



Further inequalities for the numerical radius of off-diagonal part of 2 by 2 operator matrices

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Abstract. In this paper, using a refinement of the classical Young inequality, we present some new upper weighted bounds for the numerical radius of 2×2 block matrices, with entries are bounded operators.

Keywords. Numerical radius; positive operator; operator norm

1 Introduction

Let H be a complex Hilbert with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$, and let $B(H)$ be the space of C^* -algebra of all bounded linear operators on H . For $T \in B(H)$, let T^* denote the adjoint of T . Also, $|T|$, $|T^*|$ denote the positive operators $(T^*T)^{\frac{1}{2}}$, $(TT^*)^{\frac{1}{2}}$ respectively. The numerical range and the numerical radius of T are defined by

$$W(T) := \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}$$

and

$$w(T) := \sup\{|z| : z \in W(T)\}$$

respectively. It is well known that $w(\cdot)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm, and for any $T \in B(H)$

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \quad (1.1)$$

The first inequality becomes an equality if $T^2 = 0$, and the second inequality becomes an equality if T is normal.

The spectral radius of T , denoted as $r(T)$, is defined as the radius of the smallest circle with center at origin containing the spectrum $\sigma(T)$ of the operator T . It is well known that the closure of the numerical range contains the spectrum, that is $r(T) \leq w(T)$.

In recent years, the refinement of the upper and lower bounds on the numerical radius has attracted many authors. Kittaneh in [6] and [9] gave improvements to the second inequality of (1.1). He showed that, for $T \in B(H)$, we have

$$w(T) \leq \frac{1}{2} \| |T| + |T^*| \| \leq \frac{1}{2} (\|T\| + \|T^2\|^{\frac{1}{2}}) \quad (1.2)$$

and

$$w^2(T) \leq \frac{1}{2} \| |T|^2 + |T^*|^2 \|. \quad (1.3)$$

In [4], El-Hadad and Kittaneh established a generalization of (1.3), by showing that, for all $0 \leq \alpha \leq 1$ and $r \geq 1$, we have

$$w^{2r}(T) \leq \| (1-\alpha)|T|^{2r} + \alpha|T^*|^{2r} \|. \quad (1.4)$$

Recently, Bhunia and Paul [3] established the following refinements of (1.3)

$$w^{2r}(T) \leq \left\| \frac{\alpha}{2} (|T|^{\lambda 4r} + |T^*|^{(1-\lambda)4r}) + (1-\alpha)|T|^{2r} \right\| \quad (1.5)$$

and

$$w^{2r}(T) \leq \left\| \frac{\alpha}{2} (|T|^{\lambda 4r} + |T^*|^{(1-\lambda)4r}) + (1-\alpha)|T^*|^{2r} \right\| \quad (1.6)$$

for all $r \geq 1$ and $0 \leq \alpha, \lambda \leq 1$.

A general numerical radius inequality was proved by Kittaneh, in [9] it was shown that if $A, B, C, D, S, T \in B(H)$, then for all $\alpha \in (0, 1)$,

$$w(ATB + CSD) \leq \frac{1}{2} \| A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2\alpha}D \|. \quad (1.7)$$

Shebrawi and Albadawi extended inequality (1.7), as follows,

Theorem 1.1. [13, Theorem 2.5] Let $A_i, B_i, X_i \in B(H)$ ($i = 1, 2, \dots, n$), and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $r \geq 1$,

$$w^r \left(\sum_{i=1}^n A_i X_i B_i \right) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) \right\|. \quad (1.8)$$

The purpose of this paper is to prove upper bounds for the numerical radius of an off-diagonal 2×2 operator matrices i.e. operators with the form $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ in $B(H_1 \oplus H_2)$, which extend and improve the inequalities (1.4), (1.5), (1.6), (1.8) and the existing bounds in [1, Theorem 2.11].

2 Main results

To achieve our goal, we need the following four lemmas, which can be found in [8],[15] and [5].

Lemma 2.1. (McCarthy inequality) ([15, p.20]) Let A be a positive operator in $B(H)$ and let x be in H such that $\|x\| \leq 1$. Then for all $r \geq 1$,

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle. \quad (2.1)$$

The second lemma is known as a generalized mixed Cauchy-Schwarz inequality which involves two nonnegative continuous functions.

Lemma 2.2. [8, Theorem 1] Let $A \in B(H)$, and let f and g be nonnegative continuous functions on $[0; \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $x, y \in H$,

$$|\langle Ax, y \rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|. \quad (2.2)$$

The next lemma is a consequence of the convexity of the function $f(t) = t^r$ with $r \geq 1$.

Lemma 2.3. For $i = 1, 2, \dots, n$, let a_i be positive real number. Then

$$\left(\sum_{i=1}^n a_i \right)^r \leq n^{r-1} \sum_{i=1}^n a_i^r,$$

for all $r \geq 1$.

Lemma 2.4. [5] Let $A \in B(H)$. Then

$$w\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = w(A).$$

The classical Young's inequality for non-negative real numbers says that if a and b are non-negative, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $r \geq 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^{rp}}{p} + \frac{b^{rq}}{q} \right)^{\frac{1}{r}}. \quad (2.3)$$

In [10], authors obtained a nice refinement of Young's inequality as follows,

$$a^\alpha b^{(1-\alpha)} \leq \alpha a + (1-\alpha)b - r_0(\sqrt{a} - \sqrt{b})^2, \quad (2.4)$$

where $r_0 = \min\{\alpha, 1-\alpha\}$ and $\alpha \in [0, 1]$.

Our first result reads as follows,

Theorem 2.1. Let $T = \begin{bmatrix} 0 & \sum_{i=1}^n A_i X_i B_i \\ \sum_{i=1}^n C_i Y_i D_i & 0 \end{bmatrix}$ be an operator in $B(H_1 \oplus H_2)$ where $B_i, C_i \in B(H_2)$, $A_i, D_i \in B(H_1)$, $X_i \in B(H_2, H_1)$ and $Y_i \in B(H_1, H_2)$ for all $i \in \{1, \dots, n\}$, $r \geq 1$ and let f and g be as in Lemma 2.2. Then

$$\begin{aligned} w^r(T) \leq \frac{n^{r-1}}{4} & \left[\left\| \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r) \right\| \right. \\ & \left. + \left\| \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) \right\| - \inf_{\|u_1\|^2 = \|u_2\|^2 = 1} \delta(u_1, u_2) \right] \quad (2.5) \end{aligned}$$

and

$$\begin{aligned} \delta(u_1, u_2) = & \left(\left\langle \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r) u_2, u_2 \right\rangle^{1/2} \right. \\ & \left. - \left\langle \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) u_1, u_1 \right\rangle^{1/2} \right)^2. \quad (2.6) \end{aligned}$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ a unit vector in $H_1 \oplus H_2$, then

$$\begin{aligned}
 |\langle Tx, x \rangle|^r &= \left| \left\langle \sum_{i=1}^n A_i X_i B_i x_2, x_1 \right\rangle + \left\langle \sum_{i=1}^n C_i Y_i D_i x_1, x_2 \right\rangle \right|^r \\
 &\leq \left(\sum_{i=1}^n |\langle A_i X_i B_i x_2, x_1 \rangle| + \sum_{i=1}^n |\langle C_i Y_i D_i x_1, x_2 \rangle| \right)^r \\
 &\leq (2n)^{r-1} \left(\sum_{i=1}^n |\langle A_i X_i B_i x_2, x_1 \rangle|^r + \sum_{i=1}^n |\langle C_i Y_i D_i x_1, x_2 \rangle|^r \right) \quad (\text{by Lemma 2.3}) \\
 &\leq (2n)^{r-1} \left(\sum_{i=1}^n \langle f^2(|X_i|) B_i x_2, B_i x_2 \rangle^{r/2} \langle g^2(|X_i^*|) A_i^* x_1, A_i^* x_1 \rangle^{r/2} \right. \\
 &\quad \left. + \sum_{i=1}^n \langle f^2(|Y_i|) D_i x_1, D_i x_1 \rangle^{r/2} \langle g^2(|Y_i^*|) C_i^* x_2, C_i^* x_2 \rangle^{r/2} \right) \quad (\text{by Lemma 2.2}) \\
 &= (2n)^{r-1} \left(\sum_{i=1}^n \|x_1\|^r \|x_2\|^r \left\langle B_i^* f^2(|X_i|) B_i \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle^{r/2} \left\langle A_i g^2(|X_i^*|) A_i^* \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle^{r/2} \right. \\
 &\quad \left. + \sum_{i=1}^n \|x_1\|^r \|x_2\|^r \left\langle D_i^* f^2(|Y_i|) D_i \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle^{r/2} \left\langle C_i g^2(|Y_i^*|) C_i^* \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle^{r/2} \right) \\
 &\leq (2n)^{r-1} \left(\sum_{i=1}^n \left(\frac{\|x_1\|^2 + \|x_2\|^2}{2} \right)^r \langle B_i^* f^2(|X_i|) B_i u_2, u_2 \rangle^{r/2} \langle A_i g^2(|X_i^*|) A_i^* u_1, u_1 \rangle^{r/2} \right. \\
 &\quad \left. + \sum_{i=1}^n \left(\frac{\|x_1\|^2 + \|x_2\|^2}{2} \right)^r \langle D_i^* f^2(|Y_i|) D_i u_1, u_1 \rangle^{r/2} \langle C_i g^2(|Y_i^*|) C_i^* u_2, u_2 \rangle^{r/2} \right) \quad (2.7)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{(2n)^{r-1}}{2^r} \left(\sum_{i=1}^n \langle [B_i^* f^2(|X_i|) B_i]^r u_2, u_2 \rangle^{1/2} \langle [A_i g^2(|X_i^*|) A_i^*]^r u_1, u_1 \rangle^{1/2} \right. \\
 &\quad \left. + \sum_{i=1}^n \langle [D_i^* f^2(|Y_i|) D_i]^r u_1, u_1 \rangle^{1/2} \langle [C_i g^2(|Y_i^*|) C_i^*]^r u_2, u_2 \rangle^{1/2} \right) \quad (\text{by Lemma 2.1}) \quad (2.8)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{n^{r-1}}{2} \left[\left(\sum_{i=1}^n (\langle [B_i^* f^2(|X_i|) B_i]^r u_2, u_2 \rangle + \langle [C_i g^2(|Y_i^*|) C_i^*]^r u_2, u_2 \rangle) \right)^{1/2} \right. \\
 &\quad \left. \left(\sum_{i=1}^n (\langle [D_i^* f^2(|Y_i|) D_i]^r u_1, u_1 \rangle + \langle [A_i g^2(|X_i^*|) A_i^*]^r u_1, u_1 \rangle) \right)^{1/2} \right] \quad (\text{by Cauchy-Schwartz inequality}) \\
 &\leq \frac{n^{r-1}}{2} \left[\left\langle \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r) u_2, u_2 \right\rangle^{1/2} \right. \\
 &\quad \left. \left\langle \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) u_1, u_1 \right\rangle^{1/2} \right] \quad (2.9) \\
 &\leq \frac{n^{r-1}}{2} \frac{1}{2} \left[\left\langle \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r) u_2, u_2 \right\rangle \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left\langle \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) u_1, u_1 \right\rangle \\
& - \left(\left\langle \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r) u_2, u_2 \right\rangle^{1/2} \right. \\
& \left. - \left\langle \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) u_1, u_1 \right\rangle^{1/2} \right)^2 \quad (\text{ by (2.4) }) \\
& \leq \frac{n^{r-1}}{4} \left[\left\| \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r) \right\| \right. \\
& + \left\| \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) \right\| \\
& - \inf_{\|u_1\|=\|u_2\|=1} \left(\left\langle \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r) u_2, u_2 \right\rangle^{1/2} \right. \\
& \left. - \left\langle \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) u_1, u_1 \right\rangle^{1/2} \right)^2 \\
& \leq \frac{n^{r-1}}{4} \left[\left\| \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^r + [C_i g^2(|Y_i^*|) C_i^*]^r) \right\| \right. \\
& + \left\| \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^r + [A_i g^2(|X_i^*|) A_i^*]^r) \right\| - \inf_{\|u_1\|=\|u_2\|=1} \delta(u_1, u_2) \left. \right].
\end{aligned}$$

□

Remark 1. We note here that the inequality (2.5) is a significant refinement and extension of the following inequality in [1, Theorem 2.11],

$$w^r \left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right) \leq \frac{2^{r-1}}{2} \left(\|f^{2r}(|X|) + g^{2r}(|Y^*|)\| + \|f^{2r}(|Y|) + g^{2r}(|X|)\| \right. \\
\left. - \inf_{\|(x_1, x_2)\|=1} \zeta(x_1, x_2) \right) \quad (2.10)$$

where

$$\zeta(x_1, x_2) = \left(\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))x_2, x_2 \rangle^{1/2} - \langle (f^{2r}(|Y|) + g^{2r}(|X^*|))x_1, x_1 \rangle^{1/2} \right)^2.$$

To show that, we take $n = 1$ in inequality (2.5), and we put $X_1 = X$, $Y_1 = Y$ and $A_1 = B_1 = C_1 = D_1 = I$, also we mention that $\frac{2^{r-1}}{2} \geq \frac{1}{2} > \frac{1}{4}$ and

$$\begin{aligned}
\inf_{\|x_1\|^2 + \|x_2\|^2 = 1} \zeta(x_1, x_2) &= \inf_{\|x_1\|^2 + \|x_2\|^2 = 1} \left(\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))x_2, x_2 \rangle^{1/2} \right. \\
&\quad \left. - \langle (f^{2r}(|Y|) + g^{2r}(|X^*|))x_1, x_1 \rangle^{1/2} \right)^2
\end{aligned}$$

$$\begin{aligned}
&\leq \inf_{\|x_1\|=\|x_2\|=\frac{1}{\sqrt{2}}} \left(\|x_2\| \left\langle (f^{2r}(|X|) + g^{2r}(|Y^*|)) \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle^{1/2} \right. \\
&\quad \left. - \|x_1\| \left\langle (f^{2r}(|Y|) + g^{2r}(|X^*|)) \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle^{1/2} \right)^2 \\
&\leq \frac{1}{2} \inf_{\|u_1\|=\|u_2\|=1} \left(\langle (f^{2r}(|X|) + g^{2r}(|Y^*|)) u_2, u_2 \rangle^{1/2} \right. \\
&\quad \left. - \langle (f^{2r}(|Y|) + g^{2r}(|X^*|)) u_1, u_1 \rangle^{1/2} \right)^2 \\
&\leq \frac{1}{2} \inf_{\|u_1\|=\|u_2\|=1} \delta(u_1, u_2),
\end{aligned}$$

so

$$-\frac{1}{2} \inf_{\|u_1\|=\|u_2\|=1} \delta(u_1, u_2) \leq - \inf_{\|x_1\|^2+\|x_2\|^2=1} \zeta(x_1, x_2).$$

The following corollary is an immediate consequence of theorem 2.1.

Corollary 2.2. *Let $A_i, B_i, C_i \in B(H)$ for all $i \in \{1, \dots, n\}$, $r \geq 1$ and let $0 < \alpha < 1$. Then*

$$w^r \left(\sum_{i=1}^n A_i X_i B_i \right) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^n \left([B_i^* |X_i|^{2(1-\alpha)} B_i]^r + [A_i |X_i^*|^{2\alpha} A_i^*]^r \right) \right\|. \quad (2.11)$$

Proof. Considering $g(t) = t^\alpha$, $f(t) = t^{(1-\alpha)}$ and $0 < \alpha < 1$, then by (2.5)

$$\begin{aligned}
w^r \left(\sum_{i=1}^n A_i X_i B_i \right) &= w^r \left(\begin{bmatrix} 0 & \sum_{i=1}^n A_i X_i B_i \\ \sum_{i=1}^n A_i X_i B_i & 0 \end{bmatrix} \right) \\
&\leq \frac{n^{r-1}}{4} \left[\left\| \sum_{i=1}^n ([B_i^* |X_i|^{2(1-\alpha)} B_i]^r + [A_i |X_i^*|^{2\alpha} A_i^*]^r) \right\| \right. \\
&\quad \left. + \left\| \sum_{i=1}^n ([B_i^* |X_i|^{2(1-\alpha)} B_i]^r + [A_i |X_i^*|^{2\alpha} A_i^*]^r) \right\| \right. \\
&\quad \left. - \inf_{\|x_1\|^2+\|x_2\|^2=1} \left(\left\langle \sum_{i=1}^n ([B_i^* |X_i|^{2(1-\alpha)} B_i]^r + [A_i |X_i^*|^{2\alpha} A_i^*]^r) \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle^{1/2} \right. \right. \\
&\quad \left. \left. - \left\langle \sum_{i=1}^n ([B_i^* |X_i|^{2(1-\alpha)} B_i]^r + [A_i |X_i^*|^{2\alpha} A_i^*]^r) \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle^{1/2} \right)^2 \right] \\
&= \frac{n^{r-1}}{2} \left\| \sum_{i=1}^n ([B_i^* |X_i|^{2(1-\alpha)} B_i]^r + [A_i |X_i^*|^{2\alpha} A_i^*]^r) \right\|.
\end{aligned}$$

□

We give here a weighted upper bound of the numerical radius for an off-diagonal operator in $B(H_1 \oplus H_2)$.

Theorem 2.3. Let $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ be an operator in $B(H_1 \oplus H_2)$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $r \geq 1$ and $\alpha \in [0, 1]$,

$$w^{2r}(T) \leq \frac{1}{2} \left(\left\| \frac{\alpha}{2} (f^{4r}(|X|) + g^{4r}(|Y^*|)) + (1 - \alpha)|X|^{2r} \right\| + \left\| \frac{\alpha}{2} (f^{4r}(|Y|) + g^{4r}(|X^*|)) + (1 - \alpha)|Y|^{2r} \right\| \right) \quad (2.12)$$

and

$$w^{2r}(T) \leq \frac{1}{2} \left(\left\| \frac{\alpha}{2} (f^{4r}(|X|) + g^{4r}(|Y^*|)) + (1 - \alpha)|Y^*|^{2r} \right\| + \left\| \frac{\alpha}{2} (f^{4r}(|Y|) + g^{4r}(|X^*|)) + (1 - \alpha)|X^*|^{2r} \right\| \right). \quad (2.13)$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ a unit vector in $H_1 \oplus H_2$, then

$$\begin{aligned} |\langle Tx, x \rangle| &= \alpha |\langle Tx, x \rangle| + (1 - \alpha) |\langle Tx, x \rangle| \\ &\leq \alpha |\langle Tx, x \rangle| + (1 - \alpha) \left| \left\langle \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} x, x \right\rangle \right| \\ &\leq \alpha |\langle Tx, x \rangle| + (1 - \alpha) (|\langle Xx_2, x_1 \rangle| + |\langle Yx_1, x_2 \rangle|). \end{aligned}$$

By the convexity of the function $f(t) = t^{2r}$, we find

$$\begin{aligned} |\langle Tx, x \rangle|^{2r} &\leq \alpha |\langle Tx, x \rangle|^{2r} + (1 - \alpha) (|\langle Xx_2, x_1 \rangle| + |\langle Yx_1, x_2 \rangle|)^{2r} \\ &\leq \alpha |\langle Tx, x \rangle|^{2r} + (1 - \alpha) \left(\|x_1\| \|x_2\| \left| \left\langle X \frac{x_2}{\|x_2\|}, \frac{x_1}{\|x_1\|} \right\rangle \right| + \|x_1\| \|x_2\| \left| \left\langle Y \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \right\rangle \right| \right)^{2r} \\ &\leq \alpha |\langle Tx, x \rangle|^{2r} + (1 - \alpha) \left(\frac{1}{2} \left| \left\langle X \frac{x_2}{\|x_2\|}, \frac{x_1}{\|x_1\|} \right\rangle \right| + \frac{1}{2} \left| \left\langle Y \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \right\rangle \right| \right)^{2r} \\ &\leq \alpha |\langle Tx, x \rangle|^{2r} + (1 - \alpha) \left(\frac{1}{2} \left\| X \frac{x_2}{\|x_2\|} \right\|^{2r} + \frac{1}{2} \left\| Y \frac{x_1}{\|x_1\|} \right\|^{2r} \right). \end{aligned} \quad (2.14)$$

The last inequality is an immediate consequence of Cauchy-Schwartz inequality. Now we take our matrix T and we apply (2.9) to get

$$\begin{aligned} |\langle Tx, x \rangle|^{2r} &\leq \alpha |\langle Tx, x \rangle|^{2r} + (1 - \alpha) |\langle Tx, x \rangle|^{2r} \\ &\leq \frac{\alpha}{2} \left(\left\langle (f^{4r}(|X|) + g^{4r}(|Y^*|)) \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle \right)^{1/2} \\ &\quad \left(\left\langle (f^{4r}(|Y|) + g^{4r}(|X^*|)) \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle \right)^{1/2} \\ &\quad + (1 - \alpha) |\langle Tx, x \rangle|^{2r} \quad (\text{by (2.9)}) \\ &\leq \frac{\alpha}{4} \left(\left\langle (f^{4r}(|X|) + g^{4r}(|Y^*|)) \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle \right. \\ &\quad \left. + \left\langle (f^{4r}(|Y|) + g^{4r}(|X^*|)) \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle \right) \end{aligned} \quad (2.15)$$

$$\begin{aligned}
& + (1 - \alpha) \left(\frac{1}{2} \left\| X \frac{x_2}{\|x_2\|} \right\|^{2r} + \frac{1}{2} \left\| Y \frac{x_1}{\|x_1\|} \right\|^{2r} \right) \quad (\text{by (2.14)}) \\
& \leq \frac{\alpha}{4} \left\langle (f^{4r}(|X|) + g^{4r}(|Y^*|)) \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle \quad (2.16) \\
& \quad + \frac{\alpha}{4} \left\langle (f^{4r}(|Y|) + g^{4r}(|X^*|)) \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle \\
& \quad + (1 - \alpha) \left(\frac{1}{2} \left\langle |X|^{2r} \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle + \frac{1}{2} \left\langle |Y|^{2r} \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle \right) \quad (\text{by Lemma 2.1}) \\
& \leq \left\langle \left[\frac{\alpha}{4} (f^{4r}(|X|) + g^{4r}(|Y^*|)) + \frac{(1 - \alpha)}{2} |X|^{2r} \right] \frac{x_2}{\|x_2\|}, \frac{x_2}{\|x_2\|} \right\rangle \\
& \quad + \left\langle \left[\frac{\alpha}{4} (f^{4r}(|Y|) + g^{4r}(|X^*|)) + \frac{(1 - \alpha)}{2} |Y|^{2r} \right] \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} \right\rangle \\
& \leq \left\| \frac{\alpha}{4} (f^{4r}(|X|) + g^{4r}(|Y^*|)) + \frac{(1 - \alpha)}{2} |X|^{2r} \right\| \quad (2.17) \\
& \quad + \left\| \frac{\alpha}{4} (f^{4r}(|Y|) + g^{4r}(|X^*|)) + \frac{(1 - \alpha)}{2} |Y|^{2r} \right\|.
\end{aligned}$$

Taking supremum over all unit vector $x \in H_1 \oplus H_2$, we get the inequality (2.12). By similar arguments as above we can prove (2.13). $\square$

Corollary 2.4. Let $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ be an operator in $B(H_1 \oplus H_2)$ and $r \geq 1$. Then

$$w^{2r}(T) \leq \frac{1}{2} \min_{0 \leq \alpha \leq 1} (\|\alpha |Y^*|^{2r} + (1 - \alpha) |X|^{2r}\| + \|\alpha |X^*|^{2r} + (1 - \alpha) |Y|^{2r}\|). \quad (2.18)$$

Proof. Setting $f(t) = g(t) = \sqrt{t}$ in (2.12) and (2.13), we get respectively

$$w^{2r}(T) \leq \frac{1}{2} \left(\left\| \frac{\alpha}{2} |Y^*|^{2r} + (1 - \frac{\alpha}{2}) |X|^{2r} \right\| + \left\| \frac{\alpha}{2} |X^*|^{2r} + (1 - \frac{\alpha}{2}) |Y|^{2r} \right\| \right) \quad (2.19)$$

and

$$w^{2r}(T) \leq \frac{1}{2} \left(\left\| \frac{\alpha}{2} |X|^{2r} + (1 - \frac{\alpha}{2}) |Y^*|^{2r} \right\| + \left\| \frac{\alpha}{2} |Y|^{2r} + (1 - \frac{\alpha}{2}) |X^*|^{2r} \right\| \right) \quad (2.20)$$

for all $\alpha \in [0, 1]$ and $r \geq 1$, so combining the above two inequalities we find (2.18). $\square$

If we put $Y = X = T$ in Theorem 2.3 and corollary 2.4, we obtain the following extension and refinement of [3, Theorem 2.5, Corollary 2.6, Corollary 2.7].

Corollary 2.5. Let $T \in B(H)$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $r \geq 1$ and $0 \leq \alpha \leq 1$,

$$w^{2r}(T) \leq \left\| \frac{\alpha}{2} (f^{4r}(|T|) + g^{4r}(|T^*|)) + (1 - \alpha) |T|^{2r} \right\|, \quad (2.21)$$

$$w^{2r}(T) \leq \left\| \frac{\alpha}{2} (f^{4r}(|T|) + g^{4r}(|T^*|)) + (1 - \alpha) |T^*|^{2r} \right\| \quad (2.22)$$

and

$$w^{2r}(T) \leq \min_{0 \leq \alpha \leq 1} \|\alpha |T^*|^{2r} + (1 - \alpha) |T|^{2r}\|. \quad (2.23)$$

Theorem 2.6. Let $T = \begin{bmatrix} 0 & \sum_{i=1}^n A_i X_i B_i \\ \sum_{i=1}^n C_i Y_i D_i & 0 \end{bmatrix}$ be an operator in $B(H_1 \oplus H_2)$ where $B_i, C_i \in B(H_2)$, $A_i, D_i \in B(H_1)$, $X_i \in B(H_2, H_1)$ and $Y_i \in B(H_1, H_2)$ for all $i \in \{1, \dots, n\}$, $r \geq 1$ and let f and g be as in Lemma 2.2. Then

$$w^r(T) \leq \frac{n^{r-1}}{2} \left(\alpha \left\| \sum_{i=1}^n ([B_i^* f^2(|X_i|) B_i]^{\frac{r}{2\alpha}} + [C_i g^2(|Y_i^*|) C_i^*]^{\frac{r}{2\alpha}}) \right\| \right. \\ \left. + (1-\alpha) \left\| \sum_{i=1}^n ([D_i^* f^2(|Y_i|) D_i]^{\frac{r}{2(1-\alpha)}} + [A_i g^2(|X_i^*|) A_i^*]^{\frac{r}{2(1-\alpha)}}) \right\| \right) \quad (2.24)$$

and

$$w^r(T) \leq \frac{n^{r-1}}{2} \left(\left\| \sum_{i=1}^n \left(\alpha [B_i^* f^2(|X_i|) B_i]^{\frac{r}{2\alpha}} + (1-\alpha) [C_i g^2(|Y_i^*|) C_i^*]^{\frac{r}{2(1-\alpha)}} \right) \right\| \right. \\ \left. + \left\| \sum_{i=1}^n \left(\alpha [D_i^* f^2(|Y_i|) D_i]^{\frac{r}{2\alpha}} + (1-\alpha) [A_i g^2(|X_i^*|) A_i^*]^{\frac{r}{2(1-\alpha)}} \right) \right\| \right) \quad (2.25)$$

for all $r \geq 1$, $\alpha \in]0, 1[$ with $\frac{r}{\alpha} \geq 2$ and $\frac{r}{(1-\alpha)} \geq 2$.

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ a unit vector in $H_1 \oplus H_2$, $u_1 = \frac{x_1}{\|x_1\|}$ and $u_2 = \frac{x_2}{\|x_2\|}$. Using (2.7), we find

$$| \langle Tx, x \rangle |^r \leq \frac{(2n)^{r-1}}{2^r} \left(\sum_{i=1}^n \langle [B_i^* f^2(|X_i|) B_i] u_2, u_2 \rangle^{r/2} \langle [A_i g^2(|X_i^*|) A_i^*] u_1, u_1 \rangle^{r/2} \right. \\ \left. + \sum_{i=1}^n \langle [D_i^* f^2(|Y_i|) D_i] u_1, u_1 \rangle^{r/2} \langle [C_i g^2(|Y_i^*|) C_i^*] u_2, u_2 \rangle^{r/2} \right) \\ \leq \frac{(2n)^{r-1}}{2^r} \left(\sum_{i=1}^n \left(\alpha \langle [B_i^* f^2(|X_i|) B_i] u_2, u_2 \rangle^{\frac{r}{2\alpha}} + (1-\alpha) \langle [A_i g^2(|X_i^*|) A_i^*] u_1, u_1 \rangle^{\frac{r}{2(1-\alpha)}} \right) \right. \\ \left. + \sum_{i=1}^n \left((1-\alpha) \langle [D_i^* f^2(|Y_i|) D_i] u_1, u_1 \rangle^{\frac{r}{2(1-\alpha)}} + \alpha \langle [C_i g^2(|Y_i^*|) C_i^*] u_2, u_2 \rangle^{\frac{r}{2\alpha}} \right) \right) \quad (\text{by (2.3)}) \\ \leq \frac{(2n)^{r-1}}{2^r} \left(\sum_{i=1}^n \left(\alpha \langle [B_i^* f^2(|X_i|) B_i]^{\frac{r}{2\alpha}} u_2, u_2 \rangle + (1-\alpha) \langle [A_i g^2(|X_i^*|) A_i^*]^{\frac{r}{2(1-\alpha)}} u_1, u_1 \rangle \right) \right. \\ \left. + \sum_{i=1}^n \left((1-\alpha) \langle [D_i^* f^2(|Y_i|) D_i]^{\frac{r}{2(1-\alpha)}} u_1, u_1 \rangle + \alpha \langle [C_i g^2(|Y_i^*|) C_i^*]^{\frac{r}{2\alpha}} u_2, u_2 \rangle \right) \right) \quad (\text{by Lemma 2.1}) \\ \leq \frac{(2n)^{r-1}}{2^r} \left(\sum_{i=1}^n \left(\alpha \langle [B_i^* f^2(|X_i|) B_i]^{\frac{r}{2\alpha}} u_2, u_2 \rangle + \alpha \langle [C_i g^2(|Y_i^*|) C_i^*]^{\frac{r}{2\alpha}} u_2, u_2 \rangle \right) \right. \\ \left. + \sum_{i=1}^n \left((1-\alpha) \langle [D_i^* f^2(|Y_i|) D_i]^{\frac{r}{2(1-\alpha)}} u_1, u_1 \rangle + (1-\alpha) \langle [A_i g^2(|X_i^*|) A_i^*]^{\frac{r}{2(1-\alpha)}} u_1, u_1 \rangle \right) \right) \\ \leq \frac{(2n)^{r-1}}{2^r} \left(\sum_{i=1}^n \alpha \langle ([B_i^* f^2(|X_i|) B_i]^{\frac{r}{2\alpha}} + [C_i g^2(|Y_i^*|) C_i^*]^{\frac{r}{2\alpha}}) u_2, u_2 \rangle \right.$$

$$\begin{aligned}
& + \sum_{i=1}^n (1-\alpha) \left\langle \left([D_i^* f^2(|Y_i|) D_i]^{\frac{r}{2(1-\alpha)}} + [A_i g^2(|X_i^*|) A_i^*]^{\frac{r}{2(1-\alpha)}} \right) u_1, u_1 \right\rangle \\
& \leq \frac{n^{r-1}}{2} \left(\alpha \left\| \sum_{i=1}^n [B_i^* f^2(|X_i|) B_i]^{\frac{r}{2\alpha}} + [C_i g^2(|Y_i^*|) C_i^*]^{\frac{r}{2\alpha}} \right\| \right. \\
& \quad \left. + (1-\alpha) \left\| \sum_{i=1}^n [D_i^* f^2(|Y_i|) D_i]^{\frac{r}{2(1-\alpha)}} + [A_i g^2(|X_i^*|) A_i^*]^{\frac{r}{2(1-\alpha)}} \right\| \right).
\end{aligned}$$

Similarly, we reach the second inequality. $\square$

Applying inequality (2.25) and Lemma 2.4, we obtain the following result.

Corollary 2.7. *Let T be an operator in $B(H)$. Then*

$$w^{2r}(T) \leq \inf_{0 < \alpha < 1} \left\| (1-\alpha) |T|^{\frac{r}{(1-\alpha)}} + \alpha |T^*|^{\frac{r}{\alpha}} \right\| \quad (2.26)$$

for all $r \geq 1$.

Theorem 2.8. *Let $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ be an operator in $B(H_1 \oplus H_2)$ where $s \geq 1$. Then*

$$w^s(T) \leq \frac{1}{2} \left(\max\{\|Y\|^s, \|X\|^s\} + \max\{r^{1/2}(|Y^*|^s |X|^s), r^{1/2}(|Y|^s |X^*|^s)\} - \inf_{\|x\|^2=1} \xi(x) \right) \quad (2.27)$$

and

$$\xi(x) = \left(\left\langle \begin{bmatrix} |Y|^s & 0 \\ 0 & |X|^s \end{bmatrix} x, x \right\rangle^{1/2} - \left\langle \begin{bmatrix} |X^*|^s & 0 \\ 0 & |Y^*|^s \end{bmatrix} x, x \right\rangle^{1/2} \right)^2.$$

Proof. Let x an unit vector in $H_1 \oplus H_2$, then

$$\begin{aligned}
& \left| \left\langle \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} x, x \right\rangle \right|^s = \left\langle \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} x, x \right\rangle^{s/2} \left\langle \begin{bmatrix} 0 & Y^* \\ X^* & 0 \end{bmatrix} x, x \right\rangle^{s/2} \quad (\text{by Lemma 2.2}) \\
& \leq \left\langle \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} x, x \right\rangle^{s/2} \left\langle \begin{bmatrix} 0 & Y^* \\ X^* & 0 \end{bmatrix} x, x \right\rangle^{s/2} \quad (\text{by Lemma 2.1}) \\
& \leq \left\langle \begin{bmatrix} |Y|^s & 0 \\ 0 & |X|^s \end{bmatrix} x, x \right\rangle^{1/2} \left\langle \begin{bmatrix} |X^*|^s & 0 \\ 0 & |Y^*|^s \end{bmatrix} x, x \right\rangle^{1/2} \\
& \leq \frac{1}{2} \left(\left\langle \begin{bmatrix} |Y|^s + |X^*|^s & 0 \\ 0 & |X|^s + |Y^*|^s \end{bmatrix} x, x \right\rangle \right. \\
& \quad \left. - \left(\left\langle \begin{bmatrix} |Y|^s & 0 \\ 0 & |X|^s \end{bmatrix} x, x \right\rangle^{1/2} - \left\langle \begin{bmatrix} |X^*|^s & 0 \\ 0 & |Y^*|^s \end{bmatrix} x, x \right\rangle^{1/2} \right)^2 \right) \quad (\text{by (2.4)}) \\
& \leq \frac{1}{2} \left(\max\{\||Y|^s + |X^*|^s\|, \||X|^s + |Y^*|^s\|\} - \inf_{\|x\|=1} \xi(x) \right),
\end{aligned}$$

by taking the supremum over all x , we get

$$w \left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right)^s \leq \frac{1}{2} \left(\max\{\||Y|^s + |X^*|^s\|, \||X|^s + |Y^*|^s\|\} - \inf_{\|x\|=1} \xi(x) \right).$$

It follow from [7, Corollary 2]

$$\begin{aligned} ||Y|^s + |X^*|^s|| &\leq \max\{||Y|^s||, ||X^*|^s||\} + ||Y|^{s/2}|X^*|^{s/2}|| \\ &\leq \max\{||Y||^s, ||X||^s\} + ||Y|^{s/2}|X^*|^{s/2}|| \\ &\leq \max\{||Y||^s, ||X||^s\} + r^{1/2}(|Y|^s|X^*|^s) \quad (\text{by [2, Lemma 2.5]}) \end{aligned}$$

and consequently

$$||X|^s + |Y^*|^s|| \leq \max\{||Y||^s, ||X||^s\} + r^{1/2}(|Y^*|^s|X|^s).$$

Therefore we obtain the desired inequality. $\square$

Using the above inequality and Lemma 2.4, we obtain the next result.

Corollary 2.9. [2, Theorem 2.1] *Let $T \in B(H)$. Then*

$$w(T) \leq \frac{1}{2} \left(\|T\| + r^{1/2}(|T^*||T|) \right).$$

Theorem 2.10. *Let $T = \begin{bmatrix} 0 & \sum_{i=1}^n A_i X_i B_i \\ \sum_{i=1}^n C_i Y_i D_i & 0 \end{bmatrix}$ be an operator in $B(H_1 \oplus H_2)$ where $B_i, C_i \in B(H_2)$, $A_i, D_i \in B(H_1)$, $X_i \in B(H_2, H_1)$ and $Y_i \in B(H_1, H_2)$ for all $i \in \{1, \dots, n\}$, $r \geq 1$ and let f and g be as in Lemma 2.2. Then*

$$\begin{aligned} w^r(T) &\leq \frac{n^{r-1}}{2} \left(\sum_{i=1}^n \| [B_i^* f^2(|X_i|) B_i]^r \|^{1/2} \| [A_i g^2(|X_i^*|) A_i^*]^r \|^{1/2} \right. \\ &\quad \left. + \sum_{i=1}^n \| [D_i^* f^2(|Y_i|) D_i]^r \|^{1/2} \| [C_i g^2(|Y_i^*|) C_i^*]^r \|^{1/2} \right). \quad (2.28) \end{aligned}$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ a unit vector in $H_1 \oplus H_2$, $u_1 = \frac{x_1}{\|x_1\|}$ and $u_2 = \frac{x_2}{\|x_2\|}$. Using (2.8), we find

$$\begin{aligned} |\langle Tx, x \rangle|^r &\leq \frac{(2n)^{r-1}}{2^r} \left(\sum_{i=1}^n \langle [B_i^* f^2(|X_i|) B_i]^r u_2, u_2 \rangle^{1/2} \langle [A_i g^2(|X_i^*|) A_i^*]^r u_1, u_1 \rangle^{1/2} \right. \\ &\quad \left. + \sum_{i=1}^n \langle [D_i^* f^2(|Y_i|) D_i]^r u_1, u_1 \rangle^{1/2} \langle [C_i g^2(|Y_i^*|) C_i^*]^r u_2, u_2 \rangle^{1/2} \right) \\ &\leq \frac{n^{r-1}}{2} \left(\sum_{i=1}^n \| [B_i^* f^2(|X_i|) B_i]^r \|^{1/2} \| [A_i g^2(|X_i^*|) A_i^*]^r \|^{1/2} \right. \\ &\quad \left. + \sum_{i=1}^n \| [D_i^* f^2(|Y_i|) D_i]^r \|^{1/2} \| [C_i g^2(|Y_i^*|) C_i^*]^r \|^{1/2} \right). \end{aligned}$$

$\square$

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