3j, 6s+3j+1, 6s+3j+2), & j = 1, 2, \ldots, 2k+1; \\ (10s+3j, 10s+3j+1, 10s+3j+2), & j = 1, 2, \ldots, k; \\ (11s+2, 11s+3, 11s+5). \end{cases}$$

(iii) For the case $s = 3k+2$ where $k \geq 1$, label the edges of each of the $(3k+3)$ triangles as indicated:

$$\begin{cases} (6s+3j, 6s+3j+1, 6s+3j+2), & j = 1, 2, \ldots, 2k+1; \\ (10s+3j+1, 10s+3j+2, 10s+3j+3), & j = 1, 2, \ldots, k; \\ (8s+2, 8s+3, 10s+3), (11s+2, 11s+3, 11s+5). \end{cases}$$

As all vertex labels are different, this leads us to the conclusion that $G = (\frac{n+3}{4})C_3 \cup nP_3$ is antimagic. □

The existence of antimagic labelling for the graph $G = (\frac{n+3}{4})C_3 \cup nP_3$ in cases where $n = 4s + 1$ is shown for $s = 1$ and $s = 2$ in Table 2.

Table 2: Antimagic labelling for $G = (\frac{n+3}{4})C_3 \cup nP_3$, $n = 4s + 1$.

| $s$ | $n$ | Labellings of $C_3$ | Labellings of $P_3$ |
|---|---|---|---|
| 1 | 5 | (6,11,12), (10,15,16) | (1,5),(4,8),(2,9),(3,7),(13,14) |
| 2 | 9 | (15,16,17),(18,19,23), (24,25,27) | (1,26),(2,13),(3,21),(4,12), (5,20),(6,11),(7,22),(8,10),(9,14) |

**Theorem 3.4.** *Given a positive integer $n$ where $n = 4s + 2$ for $s \geq 3$, an antimagic graph $G = \lceil \frac{n}{3} \rceil C_3 \cup nP_3$ exists.*

*Proof.* Note that for $n = 4s + 2$, we can reformulate the ordered pairs in the extended Skolem sequences into triples, as listed below:

$$\begin{cases} (4s-2r+3, 4s+r+2, 8s-r+5), & r = 1,2,\ldots,2s; \\ (4s-2r, 8s+r+6, 12s-r+6), & r = 1,2,\ldots,s-1; \\ (2s-2r, 9s+r+5, 11s-r+5), & r = 1,2,\ldots,s-2; \\ (4s+2, 6s+3, 10s+5), (4s, 6s+4, 10s+4), & (2s, 8s+6, 10s+6), \\ (1, 11s+5, 11s+6), (2, 12s+6, 12s+8). \end{cases}$$

For each 3-path indexed by $i$, we label its edges with $(i, a_i+n)$, for $i = 1, 2\ldots, n$. Notice that the highest value $a_i+n$ can reach is $12s+6$. We then define the set $B$ comprising $\{b_i+n | b_i+n \leq 2n\}$. The elements of this set are as follows:

$$\begin{aligned} B &= \{10s+4, 10s+5, 10s+6, 11s+6\} \cup \{8s-r+5 | r = 1,2,\ldots,2s\} \\ &\quad \cup \{12s-r+6 | r = 1,2,\ldots,s-1\} \cup \{11s-r+5 | r = 1,2,\ldots,s-2\} \\ &= \{6s+5, 6s+6, \ldots, 8s+4\} \cup \{10s+4, 10s+5, \ldots, 11s+4\} \\ &\quad \cup \{11s+6, 11s+7, \ldots, 12s+5\}. \end{aligned}$$

Since we are using the extended Skolem sequence, we expand our set to $B' = B \cup \{8s+5\}$. Considering that $|B| = 4s+1$, the size of the new set $B'$ becomes $|B'| = 4s+2$. The triangles in $G$ will then be labelled using all the elements in $B'$ as detailed below for each case.

(i) In the situation where $s = 3k$ where $k \geq 1$, assign labels to the edges of all the $(4k+1)$ triangles using the following pattern:

$$\begin{cases} (6s+3j+2, 6s+3j+3, 6s+3j+4), & j = 1,2,\ldots,2k; \\ (10s+3j+3, 10s+3j+4, 10s+3j+5), & j = 1,2,\ldots,k-1; \\ (11s+3j+4, 11s+3j+5, 11s+3j+6), & j = 1,2,\ldots,k-1; \\ (8s+5, 10s+4, 10s+5), (11s+3, 11s+4, 11s+6), \\ (12s+4, 12s+5, 12s+7). \end{cases}$$

(ii) For the case $s = 3k+1$ where $k \geq 1$, label the edges of all the $(4k+2)$ triangles as outlined below:

$$\begin{cases} (6s+3j+2, 6s+3j+3, 6s+3j+4), & j = 1,2,\ldots,2k+1; \\ (10s+3j+1, 10s+3j+2, 10s+3j+3), & j = 1,2,\ldots,k; \\ (11s+3j+4, 11s+3j+5, 11s+3j+6), & j = 1,2,\ldots,k; \\ (11s+3, 11s+4, 11s+6). \end{cases}$$

(iii) Given $s = 3k+2$ where $k \geq 1$, label the edges of all the $(4k+4)$ triangles as follows:

$$\begin{cases} (6s+3j+2, 6s+3j+3, 6s+3j+4), & j = 1, 2, \ldots, 2k+1; \\ (10s+3j+2, 10s+3j+3, 10s+3j+4), & j = 1, 2, \ldots, k; \\ (11s+3j+4, 11s+3j+5, 11s+3j+6), & j = 1, 2, \ldots, k; \\ (8s+4, 8s+5, 10s+4), (11s+3, 11s+4, 11s+6), & \\ (12s+5, 12s+7, 12s+8). & \end{cases}$$

Given that each vertex has a distinct label, $G = \lceil \frac{n}{3} \rceil C_3 \cup nP_3$ is thus confirmed to be antimagic. □

The existence of an antimagic labelling for the graph $G = \lceil \frac{n}{3} \rceil C_3 \cup nP_3$ is demonstrated for $n = 4s+2$ in Table 3, specifically for the cases where $s = 1$ and $s = 2$.

Table 3: Antimagic labelling for $G = \lceil \frac{n}{3} \rceil C_3 \cup nP_3$, $n = 4s+2$.

| $s$ | $n$ | Labellings of $C_3$ | Labellings of $P_3$ |
|---|---|---|---|
| 1 | 6 | (9,10,15),(11,12,17) | (1,16),(2,18),(3,8),(4,6),(5,7),(13,14) |
| 2 | 10 | (17,18,19),(20,21,24),<br>(25,26,28), (29,31,32) | (1,27),(2,30),(3,14),(4,22),(5,13),<br>(6,23),(7,12),(8,16),(9,11),(10,15) |

**Theorem 3.5.** *For $n = 4s-1$ where $s \geq 3$, there exists a graph $G = (\frac{n+1}{4})C_3 \cup nP_3$ that is antimagic.*

*Proof.* In the context of $n = 4s-1$, the ordered pairs from the extended Skolem sequences can be grouped into triples, which are outlined as follows:

$$\begin{cases} (4s-2r, 4s+r-1, 8s-r-1), & r = 1, 2, \ldots, 2s-1; \\ (4s-2r-1, 8s+r, 12s-r-1), & r = 1, 2, \ldots, s-2; \\ (2s-2r-1, 9s+r-1, 11s-r-2), & r = 1, 2, \ldots, s-2; \\ (4s-1, 6s-1, 10s-2), (2s+1, 9s-1, 11s), & \\ (2s-1, 8s, 10s-1), (1, 11s-2, 11s-1). & \end{cases}$$

We proceed to label the edges of the $i$-th 3-path using the pairs $(i, a_i+n)$, for $i = 1, 2 \ldots, n$. Note that the maximum value for $a_i+n$ is $11s-2$. The elements of $B = \{b_i+n | b_i+n \leq 11s-2\}$ can be enumerated as follows:

$$\begin{aligned} B &= \{10s-2, 10s-1\} \cup \{8s-r-1 | r = 1, 2, \ldots, 2s-1\} \\ &\quad \cup \{11s-r-2 | r = 1, 2, \ldots, s-2\} \\ &= \{6s, 6s+1, \ldots, 8s-2\} \cup \{10s-2, 10s-1, \ldots, 11s-3\}. \end{aligned}$$

Let $B' = B \cup \{8s-1\} = \{6s, 6s+1, \ldots, 8s-1\} \cup \{10s-2, 10s-1, \ldots, 11s-3\}$. Since $|B| = 3s-1$, it follows that $|B'| = 3s$. Using the elements in $B'$, we label the edges of all $s$ triangles in the following way.

(i) For $s = 3k$ where $k \geq 1$, proceed to label the edges of all $(3k)$ triangles as shown below:

$$\begin{cases} (6s+3j-3, 6s+3j-2, 6s+3j-1), & j = 1, 2, \ldots, 2k; \\ (10s+3j-5, 10s+3j-4, 10s+3j-3), & j = 1, 2, \ldots, k. \end{cases}$$

(ii) If $s = 3k + 1$ and $k \geq 1$, label each edge of the $(3k + 1)$ triangles as follows:

$$\begin{cases} (6s + 3j, 6s + 3j + 1, 6s + 3j + 2), & j = 1, 2, \ldots, 2k; \\ (10s + 3j - 4, 10s + 3j - 3, 10s + 3j - 2), & j = 1, 2, \ldots, k; \\ (8s - 2, 8s - 1, 10s - 2). \end{cases}$$

(iii) When $s = 3k+2$ with $k \geq 1$, assign the following labels to the edges of each of the $(3k+2)$ triangles:

$$\begin{cases} (6s + 3j - 3, 6s + 3j - 2, 6s + 3j - 1), & j = 1, 2, \ldots, 2k + 1; \\ (10s + 3j - 3, 10s + 3j - 2, 10s + 3j - 1), & j = 1, 2, \ldots, k; \\ (8s - 1, 10s - 2, 10s - 1). \end{cases}$$

Since the labels of all vertices are distinct, it follows that $G = (\frac{n+1}{4})C_3 \cup nP_3$ is indeed antimagic. □

In Table 4, the existence of an antimagic labelling for the graph $G = (\frac{n+1}{4})C_3 \cup nP_3$ is demonstrated for $n = 4s - 1$ with $s = 1$ and $s = 2$.

Table 4: Antimagic labelling for $G = (\frac{n+1}{4})C_3 \cup nP_3$, $n = 4s - 1$.

| $s$ | $n$ | Labellings of $C_3$ | Labellings of $P_3$ |
|---|---|---|---|
| 1 | 3 | (6,7,8) | (1,9),(2,4),(3,5) |
| 2 | 7 | (12,13,14),(15,18,19), | (1,20),(2,10),(3,16),(4,9),<br>(5,17),(6,8),(7,11) |

Building on the previously established theorems, the following demonstrates the result for the general cases.

**Theorem 3.6.** *For any integer $n \geq 9$, the graph formed by the disjoint union of $\lceil \frac{n}{3} \rceil$ copies of the graph $C_3$ and $n$ copies of the path graph $P_3$ is antimagic.*

*Proof.* The results are subsequently supported by Theorems 3.2, 3.3, 3.4, and 3.5, as well as by Proposition 2.4. □

Theorem 3.6 and Proposition 2.4 provide foundational support for the following theorem.

**Theorem 3.7.** *There exists a $G = mC_3 \cup nP_3$ which is antimagic for $n \geq 9$ and $m \geq \lceil \frac{n}{3} \rceil$.*

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