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Further inequalities for the numerical radius of off-diagonal part of 2 by 2 operator matrices

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Abstract. In this paper, using a refinement of the classical Young inequality, we present some new upper weighted bounds for the numerical radius of 2×2 block matrices, with entries are bounded operators.

Keywords. Numerical radius; positive operator; operator norm

1 Introduction

Let H be a complex Hilbert with inner product $\langle .,. \rangle$ and the corresponding norm $\|.\|$, and let $B(H)$ be the space of C^*-algebra of all bounded linear operators on H. For $T \in B(H)$, let T^* denote the adjoint of T. Also, $|T|, |T^*|$ denote the positive operators $(T^*T)^{\frac{1}{2}}$, $(TT^*)^{\frac{1}{2}}$ respectively. The numerical range and the numerical radius of T are defined by

$$W(T) := \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$$

and

$$w(T) := \sup\{|z| : z \in W(T)\}$$

respectively. It is well known that $w(.)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm, and for any $T \in B(H)$

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

The first inequality becomes an equality if $T^2 = 0$, and the second inequality becomes an equality if T is normal.

The spectral radius of T, denoted as $r(T)$, is defined as the radius of the smallest circle with center at origin containing the spectrum $\sigma(T)$ of the operator T. It is well known that the closure of the numerical range contains the spectrum, that is $r(T) \leq w(T)$.

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In recent years, the refinement of the upper and lower bounds on the numerical radius has attracted many authors. Kittaneh in [6] and [9] gave improvements to the second inequality of (1.1). He showed that, for $T \in B(H)$, we have

$$w(T) \leq \frac{1}{2}\||T| + |T^*|\| \leq \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}}) \tag{1.2}$$

and

$$w^2(T) \leq \frac{1}{2}\||T|^2 + |T^*|^2\|. \tag{1.3}$$

In [4], El-Hadad and Kittaneh established a generalization of (1.3), by showing that, for all $0 \leq \alpha \leq 1$ and $r \geq 1$, we have

$$w^{2r}(T) \leq \left\|(1-\alpha)|T|^{2r} + \alpha|T^*|^{2r}\right\|. \tag{1.4}$$

Recently, Bhunia and Paul [3] established the following refinements of (1.3)

$$w^{2r}(T) \leq \left\|\frac{\alpha}{2}(|T|^{\lambda 4r} + |T^*|^{(1-\lambda)4r}) + (1-\alpha)|T|^{2r}\right\| \tag{1.5}$$

and

$$w^{2r}(T) \leq \left\|\frac{\alpha}{2}(|T|^{\lambda 4r} + |T^*|^{(1-\lambda)4r}) + (1-\alpha)|T^*|^{2r}\right\| \tag{1.6}$$

for all $r \geq 1$ and $0 \leq \alpha, \lambda \leq 1$.
A general numerical radius inequality was proved by Kittaneh, in [9] it was shown that if $A, B, C, D, S, T \in B(H)$, then for all $\alpha \in (0,1)$,

$$w(ATB + CSD) \leq \frac{1}{2}\|A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2\alpha}D\|. \tag{1.7}$$

Shebrawi and Albadawi extended inequality (1.7), as follows,

Theorem 1.1. *[13, Theorem 2.5] Let $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, ..., n)$, and let f and g be nonnegative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then for all $r \geq 1$,*

$$w^r\left(\sum_{i=1}^{n} A_i X_i B_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}([B_i^* f^2(|X_i|)B_i]^r + [A_i g^2(|X_i^*|)A_i^*]^r)\right\|. \tag{1.8}$$

The purpose of this paper is to prove upper bounds for the numerical radius of an off-diagonal 2×2 operator matrices i.e. operators with the form $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ in $B(H_1 \oplus H_2)$, which extend and improve the inequalities (1.4), (1.5), (1.6), (1.8) and the existing bounds in [1, Theorem 2.11].

2 Main results

To achieve our goal, we need the following four lemmas, which can be found in [8],[15] and [5].

Lemma 2.1. *(McCarthy inequality)([15, p.20]) Let A be a positive operator in $B(H)$ and let x be in H such that $\|x\| \leq 1$. Then for all $r \geq 1$,*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle. \tag{2.1}$$

The second lemma is known as a generalized mixed Cauchy-Schwarz inequality which involves two nonnegative continuous functions.

Lemma 2.2. *[8, Theorem 1] Let $A \in B(H)$, and let f and g be nonnegative continuous functions on $[0;\infty)$ such that $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then for all $x, y \in H$,*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|. \tag{2.2}$$

The next lemma is a consequence of the convexity of the function $f(t) = t^r$ with $r \geq 1$.

Lemma 2.3. *For $i = 1, 2, ..., n$, let a_i be positive real number. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r,$$

for all $r \geq 1$.

Lemma 2.4. *[5] Let $A \in B(H)$. Then*

$$w\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = w(A).$$

The classical Young's inequality for non-negative real numbers says that if a and b are non-negative, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $r \geq 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^{rp}}{p} + \frac{b^{rq}}{q}\right)^{\frac{1}{r}}. \tag{2.3}$$

In [10], authors obtained a nice refinement of Young's inequality as follows,

$$a^{\alpha}b^{(1-\alpha)} \leq \alpha a + (1-\alpha)b - r_0(\sqrt{a} - \sqrt{b})^2, \tag{2.4}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$ and $\alpha \in [0,1]$.

Our first result reads as follows,

Theorem 2.1. *Let $T = \begin{bmatrix} 0 & \sum_{i=1}^{n} A_iX_iB_i \\ \sum_{i=1}^{n} C_iY_iD_i & 0 \end{bmatrix}$ be an operator in $B(H_1 \oplus H_2)$ where $B_i, C_i \in B(H_2)$, $A_i, D_i \in B(H_1)$, $X_i \in B(H_2, H_1)$ and $Y_i \in B(H_1, H_2)$ far all $i \in \{1, ..., n\}$, $r \geq 1$ and let f and g be as in Lemma 2.2. Then*

$$w^r(T) \leq \frac{n^{r-1}}{4}\left[\left\|\sum_{i=1}^{n}([B_i^* f^2(|X_i|)B_i]^r + [C_i g^2(|Y_i^*|)C_i^*]^r)\right\| + \left\|\sum_{i=1}^{n}([D_i^* f^2(|Y_i|)D_i]^r + [A_i g^2(|X_i^*|)A_i^*]^r)\right\| - \inf_{\|u_1\|^2=\|u_2\|^2=1}\delta(u_1,u_2)\right] \tag{2.5}$$

and

$$\delta(u_1,u_2) = \left(\left\langle\sum_{i=1}^{n}([B_i^* f^2(|X_i|)B_i]^r + [C_i g^2(|Y_i^*|)C_i^*]^r)u_2, u_2\right\rangle^{1/2} - \left\langle\sum_{i=1}^{n}([D_i^* f^2(|Y_i|)D_i]^r + [A_i g^2(|X_i^*|)A_i^*]^r)u_1, u_1\right\rangle^{1/2}\right)^2. \tag{2.6}$$

Proof. Let $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ a unit vector in $H_1 \oplus H_2$, then

$$
\begin{aligned}
&|\langle Tx, x\rangle|^r = \left| \left\langle \sum_{i=1}^n A_i X_i B_i x_2, x_1 \right\rangle + \left\langle \sum_{i=1}^n C_i Y_i D_i x_1, x_2 \right\rangle \right|^r \\
&\leq \left(\sum_{i=1}^n |\langle A_i X_i B_i x_2, x_1\rangle| + \sum_{i=1}^n |\langle C_i Y_i D_i x_1, x_2\rangle| \right)^r \\
&\leq (2n)^{r-1} \left(\sum_{i=1}^n |\langle A_i X_i B_i x_2, x_1\rangle|^r + \sum_{i=1}^n |\langle C_i Y_i D_i x_1, x_2\rangle|^r \right) \quad (\text{ by Lemma 2.3 }) \\
&\leq (2n)^{r-1} \left(\sum_{i=1}^n \left\langle f^2(|X_i|) B_i x_2, B_i x_2 \right\rangle^{r/2} \left\langle g^2(|X_i^*|) A_i^* x_1, A_i^* x_1 \right\rangle^{r/2} \right. \\
&\left. + \sum_{i=1}^n \left\langle f^2(|Y_i|) D_i x_1, D_i x_1 \right\rangle^{r/2} \left\langle g^2(|Y_i^*|) C_i^* x_2, C_i^* x_2 \right\rangle^{r/2} \right) \quad (\text{ by Lemma 2.2 }) \\
&= (2n)^{r-1} \left(\sum_{i=1}^n \|x_1\|^r \|x_2\|^r \left\langle B_i^* f^2(|X_i|) B_i \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle^{r/2} \left\langle A_i g^2(|X_i^*|) A_i^* \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle^{r/2} \right. \\
&\left. + \sum_{i=1}^n \|x_1\|^r \|x_2\|^r \left\langle D_i^* f^2(|Y_i|) D_i \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle^{r/2} \left\langle C_i g^2(|Y_i^*|) C_i^* \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle^{r/2} \right) \\
&\leq (2n)^{r-1} \left(\sum_{i=1}^n \left(\frac{\|x_1\|^2 + \|x_2\|^2}{2} \right)^r \left\langle B_i^* f^2(|X_i|) B_i u_2, u_2 \right\rangle^{r/2} \left\langle A_i g^2(|X_i^*|) A_i^* u_1, u_1 \right\rangle^{r/2} \right. \\
&\left. + \sum_{i=1}^n \left(\frac{\|x_1\|^2 + \|x_2\|^2}{2} \right)^r \left\langle D_i^* f^2(|Y_i|) D_i u_1, u_1 \right\rangle^{r/2} \left\langle C_i g^2(|Y_i^*|) C_i^* u_2, u_2 \right\rangle^{r/2} \right) \qquad (2.7) \\
&\leq \frac{(2n)^{r-1}}{2^r} \left(\sum_{i=1}^n \left\langle [B_i^* f^2(|X_i|) B_i]^r u_2, u_2 \right\rangle^{1/2} \left\langle [A_i g^2(|X_i^*|) A_i^*]^r u_1, u_1 \right\rangle^{1/2} \right. \\
&\left. + \sum_{i=1}^n \left\langle [D_i^* f^2(|Y_i|) D_i]^r u_1, u_1 \right\rangle^{1/2} \left\langle [C_i g^2(|Y_i^*|) C_i^*]^r u_2, u_2 \right\rangle^{1/2} \right) \quad (\text{ by Lemma 2.1 }) \qquad (2.8) \\
&\leq \frac{n^{r-1}}{2} \left[\left(\sum_{i=1}^n \left(\left\langle [B_i^* f^2(|X_i|) B_i]^r u_2, u_2 \right\rangle + \left\langle [C_i g^2(|Y_i^*|) C_i^*]^r u_2, u_2 \right\rangle \right) \right)^{1/2} \right. \\
&\left. \left(\sum_{i=1}^n \left(\left\langle [D_i^* f^2(|Y_i|) D_i]^r u_1, u_1 \right\rangle + \left\langle [A_i g^2(|X_i^*|) A_i^*]^r u_1, u_1 \right\rangle \right) \right)^{1/2} \right] (\text{ by Cauchy–Schwartz inequality }) \\
&\leq \frac{n^{r-1}}{2} \left[\left\langle \sum_{i=1}^n \left([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r \right) u_2, u_2 \right\rangle^{1/2} \right. \\
&\left. \left\langle \sum_{i=1}^n \left([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r \right) u_1, u_1 \right\rangle^{1/2} \right] \qquad (2.9) \\
&\leq \frac{n^{r-1}}{2} \frac{1}{2} \left[\left\langle \sum_{i=1}^n \left([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r \right) u_2, u_2 \right\rangle \right.
\end{aligned}
$$

$$+\left\langle \sum_{i=1}^{n}([D_i^* f^2(|Y_i|)D_i]^r + [A_i g^2(|X_i^*|)A_i^*]^r)u_1, u_1 \right\rangle$$

$$-\left(\left\langle \sum_{i=1}^{n}([B_i^* f^2(|X_i|)B_i]^r + [C_i g^2(|Y_i^*|)C_i^*]^r)u_2, u_2 \right\rangle^{1/2}\right.$$

$$\left.\left. -\left\langle \sum_{i=1}^{n}([D_i^* f^2(|Y_i|)D_i]^r + [A_i g^2(|X_i^*|)A_i^*]^r)u_1, u_1 \right\rangle^{1/2}\right)^2\right] \quad (\text{ by } (2.4)\ )$$

$$\leq \frac{n^{r-1}}{4}\left[\left\|\sum_{i=1}^{n}([B_i^* f^2(|X_i|)B_i]^r + [C_i g^2(|Y_i^*|)C_i^*]^r)\right\|\right.$$

$$+\left\|\sum_{i=1}^{n}[(D_i^* f^2(|Y_i|)D_i]^r + [A_i g^2(|X_i^*|)A_i^*]^r)\right\|$$

$$-\inf_{\|u_1\|=\|u_2\|=1}\left(\left\langle \sum_{i=1}^{n}([B_i^* f^2(|X_i|)B_i]^r + [C_i g^2(|Y_i^*|)C_i^*]^r)u_2, u_2 \right\rangle^{1/2}\right.$$

$$\left.\left. -\left\langle \sum_{i=1}^{n}([D_i^* f^2(|Y_i|)D_i]^r + [A_i g^2(|X_i^*|)A_i^*]^r)u_1, u_1 \right\rangle^{1/2}\right)^2\right]$$

$$\leq \frac{n^{r-1}}{4}\left[\left\|\sum_{i=1}^{n}([B_i^* f^2(|X_i|)B_i]^r + [C_i g^2(|Y_i^*|)C_i^*]^r)\right\|\right.$$

$$\left. +\left\|\sum_{i=1}^{n}([D_i^* f^2(|Y_i|)D_i]^r + [A_i g^2(|X_i^*|)A_i^*]^r)\right\| - \inf_{\|u_1\|=\|u_2\|=1}\delta(u_1,u_2)\right].$$

□

Remark 1. We note here that the inequality (2.5) is a significant refinement and extension of the following inequality in [1, Theorem 2.11],

$$w^r\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{2^{r-1}}{2}\left(\left\|f^{2r}(|X|) + g^{2r}(|Y^*|)\right\| + \left\|f^{2r}(|Y|) + g^{2r}(|X|)\right\|\right.$$
$$\left. - \inf_{\|(x_1,x_2)\|=1}\zeta(x_1,x_2)\right) \quad (2.10)$$

where

$$\zeta(x_1,x_2) = \left(\left\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))x_2, x_2\right\rangle^{1/2} - \left\langle (f^{2r}(|Y|) + g^{2r}(|X^*|))x_1, x_1\right\rangle^{1/2}\right)^2.$$

To show that, we take $n = 1$ in inequality (2.5), and we put $X_1 = X$, $Y_1 = Y$and $A_1 = B_1 = C_1 = D_1 = I$, also we mention that $\frac{2^{r-1}}{2} \geq \frac{1}{2} > \frac{1}{4}$ and

$$\inf_{\|x_1\|^2+\|x_2\|^2=1}\zeta(x_1,x_2) = \inf_{\|x_1\|^2+\|x_2\|^2=1}\left(\left\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))x_2, x_2\right\rangle^{1/2}\right.$$
$$\left. - \left\langle (f^{2r}(|Y|) + g^{2r}(|X^*|))x_1, x_1\right\rangle^{1/2}\right)^2$$

$$\leq \inf_{\|x_1\|=\|x_2\|=\frac{1}{\sqrt{2}}} \Bigg(\|x_2\| \left\langle (f^{2r}(|X|)+g^{2r}(|Y^*|))\frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|}\right\rangle^{1/2}$$

$$- \|x_1\| \left\langle (f^{2r}(|Y|)+g^{2r}(|X^*|))\frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|}\right\rangle^{1/2}\Bigg)^2$$

$$\leq \frac{1}{2} \inf_{\|u_1\|=\|u_2\|=1} \Big(\big\langle (f^{2r}(|X|)+g^{2r}(|Y^*|))u_2, u_2\big\rangle^{1/2}$$

$$- \big\langle (f^{2r}(|Y|)+g^{2r}(|X^*|))u_1, u_1\big\rangle^{1/2}\Big)^2$$

$$\leq \frac{1}{2} \inf_{\|u_1\|=\|u_2\|=1} \delta(u_1,u_2),$$

so

$$-\frac{1}{2} \inf_{\|u_1\|=\|u_2\|=1} \delta(u_1,u_2) \leq - \inf_{\|x_1\|^2+\|x_2\|^2=1} \zeta(x_1,x_2).$$

The following corollary is an immediate consequence of theorem 2.1.

Corollary 2.2. *Let $A_i, B_i, C_i \in B(H)$ far all $i \in \{1,...,n\}$, $r \geq 1$ and let $0<\alpha<1$. Then*

$$w^r\left(\sum_{i=1}^n A_iX_iB_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^n \left([B_i^*|X_i|^{2(1-\alpha)}B_i]^r + [A_i|X_i^*|^{2\alpha}A_i^*]^r\right)\right\|. \tag{2.11}$$

Proof. Considering $g(t)=t^{\alpha}$, $f(t)=t^{(1-\alpha)}$ and $0<\alpha<1$, then by (2.5)

$$w^r\left(\sum_{i=1}^n A_iX_iB_i\right) = w^r\left(\begin{bmatrix} 0 & \sum_{i=1}^n A_iX_iB_i \\ \sum_{i=1}^n A_iX_iB_i & 0\end{bmatrix}\right)$$

$$\leq \frac{n^{r-1}}{4}\Bigg[\left\|\sum_{i=1}^n ([B_i^*|X_i|^{2(1-\alpha)}B_i]^r + [A_i|X_i^*|^{2\alpha}A_i^*]^r)\right\|$$

$$+ \left\|\sum_{i=1}^n ([B_i^*|X_i|^{2(1-\alpha)}B_i]^r + [A_i|X_i^*|^{2\alpha}A_i^*]^r)\right\|$$

$$- \inf_{\|x_1\|^2+\|x_2\|^2=1}\Bigg(\left\langle \sum_{i=1}^n ([B_i^*|X_i|^{2(1-\alpha)}B_i]^r + [A_i|X_i^*|^{2\alpha}A_i^*]^r)\frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|}\right\rangle^{1/2}$$

$$- \left\langle \sum_{i=1}^n ([B_i^*|X_i|^{2(1-\alpha)}B_i]^r + [A_i|X_i^*|^{2\alpha}A_i^*]^r)\frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|}\right\rangle^{1/2}\Bigg)^2\Bigg]$$

$$= \frac{n^{r-1}}{2}\left\|\sum_{i=1}^n ([B_i^*|X_i|^{2(1-\alpha)}B_i]^r + [A_i|X_i^*|^{2\alpha}A_i^*]^r)\right\|.$$

□

We give here a weighted upper bound of the numerical radius for an off-diagonal operator in $B(H_1 \oplus H_2)$.

Theorem 2.3. *Let* $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ *be an operator in* $B(H_1 \oplus H_2)$ *and let* f *and* g *be nonnegative functions on* $[0,\infty)$ *which are continuous and satisfy the relation* $f(t)g(t) = t$ *for all* $t \in [0,\infty)$. *Then for all* $r \geq 1$ *and* $\alpha \in [0,1]$,

$$w^{2r}(T) \leq \frac{1}{2}\left(\left\|\frac{\alpha}{2}(f^{4r}(|X|) + g^{4r}(|Y^*|)) + (1-\alpha)|X|^{2r}\right\| + \left\|\frac{\alpha}{2}(f^{4r}(|Y|) + g^{4r}(|X^*|)) + (1-\alpha)|Y|^{2r}\right\|\right) \tag{2.12}$$

and

$$w^{2r}(T) \leq \frac{1}{2}\left(\left\|\frac{\alpha}{2}(f^{4r}(|X|) + g^{4r}(|Y^*|)) + (1-\alpha)|Y^*|^{2r}\right\| + \left\|\frac{\alpha}{2}(f^{4r}(|Y|) + g^{4r}(|X^*|)) + (1-\alpha)|X^*|^{2r}\right\|\right). \tag{2.13}$$

Proof. Let $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ a unit vector in $H_1 \oplus H_2$, then

$$\begin{aligned} |\langle Tx, x\rangle| &= \alpha|\langle Tx, x\rangle| + (1-\alpha)|\langle Tx, x\rangle| \\ &\leq \alpha|\langle Tx, x\rangle| + (1-\alpha)\left|\left\langle \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} x, x \right\rangle\right| \\ &\leq \alpha|\langle Tx, x\rangle| + (1-\alpha)\left(|\langle Xx_2, x_1\rangle| + |\langle Yx_1, x_2\rangle|\right). \end{aligned}$$

By the convexity of the function $f(t) = t^{2r}$, we find

$$\begin{aligned} &|\langle Tx, x\rangle|^{2r} \leq \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)\left(|\langle Xx_2, x_1\rangle| + |\langle Yx_1, x_2\rangle|\right)^{2r} \\ &\leq \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)\left(\|x_1\|\|x_2\|\left|\left\langle X\frac{x_2}{\|x_2\|}, \frac{x_1}{\|x_1\|}\right\rangle\right| + \|x_1\|\|x_2\|\left|\left\langle Y\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}\right\rangle\right|\right)^{2r} \\ &\leq \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)\left(\frac{1}{2}\left|\left\langle X\frac{x_2}{\|x_2\|}, \frac{x_1}{\|x_1\|}\right\rangle\right| + \frac{1}{2}\left|\left\langle Y\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}\right\rangle\right|\right)^{2r} \\ &\leq \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)\left(\frac{1}{2}\left\|X\frac{x_2}{\|x_2\|}\right\|^{2r} + \frac{1}{2}\left\|Y\frac{x_1}{\|x_1\|}\right\|^{2r}\right). \end{aligned} \tag{2.14}$$

The last inequality is an immediate consequence of Cauchy-Schwartz inequality. Now we take our matrix T and we apply (2.9) to get

$$\begin{aligned} |\langle Tx, x\rangle|^{2r} &\leq \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)|\langle Tx, x\rangle|^{2r} \\ &\leq \frac{\alpha}{2}\left(\left\langle (f^{4r}(|X|) + g^{4r}(|Y^*|))\frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|}\right\rangle^{1/2} \right. \\ &\quad \left. \left\langle (f^{4r}(|Y|) + g^{4r}(|X^*|))\frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|}\right\rangle^{1/2}\right) \\ &\quad + (1-\alpha)|\langle Tx, x\rangle|^{2r} \qquad \text{(by (2.9))} \\ &\leq \frac{\alpha}{4}\left(\left\langle (f^{4r}(|X|) + g^{4r}(|Y^*|))\frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|}\right\rangle \right. \\ &\quad \left. + \left\langle (f^{4r}(|Y|) + g^{4r}(|X^*|))\frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|}\right\rangle\right) \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
&+(1-\alpha)\left(\frac{1}{2}\left\|X\frac{x_2}{\|x_2\|}\right\|^{2r}+\frac{1}{2}\left\|Y\frac{x_1}{\|x_1\|}\right\|^{2r}\right) && \text{(by (2.14))}\\
&\leq \frac{\alpha}{4}\left\langle (f^{4r}(|X|)+g^{4r}(|Y^*|))\frac{x_2}{\|x_2\|},\frac{x_2}{\|x_2\|}\right\rangle && (2.16)\\
&+\frac{\alpha}{4}\left\langle (f^{4r}(|Y|)+g^{4r}(|X^*|))\frac{x_1}{\|x_1\|},\frac{x_1}{\|x_1\|}\right\rangle\\
&+(1-\alpha)\left(\frac{1}{2}\left\langle |X|^{2r}\frac{x_2}{\|x_2\|},\frac{x_2}{\|x_2\|}\right\rangle+\frac{1}{2}\left\langle |Y|^{2r}\frac{x_1}{\|x_1\|},\frac{x_1}{\|x_1\|}\right\rangle\right) && \text{(by Lemma 2.1)}\\
&\leq \left\langle \left[\frac{\alpha}{4}(f^{4r}(|X|)+g^{4r}(|Y^*|))+\frac{(1-\alpha)}{2}|X|^{2r}\right]\frac{x_2}{\|x_2\|},\frac{x_2}{\|x_2\|}\right\rangle\\
&+\left\langle \left[\frac{\alpha}{4}(f^{4r}(|Y|)+g^{4r}(|X^*|))+\frac{(1-\alpha)}{2}|Y|^{2r}\right]\frac{x_1}{\|x_1\|},\frac{x_1}{\|x_1\|}\right\rangle\\
&\leq \left\|\frac{\alpha}{4}(f^{4r}(|X|)+g^{4r}(|Y^*|))+\frac{(1-\alpha)}{2}|X|^{2r}\right\| && (2.17)\\
&+\left\|\frac{\alpha}{4}(f^{4r}(|Y|)+g^{4r}(|X^*|))+\frac{(1-\alpha)}{2}|Y|^{2r}\right\|.
\end{aligned}
$$

Taking supremum over all unit vector $x \in H_1 \oplus H_2$, we get the inequality (2.12). By similar arguments as above we can prove (2.13). □

Corollary 2.4. *Let $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ be an operator in $B(H_1 \oplus H_2)$ and $r \geq 1$. Then*

$$
w^{2r}(T) \leq \frac{1}{2}\min_{0\leq\alpha\leq 1}\left(\left\|\alpha|Y^*|^{2r}+(1-\alpha)|X|^{2r}\right\|+\left\|\alpha|X^*|^{2r}+(1-\alpha)|Y|^{2r}\right\|\right). \tag{2.18}
$$

Proof. Setting $f(t) = g(t) = \sqrt{t}$ in (2.12) and (2.13), we get respectively

$$
w^{2r}(T) \leq \frac{1}{2}\left(\left\|\frac{\alpha}{2}|Y^*|^{2r}+(1-\frac{\alpha}{2})|X|^{2r}\right\|+\left\|\frac{\alpha}{2}|X^*|^{2r}+(1-\frac{\alpha}{2})|Y|^{2r}\right\|\right) \tag{2.19}
$$

and

$$
w^{2r}(T) \leq \frac{1}{2}\left(\left\|\frac{\alpha}{2}|X|^{2r}+(1-\frac{\alpha}{2})|Y^*|^{2r}\right\|+\left\|\frac{\alpha}{2}|Y|^{2r}+(1-\frac{\alpha}{2})|X^*|^{2r}\right\|\right) \tag{2.20}
$$

for all $\alpha \in [0, 1]$ and $r \geq 1$, so combining the above two inequalities we find (2.18). □

If we put $Y = X = T$ in Theorem 2.3 and corollary 2.4, we obtain the following extension and refinement of [3, Theorem 2.5, Corollary 2.6, Corollary 2.7].

Corollary 2.5. *Let $T \in B(H)$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $r \geq 1$ and $0 \leq \alpha \leq 1$,*

$$
w^{2r}(T) \leq \left\|\frac{\alpha}{2}(f^{4r}(|T|)+g^{4r}(|T^*|))+(1-\alpha)|T|^{2r}\right\|, \tag{2.21}
$$

$$
w^{2r}(T) \leq \left\|\frac{\alpha}{2}(f^{4r}(|T|)+g^{4r}(|T^*|))+(1-\alpha)|T^*|^{2r}\right\| \tag{2.22}
$$

and

$$
w^{2r}(T) \leq \min_{0\leq\alpha\leq 1}\left\|\alpha|T^*|^{2r}+(1-\alpha)|T|^{2r}\right\|. \tag{2.23}
$$

Theorem 2.6. *Let* $T = \begin{bmatrix} 0 & \sum_{i=1}^n A_iX_iB_i \\ \sum_{i=1}^n C_iY_iD_i & 0 \end{bmatrix}$ *be an operator in* $B(H_1 \oplus H_2)$ *where* $B_i, C_i \in B(H_2)$, $A_i, D_i \in B(H_1)$, $X_i \in B(H_2, H_1)$ *and* $Y_i \in B(H_1, H_2)$ *far all* $i \in \{1, ..., n\}$, $r \geq 1$ *and let* f *and* g *be as in Lemma 2.2. Then*

$$w^r(T) \leq \frac{n^{r-1}}{2}\left(\alpha\left\|\sum_{i=1}^n \left([B_i^* f^2(|X_i|)B_i]^{\frac{r}{2\alpha}} + [C_i g^2(|Y_i^*|)C_i^*]^{\frac{r}{2\alpha}}\right)\right\| + (1-\alpha)\left\|\sum_{i=1}^n \left([D_i^* f^2(|Y_i|)D_i]^{\frac{r}{2(1-\alpha)}} + [A_i g^2(|X_i^*|)A_i^*]^{\frac{r}{2(1-\alpha)}}\right)\right\|\right) \quad (2.24)$$

and

$$w^r(T) \leq \frac{n^{r-1}}{2}\left(\left\|\sum_{i=1}^n \left(\alpha[B_i^* f^2(|X_i|)B_i]^{\frac{r}{2\alpha}} + (1-\alpha)[C_i g^2(|Y_i^*|)C_i^*]^{\frac{r}{2(1-\alpha)}}\right)\right\| + \left\|\sum_{i=1}^n \left(\alpha[D_i^* f^2(|Y_i|)D_i]^{\frac{r}{2\alpha}} + (1-\alpha)[A_i g^2(|X_i^*|)A_i^*]^{\frac{r}{2(1-\alpha)}}\right)\right\|\right) \quad (2.25)$$

for all $r \geq 1$, $\alpha \in]0,1[$ *with* $\frac{r}{\alpha} \geq 2$ *and* $\frac{r}{(1-\alpha)} \geq 2$.

Proof. Let $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ a unit vector in $H_1 \oplus H_2$, $u_1 = \frac{x_1}{\|x_1\|}$ and $u_2 = \frac{x_2}{\|x_2\|}$. Using (2.7), we find

$$\begin{aligned}
|\langle Tx, x\rangle|^r &\leq \frac{(2n)^{r-1}}{2^r}\left(\sum_{i=1}^n \langle [B_i^* f^2(|X_i|)B_i]u_2, u_2\rangle^{r/2}\langle [A_i g^2(|X_i^*|)A_i^*]u_1, u_1\rangle^{r/2}\right.\\
&\quad \left. + \sum_{i=1}^n \langle [D_i^* f^2(|Y_i|)D_i]u_1, u_1\rangle^{r/2}\langle [C_i g^2(|Y_i^*|)C_i^*]u_2, u_2\rangle^{r/2}\right)\\
&\leq \frac{(2n)^{r-1}}{2^r}\left(\sum_{i=1}^n \left(\alpha\langle [B_i^* f^2(|X_i|)B_i]u_2, u_2\rangle^{\frac{r}{2\alpha}} + (1-\alpha)\langle [A_i g^2(|X_i^*|)A_i^*]u_1, u_1\rangle^{\frac{r}{2(1-\alpha)}}\right)\right.\\
&\quad \left. + \sum_{i=1}^n \left((1-\alpha)\langle [D_i^* f^2(|Y_i|)D_i]u_1, u_1\rangle^{\frac{r}{2(1-\alpha)}} + \alpha\langle [C_i g^2(|Y_i^*|)C_i^*]u_2, u_2\rangle^{\frac{r}{2\alpha}}\right)\right) \quad (\text{ by } (2.3)\ )\\
&\leq \frac{(2n)^{r-1}}{2^r}\left(\sum_{i=1}^n \left(\alpha\langle [B_i^* f^2(|X_i|)B_i]^{\frac{r}{2\alpha}}u_2, u_2\rangle + (1-\alpha)\langle [A_i g^2(|X_i^*|)A_i^*]^{\frac{r}{2(1-\alpha)}}u_1, u_1\rangle\right)\right.\\
&\quad \left. + \sum_{i=1}^n \left((1-\alpha)\langle [D_i^* f^2(|Y_i|)D_i]^{\frac{r}{2(1-\alpha)}}u_1, u_1\rangle + \alpha\langle [C_i g^2(|Y_i^*|)C_i^*]^{\frac{r}{2\alpha}}u_2, u_2\rangle\right)\right) \quad (\text{ by Lemma } 2.1)\\
&\leq \frac{(2n)^{r-1}}{2^r}\left(\sum_{i=1}^n \left(\alpha\langle [B_i^* f^2(|X_i|)B_i]^{\frac{r}{2\alpha}}u_2, u_2\rangle + \alpha\langle [C_i g^2(|Y_i^*|)C_i^*]^{\frac{r}{2\alpha}}u_2, u_2\rangle\right)\right.\\
&\quad \left. + \sum_{i=1}^n \left((1-\alpha)\langle [D_i^* f^2(|Y_i|)D_i]^{\frac{r}{2(1-\alpha)}}u_1, u_1\rangle + (1-\alpha)\langle [A_i g^2(|X_i^*|)A_i^*]^{\frac{r}{2(1-\alpha)}}u_1, u_1\rangle\right)\right)\\
&\leq \frac{(2n)^{r-1}}{2^r}\left(\sum_{i=1}^n \alpha\left\langle \left([B_i^* f^2(|X_i|)B_i]^{\frac{r}{2\alpha}} + [C_i g^2(|Y_i^*|)C_i^*]^{\frac{r}{2\alpha}}\right)u_2, u_2\right\rangle\right.
\end{aligned}$$

$$+\sum_{i=1}^{n}(1-\alpha)\langle\Big([D_i^*f^2(|Y_i|)D_i]^{\frac{r}{2(1-\alpha)}}+[A_ig^2(|X_i^*|)A_i^*]^{\frac{r}{2(1-\alpha)}}\Big)u_1,u_1\rangle\Big)$$
$$\le \frac{n^{r-1}}{2}\left(\alpha\left\|\sum_{i=1}^{n}[B_i^*f^2(|X_i|)B_i]^{\frac{r}{2\alpha}}+[C_ig^2(|Y_i^*|)C_i^*]^{\frac{r}{2\alpha}}\right\|\right.$$
$$\left.+(1-\alpha)\left\|\sum_{i=1}^{n}[D_i^*f^2(|Y_i|)D_i]^{\frac{r}{2(1-\alpha)}}+[A_ig^2(|X_i^*|)A_i^*]^{\frac{r}{2(1-\alpha)}}\right\|\right).$$

Similarly, we reach the second inequality. □

Applying inequality (2.25) and Lemma 2.4, we obtain the following result.

Corollary 2.7. *Let T be an operator in $B(H)$. Then*

$$w^{2r}(T)\le \inf_{0<\alpha<1}\left\|(1-\alpha)|T|^{\frac{r}{(1-\alpha)}}+\alpha|T^*|^{\frac{r}{\alpha}}\right\| \tag{2.26}$$

for all $r\ge 1$.

Theorem 2.8. *Let $T=\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}$ be an operator in $B(H_1\oplus H_2)$ where $s\ge 1$. Then*

$$w^s(T)\le\frac{1}{2}\left(\max\{\|Y\|^s,\|X\|^s\}+\max\{r^{1/2}(|Y^*|^s|X|^s),r^{1/2}(|Y|^s|X^*|^s)\}-\inf_{\|x\|^2=1}\xi(x)\right) \tag{2.27}$$

and

$$\xi(x)=\left(\left\langle\begin{bmatrix}|Y|^s & 0\\ 0 & |X|^s\end{bmatrix}x,x\right\rangle^{1/2}-\left\langle\begin{bmatrix}|X^*|^s & 0\\ 0 & |Y^*|^s\end{bmatrix}x,x\right\rangle^{1/2}\right)^2.$$

Proof. Let x an unit vector in $H_1\oplus H_2$, then

$$\left|\left\langle\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}x,x\right\rangle\right|^s=\left\langle\left|\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right|x,x\right\rangle^{s/2}\left\langle\left|\begin{bmatrix}0 & Y^*\\ X^* & 0\end{bmatrix}\right|x,x\right\rangle^{s/2} \quad \text{(by Lemma 2.2)}$$
$$\le\left\langle\left|\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right|^s x,x\right\rangle^{1/2}\left\langle\left|\begin{bmatrix}0 & Y^*\\ X^* & 0\end{bmatrix}\right|^s x,x\right\rangle^{1/2} \quad \text{(by Lemma 2.1)}$$
$$\le\left\langle\begin{bmatrix}|Y|^s & 0\\ 0 & |X|^s\end{bmatrix}x,x\right\rangle^{1/2}\left\langle\begin{bmatrix}|X^*|^s & 0\\ 0 & |Y^*|^s\end{bmatrix}x,x\right\rangle^{1/2}$$
$$\le\frac{1}{2}\left(\left\langle\begin{bmatrix}|Y|^s+|X^*|^s & 0\\ 0 & |X|^s+|Y^*|^s\end{bmatrix}x,x\right\rangle\right.$$
$$\left.-\left(\left\langle\begin{bmatrix}|Y|^s & 0\\ 0 & |X|^s\end{bmatrix}x,x\right\rangle^{1/2}-\left\langle\begin{bmatrix}|X^*|^s & 0\\ 0 & |Y^*|^s\end{bmatrix}x,x\right\rangle^{1/2}\right)^2\right) \quad \text{(by (2.4))}$$
$$\le\frac{1}{2}\left(\max\{\||Y|^s+|X^*|^s\|,\||X|^s+|Y^*|^s\|\}-\inf_{\|x\|=1}\xi(x)\right),$$

by taking the supremum over all x, we get

$$w\left(\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right)^s\le\frac{1}{2}\left(\max\{\||Y|^s+|X^*|^s\|,\||X|^s+|Y^*|^s\|\}-\inf_{\|x\|=1}\xi(x)\right).$$

It follow from [7, Corollary 2]

$$\begin{aligned}\||Y|^s+|X^*|^s\| &\le \max\{\||Y|^s\|,\||X^*|^s\|\}+\||Y|^{s/2}|X^*|^{s/2}\|\\ &\le \max\{\|Y\|^s,\|X\|^s\}+\||Y|^{s/2}|X^*|^{s/2}\|\\ &\le \max\{\|Y\|^s,\|X\|^s\}+r^{1/2}(|Y|^s|X^*|^s) \qquad (\text{ by [2, Lemma 2.5] })\end{aligned}$$

and consequently

$$\||X|^s+|Y^*|^s\| \le \max\{\|Y\|^s,\|X\|^s\}+r^{1/2}(|Y^*|^s|X|^s).$$

Therefore we obtain the desired inequality. □

Using the above inequality and Lemma 2.4, we obtain the next result.

Corollary 2.9. *[2, Theorem 2.1] Let $T \in B(H)$. Then*

$$w(T) \le \frac{1}{2}\left(\|T\|+r^{1/2}(|T^*||T|)\right).$$

Theorem 2.10. *Let $T=\begin{bmatrix} 0 & \sum_{i=1}^n A_iX_iB_i \\ \sum_{i=1}^n C_iY_iD_i & 0\end{bmatrix}$ be an operator in $B(H_1\oplus H_2)$ where $B_i, C_i \in B(H_2)$, $A_i, D_i \in B(H_1)$, $X_i \in B(H_2,H_1)$ and $Y_i \in B(H_1,H_2)$ far all $i \in \{1,...,n\}$, $r\ge 1$ and let f and g be as in Lemma 2.2. Then*

$$w^r(T) \le \frac{n^{r-1}}{2}\left(\sum_{i=1}^n \|[B_i^*f^2(|X_i|)B_i]^r\|^{1/2}\|[A_ig^2(|X_i^*|)A_i^*]^r\|^{1/2} + \sum_{i=1}^n \|[D_i^*f^2(|Y_i|)D_i]^r\|^{1/2}\|[C_ig^2(|Y_i^*|)C_i^*]^r\|^{1/2}\right). \tag{2.28}$$

Proof. Let $x=\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right]$ a unit vector in $H_1\oplus H_2$, $u_1=\frac{x_1}{\|x_1\|}$ and $u_2=\frac{x_2}{\|x_2\|}$. Using (2.8), we find

$$\begin{aligned}|\langle Tx,x\rangle|^r &\le \frac{(2n)^{r-1}}{2^r}\left(\sum_{i=1}^n \left\langle [B_i^*f^2(|X_i|)B_i]^r u_2,u_2\right\rangle^{1/2}\left\langle [A_ig^2(|X_i^*|)A_i^*]^r u_1,u_1\right\rangle^{1/2}\right.\\ &\left.+\sum_{i=1}^n \left\langle [D_i^*f^2(|Y_i|)D_i]^r u_1,u_1\right\rangle^{1/2}\left\langle [C_ig^2(|Y_i^*|)C_i^*]^r u_2,u_2\right\rangle^{1/2}\right)\\ &\le \frac{n^{r-1}}{2}\left(\sum_{i=1}^n \|[B_i^*f^2(|X_i|)B_i]^r\|^{1/2}\|[A_ig^2(|X_i^*|)A_i^*]^r\|^{1/2}\right.\\ &\left.+\sum_{i=1}^n \|[D_i^*f^2(|Y_i|)D_i]^r\|^{1/2}\|[C_ig^2(|Y_i^*|)C_i^*]^r\|^{1/2}\right).\end{aligned}$$

□

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