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Riemann solitons on Lorentzian generalized symmetric spaces

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Abstract. In this paper, we classify Riemann solitons on simply-connected 4-dimensional Lorentzian generalized symmetric spaces up to isometry. Then it is proved which of them is the gradient soliton. Also, we prove none of the potential vector fields of Riemann solitons are Killing vector fields.

Keywords. Conformal vector fields, generalized symmetric spaces, pseudo-Riemannian metrics.

1 Introduction

Hamilton [34] introduced the Ricci flow on a pseudo-Riemannian manifold (M, g) with the Ricci tensor S as follows

$$\frac{\partial}{\partial t} g = -2S. \tag{1.1}$$

Special solution to (1.1) is Ricci soliton [16] which is given by

$$\mathcal{L}_W g + S + \lambda g = 0, \tag{1.2}$$

for some constant λ and vector field W where $\mathcal{L}_W$ denotes the Lie derivative along W.

A Ricci soliton is a generalization of Einstein's metric and has applications in physics [1, 23, 24, 32, 38]. The pseudo-Riemannian geometry has more interesting properties than the Riemannian state and Ricci solitons have been studied on it [13, 17]. For example, in dimension three, there are some Riemannian homogeneous Ricci solitons [7, 41], but there are no left-invariant Riemannian Lie groups that admit Ricci solitons [31, 35, 42]. Also, there are several examples of the left-invariant three-dimensional Lorentzian Ricci solitons [13]. Many authors have generalized the Ricci flow and introduced new geometric flows. For instance, on manifold (M, g) with Riemann curvature tensor R the Riemann flow introduced by Udrişte [46, 47] is as follows

$$\frac{\partial}{\partial t} G(t) = -2R(g(t)),$$

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where $G = \frac{1}{2} g \odot g$ where for two $(0,2)$-tensors ω and θ product $\odot$ is defined by

$$\begin{aligned}(\omega \odot \theta)(Z_1, Z_2, Z_3, Z_4) &= \omega(Z_1, Z_4)\theta(Z_2, Z_3) + \omega(Z_2, Z_3)\theta(Z_1, Z_4) \\ &\quad -\omega(Z_1, Z_3)\theta(Z_2, Z_4) - \omega(Z_2, Z_4)\theta(Z_1, Z_3),\end{aligned}$$

for all vector fields Z_1, Z_2, Z_3, Z_4. Manifold (M, g) is said to be a Riemann soliton (or RS) [36] and denoted by (M^n, g, μ, Y) if it admits

$$2R + \mu g \odot g + g \odot \mathcal{L}_Y g = 0, \tag{1.3}$$

for some vector field Y and constant μ. If $\mu > 0$ or $\mu < 0$ or $\mu = 0$ then the Riemann soliton is called expanding or shrinking or steady. A Riemann soliton is labeled as a gradient Riemann soliton if $Y = \text{grad} h$ for a smooth map h and the RS becomes

$$2R + \mu g \odot g + 2g \odot \nabla^2 h = 0.$$

When μ is a smooth function then RS and gradient RS is said to be an almost RS and an almost gradient RS, respectively. The RS is corresponding to the Riemann flow as a fixed point. From (1.3), we get

$$\begin{aligned}2R(Z_1, Z_2, Z_3, Z_4) &= -2\mu\left[g(Z_1, Z_4)g(Z_2, Z_3) - g(Z_1, Z_3)g(Z_2, Z_4)\right] \\ &\quad - \left[g(Z_1, Z_4)\mathcal{L}_V g(Z_2, Z_3) + g(Z_2, Z_3)\mathcal{L}_V g(Z_1, Z_4)\right] \\ &\quad + \left[g(Z_1, Z_3)\mathcal{L}_V g(Z_2, Z_4) + g(Z_2, Z_4)\mathcal{L}_V g(Z_1, Z_3)\right].\end{aligned} \tag{1.4}$$

Contracting Z_1 and Z_4 in the last equation, we have

$$2S(Z_2, Z_3) = -2((n-1)\mu + \text{div}V)g(Z_2, Z_3) - (n-2)\mathcal{L}_V g(Z_2, Z_3). \tag{1.5}$$

A lot of studies have been done on RSs on manifolds. For instance, Biswas et al. [10] investigated RS on a almost co-Kahler 3-dimensional manifold, Venkatesha et al. [30, 48] studied RS on contact geometry and almost Kenmotsu manifolds, and K. De and U. C. De [25] studied almost RS on para-Sasakian manifolds. For the Ricci solitons on the Walker manifolds we refer the reader to [14, 15]. Also, see [8, 11, 26].

We use the abbreviation GSS to represent generalized symmetric spaces, and FGSS for four-dimensional GSS. In a paper by Cerny and Kowalski [22], pseudo-Riemannian FGSS were classified into four classes denoted as A, B, C, and D. With the exception of type C, which is Lorentzian, the associated pseudo-Riemannian metrics have signatures of $(4, 0)$, $(2, 2)$, and $(0, 4)$. The geometric properties of these spaces, such as the Levi-Civita connection, curvature tensor, and Ricci tensor, were calculated in [20, 29]. Batat and Onda [9] conducted a study on algebraic Ricci solitons on pseudo-Riemannian FGSS. Numerous research works have been carried out on the geometric structures of GGSSs, including studies on homogeneous geodesics [28], curvature properties [20], harmonicity properties of invariant vector fields [19], Ricci solitons [12], and Ricci bi-conformal vector fields [2]. Also, see [33, 37, 39, 40, 44, 45, 18].

Motivated by mentioned works, we study the RSs on Lorentzian FGSSs up to isometry. The article is arranged as follows. We recall some essential notions about simply-connected non-symmetric Lorentzian FGSSs which be used throughout this paper in Section 2. In the Section 3, the main results and their proof are presented.

2 Preliminaries

Let (M, g) be a pseudo-Riemannian connected manifold and $x \in M$. If an isometry s_x of M is such that x is fixed isolated point, then it is called symmetry at the point x.

A family of isometries denoted by $\{s_x | x \in M\}$ is termed a regular s structure if it satisfies the condition $s_x \circ s_y = s_{s_x(y)} \circ s_x$ for all $x, y \in M$, where s_x represents a symmetry for all $x \in M$, and the mapping $M \times M \to M$, given by $(x, y) \mapsto s_x(y)$, is smooth. (M, g) that possesses at least one regular s structure is referred to as a GSS. In the context of GSS, if (M, g) is a GSS, it is homogeneous, and can be identified with $(G/H, g)$, where G is a subgroup of the isometry group $I(M)$ that acts transitively on M, and H denotes the isotropy group at a fixed point $x \in M$. A FGSS of type C is the underlying homogeneous space G/H, where

$$G = \begin{pmatrix} e^{-w} & 0 & 0 & x \\ 0 & e^{w} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A FGSS of type C is the space $\mathbb{R}^4(x, y, z, w)$ with the metric

$$g = \epsilon(e^{-2w}dx^2 + e^{2w}dy^2) + dzdw, \tag{2.1}$$

where $\epsilon = \pm 1$. The possible signatures are $(1, 3), (3, 1)$.

Now, let $(M = G/H, g)$ be a non-symmetric simply-connected FGSS of type C with coordinates (x, y, z, w). Let ∇ be the Levi-Civita connection of (M, g) and R be its curvature tensor. The Ricci tensor is defined by

$$S(X_1, X_2) = \text{trace}(X_3 \to R(X_1, X_3)X_2). \tag{2.2}$$

Let

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = \frac{\partial}{\partial z}, \quad \partial_4 = \frac{\partial}{\partial w}$$

be the coordinates vector fields on M. From [12], ∇ of M with respect to $\{\partial_i\}_{i=1}^4$ is given by

$$\nabla_{\partial_i}\partial_j = \begin{pmatrix} 2\epsilon e^{-2w}\partial_3 & 0 & 0 & -\partial_1 \\ 0 & -2\epsilon e^{2w}\partial_3 & 0 & \partial_2 \\ 0 & 0 & 0 & 0 \\ -\partial_1 & \partial_2 & 0 & 0 \end{pmatrix}, \tag{2.3}$$

and

$$\begin{aligned} R(\partial_2, \partial_4)\partial_4 &= \partial_2, & R(\partial_4, \partial_2)\partial_2 &= 2\epsilon e^{2w}\partial_3, \\ R(\partial_4, \partial_1)\partial_4 &= -\partial_1, & R(\partial_1, \partial_4)\partial_1 &= -2\epsilon e^{-2w}\partial_3. \end{aligned}$$

Hence, the Riemann curvature tensor is characterized by non-zero elements

$$\begin{aligned} R_{1414} = -R_{1441} = -R_{4114} = R_{4141} = -\epsilon e^{-2w}, \\ R_{2424} = -R_{2442} = -R_{4224} = R_{4242} = -\epsilon e^{2w} \end{aligned} \tag{2.4}$$

and the Ricci tensor is given by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Along an arbitrary vector field $X = X_i\partial_i$ on manifold where $X_i = X_i(x, y, z, w)$, $i = 1, 2, 3, 4$ are smooth maps, we have

$$\begin{cases}(\mathcal{L}_Xg)_{11} = 2\epsilon e^{-2w}(\partial_1X_1 - X_4), & (\mathcal{L}_Xg)_{12} = \epsilon(e^{2w}\partial_1X_2 + e^{-2w}\partial_2X_1),\\ (\mathcal{L}_Xg)_{13} = \frac{1}{2}\partial_1X_4 + \epsilon e^{-2w}\partial_3X_1, & (\mathcal{L}_Xg)_{14} = \frac{1}{2}\partial_1X_3 + \epsilon e^{-2w}\partial_4X_1,\\ (\mathcal{L}_Xg)_{22} = 2\epsilon e^{2w}(X_4 + \partial_2X_2), & (\mathcal{L}_Xg)_{23} = \frac{1}{2}\partial_2X_4 + \epsilon e^{2w}\partial_3X_2,\\ (\mathcal{L}_Xg)_{24} = \frac{1}{2}\partial_2X_3 + \epsilon e^{2w}\partial_4X_2, & (\mathcal{L}_Xg)_{33} = \partial_3X_4,\\ (\mathcal{L}_Xg)_{34} = \frac{1}{2}(\partial_3X_3 + \partial_4X_4), & (\mathcal{L}_Xg)_{44} = \partial_4X_3.\end{cases} \tag{2.5}$$

3 RS on FGSS of type C

Now, we classify all RSs on non-symmetric simply-connected FGSSs of type C. Using (1.4), (M, g, μ, X) is a Riemann soliton if and only if

$$\begin{aligned}
2R_{1414} &= g_{11}(\mathcal{L}_Xg)_{44},\\
2R_{2424} &= g_{22}(\mathcal{L}_Xg)_{44},\\
2R_{3434} &= -2\mu g_{34}^2 - 2g_{34}(\mathcal{L}_Xg)_{34},\\
2R_{1212} &= 2\mu g_{11}g_{22} + g_{11}(\mathcal{L}_Xg)_{22} + g_{22}(\mathcal{L}_Xg)_{11},\\
2R_{1313} &= g_{11}(\mathcal{L}_Xg)_{33},\\
2R_{1214} &= g_{11}(\mathcal{L}_Xg)_{24},\\
2R_{1213} &= g_{11}(\mathcal{L}_Xg)_{23},\\
2R_{1223} &= -g_{22}(\mathcal{L}_Xg)_{13},\\
2R_{1224} &= -g_{22}(\mathcal{L}_Xg)_{14},\\
2R_{1314} &= 2\mu g_{11}g_{34} + g_{11}(\mathcal{L}_Xg)_{34} + g_{34}(\mathcal{L}_Xg)_{11},\\
2R_{1324} &= g_{34}(\mathcal{L}_Xg)_{12},\\
2R_{1334} &= g_{34}(\mathcal{L}_Xg)_{13}.
\end{aligned} \tag{3.1}$$

Applying (2.1), (2.4), and (2.5) in the equations (3.1), we have

$$\partial_4X_3 = -2, \tag{3.2}$$

$$\frac{1}{2}(\partial_3X_3 + \partial_4X_4) = -\mu, \tag{3.3}$$

$$\mu + \partial_2X_2 + \partial_1X_1 = 0, \tag{3.4}$$

$$\partial_3X_4 = 0, \tag{3.5}$$

$$\frac{1}{2}\partial_2X_3 + \epsilon e^{2w}\partial_4X_2 = 0, \tag{3.6}$$

$$\frac{1}{2}\partial_2X_4 + \epsilon e^{2w}\partial_3X_2 = 0, \tag{3.7}$$

$$\frac{1}{2}\partial_1X_4 + \epsilon e^{-2w}\partial_3X_1 = 0, \tag{3.8}$$

$$\frac{1}{2}\partial_1X_3 + \epsilon e^{-2w}\partial_4X_1 = 0, \tag{3.9}$$

$$2\mu + \frac{1}{2}(\partial_3X_3 + \partial_4X_4) + 2(\partial_1X_1 - X_4) = 0, \tag{3.10}$$

$$e^{2w}\partial_1X_2 + e^{-2w}\partial_2X_1 = 0. \tag{3.11}$$

Equations (3.2) and (3.5) imply that

$$X_3 = -2w + G(x,y,z), \quad X_4 = F(x,y,w), \tag{3.12}$$

for smooth maps F and G. Deriving equations (3.6) and (3.9) with respect to w, one gets

$$\partial^2_{44} X_1 - 2\partial_4 X_1 = 0, \tag{3.13}$$
$$\partial^2_{44} X_2 + 2\partial_4 X_2 = 0. \tag{3.14}$$

Solving these equations and using equations (3.6) and (3.9), we deduce

$$X_1 = H(x,y,z) - \frac{1}{4\epsilon} e^{2w} \partial_1 G(x,y,z), \tag{3.15}$$
$$X_2 = K(x,y,z) + \frac{1}{4\epsilon} e^{-2w} \partial_2 G(x,y,z) \tag{3.16}$$

for some smooth functions H and K. Utilizing (3.15) and (3.16) in (3.11), we conclude $e^{2w}\partial_1 K(x,y,z) + e^{-2w}\partial_2 H(x,y,z) = 0$. Hence $\partial_1 K(x,y,z) = 0$ and $\partial_2 H(x,y,z) = 0$. Therefore,

$$X_1 = L(x,z) - \frac{1}{4\epsilon} e^{2w} \partial_1 G(x,y,z), \tag{3.17}$$
$$X_2 = I(y,z) + \frac{1}{4\epsilon} e^{-2w} \partial_2 G(x,y,z) \tag{3.18}$$

for some smooth functions L and I. Deriving (3.7) and (3.8) with respect to z, it follows $\partial^2_{33} X_1 = 0$ and $\partial^2_{33} X_2 = 0$. It follows that

$$L(x,y) = L_1(x)z + L_2(x), \quad \partial_1 G(x,y,z) = G_1(x,y)z + G_2(x,y), \tag{3.19}$$
$$I(x,y) = I_1(y)z + I_2(y), \quad \partial_2 G(x,y,z) = G_3(x,y)z + G_4(x,y), \tag{3.20}$$

and

$$X_1 = L_1(x)z + L_2(x) - \frac{1}{4\epsilon} e^{2w} \left(G_1(x,y)z + G_2(x,y) \right), \tag{3.21}$$
$$X_2 = I_1(y)z + I_2(y) + \frac{1}{4\epsilon} e^{-2w} \left(G_3(x,y)z + G_4(x,y) \right). \tag{3.22}$$

Equations (3.19) and (3.20) yield

$$\partial_2 G_1(x,y) = \partial_1 G_3(x,y), \quad \partial_2 G_2(x,y) = \partial_1 G_4(x,y). \tag{3.23}$$

Applying (3.21) and (3.22) in (3.7) and (3.8), we infer

$$\frac{1}{2}\partial_1 F(x,y,w) + \epsilon e^{-2w} \left(L_1(x) - \frac{1}{4\epsilon} e^{2w} G_1(x,y) \right) = 0, \tag{3.24}$$
$$\frac{1}{2}\partial_2 F(x,y,w) + \epsilon e^{2w} \left(I_1(y) + \frac{1}{4\epsilon} e^{-2w} G_3(x,y) \right) = 0. \tag{3.25}$$

Deriving (3.24) and (3.25) with respect to y and x, respectively, we find

$$\frac{1}{2}\partial^2_{12} F(x,y,w) - \frac{1}{4}\partial_2 G_1(x,y) = 0, \tag{3.26}$$
$$\frac{1}{2}\partial^2_{21} F(x,y,w) + \frac{1}{4}\partial_1 G_3(x,y) = 0. \tag{3.27}$$

Equations (3.26) and (3.27) lead to $-\partial_2 G_1(x,y) = \partial_1 G_3(x,y)$. Then (3.23) leads to $\partial_2 G_1(x,y) = \partial_1 G_3(x,y) = 0$ and $G_1(x,y) = G_5(x)$ and $G_3(x,y) = G_6(y)$ for some smooth functions G_5 and G_6. Using (3.3), (3.4), and (3.10), we have

$$\partial_1 X_1 - \partial_2 X_2 - 2X_4 = 0. \tag{3.28}$$

Upon differentiating equation (3.28) with respect to z it follows

$$\partial^2_{13} X_1 - \partial^2_{23} X_2 = 0. \tag{3.29}$$

Substiutiting (3.21) and (3.22) in (3.29), we get

$$L_1'(x) - \frac{1}{4\epsilon} e^{2w} G_5'(x) - I_1'(y) - \frac{1}{4\epsilon} e^{-2w} G_6'(y) = 0. \tag{3.30}$$

The last equation yields $L_1'(x) = I_1'(y)$ and $G_5'(x) = G_6'(y) = 0$, then

$$L_1(x) = a_1 x + a_2, \quad I_1(y) = a_1 y + a_3, \quad G_5(x) = a_4, \quad G_6(y) = a_5 \tag{3.31}$$

for some constants a_1, a_2, a_3, a_4 and a_5. Upon differentiating equation (3.28) regarding to x it follows that

$$L_2''(x) - \frac{1}{4\epsilon} e^{2w} \partial^2_{11} G_2(x,y) - \frac{1}{4\epsilon} e^{-2w} \partial^2_{12} G_4(x,y) + 4\epsilon e^{-2w}(a_1 x + a_2) - a_4 = 0. \tag{3.32}$$

Equation (3.32) leads to

$$\partial^2_{11} G_2(x,y) = 0, \quad \partial^2_{12} G_4(x,y) = 16(a_1 x + a_2), \quad L_2''(x) = a_4. \tag{3.33}$$

Upon differentiating equation (3.28) regarding to y, we arrive at

$$-I_2''(y) - \frac{1}{4\epsilon} e^{2w} \partial^2_{12} G_2(x,y) - \frac{1}{4\epsilon} e^{-2w} \partial^2_{22} G_4(x,y) + 4\epsilon e^{2w}(a_1 y + a_3) + a_5 = 0. \tag{3.34}$$

Equation (3.34) leads to

$$\partial^2_{12} G_2(x,y) = 16(a_1 y + a_3), \quad \partial^2_{22} G_4(x,y) = 0, \quad I_2''(y) = a_5. \tag{3.35}$$

From (3.23), (3.33), and (3.35), we conclude $a_1 = a_2 = a_3 = 0$

$$\begin{aligned} G_2(x,y) &= 16(\frac{1}{2} a_1 y^2 + a_3 y + a_6)x + 8a_2 y^2 - a_7 y + a_8, \\ G_4(x,y) &= 16(\frac{1}{2} a_1 x^2 + a_2 x + a_9)y + 8a_3 x^2 - a_7 x + a_{10}, \\ L_2(x) &= \frac{1}{2} a_4 x^2 + a_{11} x + a_{12}, \quad I_2(y) = \frac{1}{2} a_5 y^2 + a_{13} y + a_{14}, \end{aligned}$$

for some constants $a_6, \ldots, a_{14}$. Also, from (3.24) and (3.25), we deduce

$$F(x,y,w) = -2\epsilon e^{-2w}(\frac{1}{2} a_1 x^2 + a_2 x) + \frac{1}{2} a_4 x - 2\epsilon e^{2w}(\frac{1}{2} a_1 y^2 + a_3 y) - \frac{1}{2} a_5 y + F_1(w) \tag{3.36}$$

for some smooth function F_1. Applying (3.28), it follows that

$$F_1(w) = \frac{1}{2}\left(a_{11} - a_{13} - 4\epsilon a_6 e^{2w} - 4\epsilon a_9 e^{-2w}\right). \tag{3.37}$$

Equation (3.3) implies that

$$\mu = -\frac{1}{2}\left(\partial_3 G(x,y,z) + 4\epsilon e^{-2w}(\frac{1}{2}a_1x^2 + a_2x + a_9) - 4\epsilon e^{2w}(\frac{1}{2}a_1y^2 + a_3y + a_6)\right). \tag{3.38}$$

Equation (3.4) implies that

$$\begin{aligned}\mu = -(2a_1z + a_4x + a_{11} + a_5y + a_{13} + 4\epsilon e^{-2w}(\frac{1}{2}a_1x^2 + a_2x + a_9) \\ -4\epsilon e^{2w}(\frac{1}{2}a_1y^2 + a_3y + a_6)).\end{aligned} \tag{3.39}$$

Compare (3.38) and (3.39) gives $a_1 = a_2 = a_3 = a_6 = a_9 = 0$ and $\partial_3 G(x,y,z) = 2(a_4x + a_{11} + a_5y + a_{13})$. Applying (3.39) in (3.10) we get $a_{11} = 0$. Using $\partial_3 G(x,y,z) = 2(a_4x + a_5y + a_{13})$, (3.19), and (3.20), we obtain $a_4 = a_5 = 0$,

$$G(x,y,z) = -a_7xy + a_8x + a_{10}y + 2a_{13}z + a_{14} \tag{3.40}$$

for some constant a_{14}.

Hence, we have the next consequence:

Theorem 3.1. *A FGSS of type C admit almost Riemann soliton (M,g,μ,X) with $X = X_i\partial_i$ if and only if $\mu = b_1$ and*

$$\begin{cases} X_1 = b_2 - \frac{1}{4\epsilon}e^{2w}(b_3y + b_4), \\ X_2 = -b_1y + b_5 + \frac{1}{4\epsilon}e^{-2w}(b_3x + b_6), \\ X_3 = -2w + b_3xy + b_4x + b_6y - 2b_1z + b_7, \\ X_4 = \frac{1}{2}b_1, \end{cases} \tag{3.41}$$

where $b_1, b_2, \ldots, b_7 \in \mathbb{R}$.

Theorem (3.1) gives:

Corollary 3.2. *Any almost Riemann soliton on FGSS of type C is a Riemann soliton.*

Now, we investigate which of RSs are gradient RS. For this end, we consider the potential vector field of a gradient RS is $X = \nabla f$ for some smooth function f on a FGSS of type C. On a FGSS, we have

$$\nabla f = \epsilon e^{2w}(\partial_1 f)\partial_1 + \epsilon e^{-2w}(\partial_2 f)\partial_2 + 2(\partial_4 f)\partial_3 + 2(\partial_3 f)\partial_4. \tag{3.42}$$

From (3.41) and (3.42), we have

$$\partial_1 f = \epsilon b_2 e^{-2w} - \frac{1}{4}(b_3y + b_4), \tag{3.43}$$

$$\partial_2 f = \epsilon e^{2w}(-b_1y + b_5) + \frac{1}{4}(b_3x + b_6), \tag{3.44}$$

$$\partial_3 f = \frac{1}{4}b_1, \tag{3.45}$$

$$\partial_4 f = \frac{1}{2}(-2w + b_3xy + b_4x + b_6y - 2b_1z + b_7). \tag{3.46}$$

By taking derivative of the equations (3.43) and (3.46) with respect to w and x, respectively, we obtain $-2\epsilon b_2 e^{-2w} = \frac{1}{2}(b_3 y + b_4)$ which implies that $b_2 = b_3 = b_4 = 0$. Also, taking derivative of (3.44) and (3.46) with respect to w and y, respectively, we deduce $\epsilon e^{2w}(-b_1 y + b_5) = \frac{1}{2}b_6$ which yields to $b_1 = b_5 = b_6 = 0$. Thus

$$\partial_1 f = \partial_2 f = \partial_3 f = 0, \text{ and } \partial_4 f = \frac{b_7}{2}, \tag{3.47}$$

and

$$f = \frac{b_7}{2} w + b_8, \tag{3.48}$$

where $b_8 \in \mathbb{R}$. Therefore, we have:

Corollary 3.3. *A FGSS of type C admits gradient RS $(M, g, \mu, \nabla f)$ if and only if $f = \frac{a}{2}w + b$, where $a, b \in \mathbb{R}$.*

Remark 1. A Killing vector field on (M, g) is a vector field X such that

$$\mathcal{L}_X g = 0.$$

Hence, from Theorem 3.1 we deduce that any potential vector fields of almost Riemann soliton on FGSS of type C is not Killing vector field because $(\mathcal{L}_X g)_{44} = \partial_4 X_3 = -2 \neq 0$. Also, is not conformal vector field.

Remark 2. A Ricci collineation vector field [21] is the vector field X such that $\mathcal{L}_X S = 0$. The Lie-derivative of S regarding to $X = X_i\partial_i$ on FGSS of type C is represented by

$$\mathcal{L}_X S = \begin{pmatrix} 0 & 0 & 0 & -2\partial_1 X_4 \\ 0 & 0 & 0 & -2\partial_2 X_4 \\ 0 & 0 & 0 & -2\partial_3 X_4 \\ -2\partial_1 X_4 & -2\partial_2 X_4 & -2\partial_3 X_4 & -4\partial_4 X_4 \end{pmatrix}. \tag{3.49}$$

Using Theorem 3.1, the potential vector fields of Riemann solitons on FGSS of type C become Ricci collineation vector field.

Remark 3. Ricci bi-conformal vector field [27] on a pseudo-Riemannian manifold (M, g) is a vector field X such that

$$\mathcal{L}_X g = \alpha g + \beta S, \qquad \mathcal{L}_X S = \alpha S + \beta g, \tag{3.50}$$

for smooth maps β and α. Also, see [2, 3, 4, 5, 6, 43]. Using Theorem 3.1, $\mathcal{L}_X S = 0$ which leads to $\alpha = \beta = 0$. But $\mathcal{L}_X g \neq 0$. Therefore, the potential vector fields of Riemann solitons on FGSSs of type C are not Ricci bi-conformal vector fields.

Remark 4. If a vector field X will be a potential vector field of a Ricci soliton on FGSSs of type C then the equation (1.2) implies that it satisfies in the following system.

$$\begin{cases} 2\epsilon e^{-2w}(\partial_1 X_1 - X_4) = \epsilon\lambda e^{-2w}, & \epsilon(e^{2w}\partial_1 X_2 + e^{-2w}\partial_2 X_1) = 0, \\ \frac{1}{2}\partial_1 X_4 + \epsilon e^{-2w}\partial_3 X_1 = 0, & \frac{1}{2}\partial_1 X_3 + \epsilon e^{-2w}\partial_4 X_1 = 0, \\ 2\epsilon e^{2w}(X_4 + \partial_2 X_2) = \epsilon\lambda e^{2w}, & \frac{1}{2}\partial_2 X_4 + \epsilon e^{2w}\partial_3 X_2 = 0, \\ \frac{1}{2}\partial_2 X_3 + \epsilon e^{2w}\partial_4 X_2 = 0, & \partial_3 X_4 = 0, \\ \frac{1}{2}(\partial_3 X_3 + \partial_4 X_4) = \lambda, & \partial_4 X_3 = 2. \end{cases} \tag{3.51}$$

Applying Theorem 3.1, we conclude that any potential vector fields of almost Riemann soliton on FGSS of type C is not potential vector fields of Ricci soliton because $\partial_4 X_3 = -2 \neq 2$.

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