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Inequalities for generalized normalized δ-Casorati curvatures of quasi bi-slant submanifolds of generalized complex space forms

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Abstract. In this article, we establish sharp inequalities involving generalized normalized δ-Casorati curvatures for quasi bi-slant submanifolds in generalized complex space forms and characterize the submanifolds for which the equality holds. In addition, we've extended the same inequalities to other types of submanifolds within the same geometric space. These include slant, invariant, anti-invariant, semi-slant, hemi-slant and bi-slant submanifolds.

Keywords. Generalized complex space forms, quasi bi-slant submanifolds, δ-Casorati curvature, shape operator

1 Introduction

Curvature invariants are the most important Riemannian invariants and the most natural ones in Riemannian geometry. Curvature invariants also play a key role in physics. For instance, the motion of a body in a gravitational field is determined by the curvature of spacetime, according to Einstein. All sorts of shapes, from soap bubbles to red blood cells, are determined by various curvatures [37]. In 1956, Nash [35] proved his famous embedding theorem.

Theorem 1.1. *Every Riemannian n-manifold can be isometrically embedded in a Euclidean m-space with dimension $m = \frac{n}{2}(n+1)(3n+11)$.*

This embedding theorem was aimed for in the hope that if each Riemannian manifold could always be regarded as an Euclidean submanifold, then it could yield the opportunity to use help from extrinsic geometry. However, this hope was not materialized for many years (see [25]). One important reason is that at that time, there did not exist general optimal relationships between known intrinsic invariants and main extrinsic invariants for *arbitrary* Euclidean submanifolds, except the three fundamental equations of Gauss, Codazzi and Ricci. This leads to the following fundamental problem in the theory of submanifolds (see [16, 17]).

Problem Find the relationship between extrinsic invariants and intrinsic invariants of a submanifold and find their applications. To provide answers to this problem, Chen, in the 1990 introduced his δ-invariants (also known as Chen invariants). Chen was able to establish optimal

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inequalities involving the δ-invariants and the squared mean curvature of submanifolds [13, 18]. Also, Chen discovered sharp inequalities involving Ricci curvature and squared mean curvature [16, 17], known as Chen-Ricci's inequality. During the last 25 years, these inequalities have been studied by many authors in various settings (see ([14, 15, 20, 21, 30, 32, 33, 34, 40, 41, 46]). Casorati curvature, an extrinsic invariant of submanifolds within a Riemannian manifold, was first introduced by Casorati [10]. It is defined as the normalized square length of the second fundamental form. This concept expands upon the notion of principal directions for hypersurfaces within a Riemannian manifold. The geometric significance and importance of Casorati curvature have been extensively discussed by notable geometers([22, 23, 27]). Consequently, it has garnered attention from geometers aiming to derive optimal inequalities for Casorati curvatures across various ambient spaces ([5, 6, 19, 26, 28, 29, 31, 36, 43, 45]). On the other hand, in the theory of submanifolds, the notion of slant submanifolds was introduced by Chen [11] as a natural generalization of holomorphic immersions and totally real immersions. In the course of time, this interesting notion has been studied broadly by several geometers ([3, 12, 41]). We note that invariant and anti-invariant ([47]) submanifolds are special cases of slant submanifolds with slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. Slant submanifolds that are neither invariant nor anti-invariant are referred to as proper slant submanifolds. As a generalisation of slant submanifolds, there are several kinds of submanifolds, semi-slant submanifolds ([38]), hemi-slant submanifolds ([4]), bi-slant submanifolds ([8, 9]), quasi bi-slant submanifolds ([1, 39]) and point wise quasi bi-slant submanifolds [2].

Thus it is worthwhile to study relationships between intrinsic and extrinsic invariants of submanifolds in a generalized space. This article explores such relationships for various types of submanifolds, including slant, totally real, and invariant ones, within generalized complex space forms, complex space forms, and RK-manifolds. The paper's structure is as follows: In Section 2, we provide preliminary information about generalized complex space forms and their submanifolds. We also provide information about Casorati curvature and this section lays the foundation for understanding the subsequent discussions. Section 3 focuses on establishing fundamental inequalities for quasi bi-slant submanifolds within generalized complex space forms. Various subcases are presented succinctly within a tabular form. In section 4 we summarizes the key implications of our results and out line several possible directions for future research in the theory of submanifolds.

2 Preliminaries

Let M be an almost Hermitian manifold equipped with an almost complex structure $\breve{J}$ and a Riemannian metric $\breve{g}$. If $\breve{J}$ satisfies the condition of integrability, i.e., the Nijenhuis tensor $[\breve{J}, \breve{J}]$ vanishes, then M is termed a Hermitian manifold. The fundamental 2-form Λ of M is defined as follows:

$$\Lambda(X_1, X_2) = \breve{g}(X_1, \breve{J}X_2),$$

where X_1 and X_2 are elements of the tangent space TM. An almost Hermitian manifold is called a Käehler manifold if the the fundamental 2-form Λ is closed and $\breve{\nabla}_{X_1}\breve{J}X_2 = 0$ is satisfied for any $X_1, X_2 \in \mathcal{X}(M)$, where $\breve{\nabla}$ denote the Levi-Cevita connection on the manifold M. Moreover, a complex space form with constant holomorphic sectional curvature c is denoted by $M(c)$.

Gray ([24]) introduced the notion of constant type for a nearly Kähler manifold, which led to definitions of RK-manifolds of constant holomorphic sectional curvature c and constant type ([44]) and generalized complex space forms([42]). An RK-manifold M is an almost Hermitian manifold for which the curvature tensor $\breve{R}$ is $\breve{J}$-invariant, i.e.,

$$\breve{R}(\breve{J}X_1, \breve{J}X_2, \breve{J}X_3, \breve{J}X_4) = \breve{R}(X_1, X_2, X_3, X_4),$$

for all vector fields $X_1, X_2, X_3, X_4 \in TM$. An almost Hermitian manifold M is said to have (pointwise) constant type if for each $x \in M$ and for all vector fields $X_1, X_2, X_3 \in T_xM$, such that

$$\begin{aligned} \breve{g}(X_1, X_2) &= \breve{g}(X_1, X_3) = \breve{g}(X_1, \breve{J}X_2) = \breve{g}(X_1, \breve{J}X_3) = 0, \\ \breve{g}(X_1, X_1) &= \breve{g}(X_2, X_2) = \breve{g}(X_3, X_3) = 1, \end{aligned}$$

we have

$$\begin{aligned} \breve{R}(X_1, X_2, X_1, X_2) - \breve{R}(X_1, X_2, \breve{J}X_1, \breve{J}X_2) &= \breve{R}(X_1, X_3, X_1, X_3) \\ &\quad -\breve{R}(X_1, X_3, \breve{J}X_1, \breve{J}X_3). \end{aligned}$$

An RK-manifold M has (pointwise) constant type if and only if there is a differentiable function $\breve{\alpha}$ on M such that

$$\begin{aligned} \breve{R}(X_1, X_2, X_1, X_2) - \breve{R}(X_1, X_2, \breve{J}X_1, \breve{J}X_2) &= \breve{\alpha}\{\breve{g}(X_1, X_1)\breve{g}(X_2, X_2) \\ &- \breve{g}^2(X_1, X_2) - \breve{g}^2(X_1, \breve{J}X_2)\}, \end{aligned}$$

for all vector fields $X_1, X_2 \in TM$. Furthermore, M has a global constant type if $\breve{\alpha}$ is constant. The function $\breve{\alpha}$ is called the constant type of M. An RK-manifold of constant holomorphic sectional curvature c and constant type $\breve{\alpha}$ is called a generalized complex space form, denoted by $M(c, \breve{\alpha})$. The curvature tensor $\breve{R}$ of $M(c, \breve{\alpha})$ has the following expression([34]);

$$\begin{aligned} 4\breve{R}(X_1, X_2, X_3, X_4) &= (c + 3\breve{\alpha})\{\breve{g}(X_1, X_3)\breve{g}(X_2, X_4) - \breve{g}(X_1, X_4)\breve{g}(X_2, X_3)\} \\ &\quad +(c - \breve{\alpha})\{\breve{g}(\breve{J}X_1, X_3)\breve{g}(\breve{J}X_2, X_4) - \breve{g}(\breve{J}X_1, X_4) \\ &\quad \breve{g}(\breve{J}X_2, X_3) + 2\breve{g}(X_1, \breve{J}X_2)\breve{g}(X_3, \breve{J}X_4)\}, \end{aligned} \tag{2.1}$$

for all vector fields $X_1, X_2, X_3, X_4 \in TM$.

If $c = \breve{\alpha}$, then $M(c, \breve{\alpha})$ is a space of constant curvature. A complex space form $M(c)$ (i.e., a Käehler manifold of constant holomorphic sectional curvature c) belongs to the class of almost Hermitian manifold $M(c, \breve{\alpha})$ (with constant type zero).
Let N be an m-dimensional submanifold of an n-dimensional generalized complex space form $M(c, \breve{\alpha})$, the Gauss and Weingarten formulas are defined by;

$$\breve{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \sigma(X_1, X_2),$$

and

$$\breve{\nabla}_{X_1} \xi = -A_\xi X_1 + \nabla^\perp_{X_1} \xi,$$

respectively, for each $X_1, X_2 \in \mathcal{X}(N)$ and for the normal vector field ξ of N, where $\breve{\nabla}$, ∇ and $\nabla^\perp$ are Riemannian, induced Riemannian and induced normal connections in M , N and the normal bundle $T^\perp M$ of M respectively and σ and A_ξ are denoted as the second fundamental form and shape operator and are related as,

$$\breve{g}(\sigma(X_1, X_2), \xi) = \breve{g}(A_\xi X_1, X_2).$$

Now, for any $X_1 \in \mathcal{X}(N)$ and for the normal vector field ξ of N, we have:

$$\breve{J}X_1 = PX_1 + FX_1,$$

$$\breve{J}\xi = t\xi + f\xi,$$

where $PU(t\xi)$ and $FU(f\xi)$ are tangential to N and normal to N, respectively. Similarly, the equations of Gauss is given by:

$$\begin{aligned}\breve{R}(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) + \breve{g}(\sigma(X_1, X_4), \sigma(X_2, X_3)) \\ &\quad -\breve{g}(\sigma(X_1, X_3), \sigma(X_2, X_4)), \end{aligned} \tag{2.2}$$

for all X_1, X_2, X_3, X_4 tangent to N, where $\breve{R}$ and R are curvature tensors of M and N respectively. The Mean curvature H at $x \in N$ is given by,

$$H = \frac{1}{m}\text{trace}(\sigma), \tag{2.3}$$

Also, set

$$\sigma_{ij}^{\gamma} = \breve{g}(\sigma(e_i, e_j), e_\gamma), \quad i, j \in \{1, \ldots, m\}, \quad \gamma \in \{m+1, \ldots, n\},$$

and

$$\|\sigma\|^2 = \sum_{i,j=1}^{m} \breve{g}(\sigma(e_i, e_j), \sigma(e_i, e_j)), \tag{2.4}$$

and the squared norm of second fundamental form σ denoted by $\mathcal{C}$ is defined as

$$\mathcal{C} = \frac{1}{m} \sum_{\gamma=m+1}^{n} \sum_{i,j=1}^{m} \left(\sigma_{ij}^{\gamma}\right)^2, \tag{2.5}$$

known as Casorati curvature of the submanifold ([28]).

Let L be a subspace of T_xM of dimension $k \geq 2$, and $\{e_1, \ldots, e_k\}$ an orthonormal basis of L. Define $\tau(L)$ as the scalar curvature of the k-plane section L by

$$\tau(L) = \sum_{i<j} K(e_i \wedge e_j), \quad i, j = 1, \ldots, k.$$

Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_xM, we denote by $\tau_{1\ldots k}$ the scalar curvature of the k-plane section spanned by $e_1, \ldots, e_k$. The scalar curvature $\tau(x)$ of M at x is the scalar curvature of the tangent space of M at p. If L is a 2-plane section, then $\tau(L)$ reduces to the sectional curvature $K(L)$ of the plane section L. If $K(\pi)$ is the sectional curvature of M for a plane section π in T_xM, where $x \in M$, then the scalar curvature $\tau(x)$ and normalized scalar curvature $\rho(x)$ at x are defined respectively by

$$\tau(x) = \sum_{i<j} K_{ij}, \qquad \rho(x) = \frac{2\tau}{m(m-1)},$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $x \in M$ and the Casorati curvature $\mathcal{C}$ of the subspace L is as follows [28]

$$\mathcal{C}(L) = \frac{1}{k} \sum_{\gamma=n+1}^{m} \sum_{i,j=1}^{n} \left(h_{ij}^{\gamma}\right)^2.$$

A point $x \in N$ is said to be an *invariantly quasi-umbilical point* if there exist $n-m$ mutually orthogonal unit normal vectors $\xi_{m+1}, \ldots, \xi_n$ such that the shape operators with respect to all directions ξ_γ have an eigenvalue of multiplicity $m-1$ and that for each ξ_γ the distinguished eigen direction is the same. The submanifold is said to be an invariantly quasi-umbilical submanifold if each of its points is an invariantly quasi-umbilical point [7].

The normalized δ-Casorati curvature $\delta_c(m-1)$ and $\widehat{\delta}_c(m-1)$ are defined as[28]

$$[\delta_c(m-1)]_x = \frac{1}{2}\mathcal{C}_x + \frac{m+1}{2m} inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_xN\}, \tag{2.6}$$

and

$$[\widehat{\delta}_c(m-1)]_x = 2\mathcal{C}_x + \frac{2m-1}{2m} sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_xN\}. \tag{2.7}$$

For a positive real number $\nu \neq m(m-1)$, put

$$\beta(\nu) = \frac{1}{m\nu}(m-1)(m+\nu)(m^2-m-\nu), \tag{2.8}$$

then the generalized normalized δ-Casorati curvatures $\delta_c(\nu; m-1)$ and $\widehat{\delta}_c(\nu; m-1)$ are given as

$$[\delta_c(\nu; m-1)]_x = \nu\mathcal{C}_p + \beta(\nu) inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_xN\},$$

if $0 < \nu < m^2 - m$, and

$$[\widehat{\delta}_c(\nu; m-1)]_x = \nu\mathcal{C}_x + \beta(\nu) sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_xN\},$$

if $\nu > m(m-1)$.

Definition 1. ([1]) Let N be isometrically immersed submanifold in Käehler manifold M. Then N is called quasi bi-slant submanifold if there exists distributions Δ, Δ_1 and Δ_2 such that:
(1) The tangent bundle TN can be decomposed orthogonally as:

$$TN = \Delta \oplus \Delta_1 \oplus \Delta_2,$$

(2) The distribution Δ is invariant under the complex structure $\breve{J}$, i.e.,

$$\breve{J}(\Delta) = \Delta,$$

(3) The transformed distribution $\breve{J}(\Delta_1)$ is orthogonal to the distribution Δ_2, i.e.,

$$\breve{J}(\Delta_1) \perp \Delta_2,$$

(4) For any non-zero vector field $X_1 \in (\Delta_1)_x$, where x is a point in N, the angle θ_1 between $\breve{J}X_1$ and $(\Delta_1)_x$ remains constant and does not depend on the specific choice of x and X_1.
(5) For any non-zero vector field $Z_1 \in (\Delta_2)_y$, where y is a point in N, the angle θ_2 between $\breve{J}Z_1$ and $(\Delta)_y$ remains constant and does not depend on the specific choice of y and Z_1.

Remark 1. Based on the dimensions of the distributions and the values of the slant angles θ_1 and θ_2, different cases can be identified:
(i) If $\dim(\Delta) \neq 0$, $\dim(\Delta_1) = 0$, and $\dim(\Delta_2) = 0$, the submanifold N is classified as an invariant submanifold.
(ii) If $\dim(\Delta) \neq 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) = 0$, the submanifold N is classified as a proper semi-slant submanifold.
(iii) If $\dim(\Delta) = 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) = 0$, the submanifold N is classified as a slant submanifold with a slant angle of θ_1.
(iv) If $\dim(\Delta) = 0$, $\dim(\Delta_1) = 0$, and $\dim(\Delta_2) \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, the submanifold N is classified as a slant submanifold with a slant angle of θ_2.
(v) If $\dim(\Delta) = 0$, $\dim(\Delta_1) \neq 0$, $\theta_1 = \frac{\pi}{2}$, and $\dim(\Delta_2) = 0$, the submanifold N is classified as an anti-invariant submanifold.
(vi) If $\dim(\Delta) \neq 0$, $\dim(\Delta_1) \neq 0$, $\theta_1 = \frac{\pi}{2}$, and $\dim(\Delta_2) = 0$, the submanifold N is classified as semi-invariant submanifold.
(vii) If $\dim(\Delta) = 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) \neq 0$, $\theta_2 = \frac{\pi}{2}$, the submanifold N is classified as a hemi-slant submanifold.
(viii) If $\dim(\Delta) = 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, the submanifold N is classified as proper bi-slant submanifold.
(ix) If $\dim(\Delta) \neq 0$ and $0 < \theta_1 = \theta_2 < \frac{\pi}{2}$, then submanifold N is classified as a proper semi-slant submanifold.
(x) If $\dim(\Delta) \neq 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, the submanifold N is classified as proper quasi bi-slant submanifold.

Thus quasi bi-slant submanifolds are generalisation of invariant, anti-invariant, slant, semi-slant, hemi-slant and bi-slant submanifolds.

3 Main Results

In this section we obtain inequalities for generalized normalized δ-Casorati curvature of quasi bi-slant submanifolds of generalized complex space forms.

Theorem 3.1. *Let N be an m-dimensional quasi bi-slant submanifold of a n-dimensional generalized complex space form $M(c, \breve{\alpha})$, then*
(i) The generalized normalized δ-Casorati curvature $\delta_c(\nu; m-1)$ satisfies

$$\rho \leq \frac{\delta_c(\nu; m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}(d_1 + d_2 cos^2\theta_1 + d_3 cos^2\theta_2), \tag{3.1}$$

for any real number ν such that $0 < \nu < m(m-1)$.
(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(\nu; m-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta}_c(\nu; m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}(d_1 + d_2 cos^2\theta_1 + d_3 cos^2\theta_2), \tag{3.2}$$

for any real number $\nu > m(m-1)$.

Moreover , the equality holds in (3.1) and (3.2) iff N is an invariantly quasi-umbilical submanifold with trivial normal connection in $M(c, \breve{\alpha})$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_m\}$ and normal orthonormal frame $\{e_{m+1}, \ldots, e_n\}$, the shape oper-

ator $A_r \equiv A_{e_r}$, $r \in \{m+1, \ldots, n\}$, *take the following form*

$$A_{m+1} = \begin{bmatrix} \beta & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{m(m-1)}{\nu}\beta \end{bmatrix}, \qquad A_{m+2} = A_{m+3} = \cdots = A_n = 0. \tag{3.3}$$

Proof. Let $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_n\}$ be the orthonormal basis of T_xN and $T_x^{\perp}N$ respectively at any point $x \in N$. Putting $X_1 = X_4 = e_i$, $X_2 = X_3 = e_j$, $i \neq j$ from equation (2.1), we have

$$4\breve{R}(e_i, e_j, e_i, e_j) = (c + 3\breve{\alpha})\{m^2 - m\} + 3(c - \breve{\alpha})\{\sum_{i,j=1}^{m} \breve{g}^2(\breve{J}e_i, e_j). \tag{3.4}$$

Let $dim(N) = m = 2d_1 + 2d_2 + 2d_3$, we consider an adopted quasi bi-slant orthonormal frames

$$\begin{aligned} &e_1,\ e_2 = \breve{J}e_1, \ldots, e_{2d_1-1},\ e_{2d_1} = \breve{J}e_{2d_1-1}, \\ &e_{2d_1+1},\ e_{2d_1+2} = \sec^2\theta_1 \breve{J}e_{2d_1+1}, \ldots, \\ &e_{2d_1+2d_2-1},\ e_{2d_1+2d_2} = \sec^2\theta_1 \breve{J}e_{2d_1+2d_2-1}, \\ &e_{2d_1+2d_2+1}, e_{2d_1+2d_2+2} = \sec^2\theta_2 \breve{J}e_{2d_1+2d_2+1}, \ldots, \\ &e_{2d_1+2d_2+2d_3-1},\ e_{2d_1+2d_2+2d_3} = \sec^2\theta_2 \breve{J}e_{2d_1+2d_2+2d_3-1}. \end{aligned}$$

Clearly, we have

$$\breve{g}(\breve{J}e_j, e_{j+1}) = \begin{cases} 1 & \text{for } j \in \{1, ..., 2d_1 - 1\} \\ \cos^2\theta_1 & \text{for } j \in \{2d_1 + 1, \ldots, 2d_1 + 2d_2 - 1\} \\ \cos^2\theta_2 & \text{for } j \in \{2d_1 + 2d_2 + 1, \ldots, 2d_1 + 2d_2 + 2d_3 - 1\}, \end{cases} \tag{3.5}$$

and hence

$$\sum_{i,j=1}^{m} \breve{g}^2(\breve{J}e_i, e_j) = 2(d_1 + d_2\cos^2\theta_1 + d_3\cos^2\theta_2). \tag{3.6}$$

Substituting foregoing equation in (3.4), we get

$$\begin{aligned} \breve{R}(e_i, e_j, e_i, e_j) &= \frac{(c + 3\breve{\alpha})}{4}\{m(m-1)\} \\ &\quad + \frac{6(c - \breve{\alpha})}{4}(d_1 + d_2 cos^2\theta_1 + d_3 cos^2\theta_2). \end{aligned} \tag{3.7}$$

On the other hand from (2.2), (2.4) and (2.3), we get

$$\breve{R}(e_i, e_j, e_i, e_j) = 2\tau + \|\sigma\|^2 - m^2\|H\|^2. \tag{3.8}$$

From (3.7) and (3.8), we get

$$2\tau = m^2\|H\|^2 - \|\sigma\|^2 + \frac{c + 3\breve{\alpha}}{4}\{m(m-1)\}$$

$$+6(\frac{c-\breve{\alpha}}{4})\{(d_1+d_2cos^2\theta_1+d_3cos^2\theta_2)\}. \tag{3.9}$$

Consider a quadratic polynomial $\mathcal{P}$ in the components of the second fundamental form

$$\begin{aligned}\mathcal{P} &= \nu\mathcal{C}+\beta(\nu)\mathcal{C}(L)-2\tau+m(m-1)\frac{(c+3\breve{\alpha})}{4}\\ &+6(\frac{c-\breve{\alpha}}{4})\{(d_1+d_2cos^2\theta_1+d_3cos^2\theta_2)\},\end{aligned} \tag{3.10}$$

where L is the hyperplane of T_pN. Without loss of generality, we suppose that L is spanned by $e_1,\ldots,e_{m-1}$, it follows from (3.10) that

$$\mathcal{P}=\frac{m+\nu}{m}\sum_{\gamma=m+1}^{n}\sum_{i,j=1}^{m}(\sigma_{ij}^{\gamma})^2+\frac{\beta(\nu)}{m-1}\sum_{\gamma=m+1}^{n}\sum_{i,j=1}^{m-1}(\sigma_{ij}^{\gamma})^2-\sum_{\gamma=m+1}^{n}\left(\sum_{i=1}^{m}\sigma_{ii}^{\gamma}\right)^2,$$

which can be easily written as

$$\begin{aligned}\mathcal{P} &= \sum_{\gamma=m+1}^{n}\sum_{i=1}^{m-1}\left[\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right)(\sigma_{ii}^{\gamma})^2+\frac{2(m+\nu)}{m}(\sigma_{im}^{\gamma})^2\right]\\ &+\sum_{m+1}^{m}\left[2\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right)\sum_{(i<j)=1}^{m}(\sigma_{ij}^{\gamma})^2-2\sum_{(i<j)=1}^{m}\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma}+\frac{\nu}{m}(\sigma_{mm}^{\gamma})^2\right].\end{aligned} \tag{3.11}$$

From (3.11), we can see that the critical points

$$\sigma^c=(\sigma_{11}^{m+1},\sigma_{12}^{m+1},\ldots,\sigma_{mm}^{m+1},\ldots,\sigma_{11}^{n},\ldots,\sigma_{nn}^{m}),$$

of $\mathcal{P}$ are the solutions of the following system of homogenous equations:

$$\begin{cases}\frac{\partial\mathcal{P}}{\partial\sigma_{ii}^{\gamma}}=2\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right)(\sigma_{ii}^{\gamma})-2\sum_{t=1}^{n}\sigma_{tt}^{\gamma}=0\\ \frac{\partial\mathcal{P}}{\partial\sigma_{mm}^{\gamma}}=\frac{2\nu}{m}\sigma_{mm}^{\gamma}-2\sum_{t=1}^{m-1}\sigma_{tt}^{\gamma}=0\\ \frac{\partial\mathcal{P}}{\partial\sigma_{ij}^{\gamma}}=4\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right)(\sigma_{ij}^{\gamma})=0\\ \frac{\partial\mathcal{P}}{\partial\sigma_{im}^{\gamma}}=4(\frac{m+\nu}{m})(\sigma_{im}^{\gamma})=0,\end{cases} \tag{3.12}$$

where $i,j=\{1,2,\ldots,m-1\}, i\neq j$, and $\gamma\in\{m+1,m+2,\ldots,n\}$.

Hence, every solution σ^c has $\sigma_{ij}^{\gamma}=0$ for $i\neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of $\mathcal{P}$ is of the following form

$$\mathcal{H}(\mathcal{P})=\begin{pmatrix}H_1 & O & O\\ O & H_2 & O\\ O & O & H_3\end{pmatrix},$$

where

$$H_1=\begin{pmatrix}2(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1})-2 & -2 & \ldots & -2 & -2\\ -2 & 2(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1})-2 & \ldots & -2 & -2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -2 & -2 & \ldots & 2(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1})-2 & -2\\ -2 & -2 & \ldots & -2 & \frac{2\nu}{m}\end{pmatrix},$$

H_2 and H_3 are the diagonal matrices and O is the null matrix of the respective dimensions. H_2 and H_3 are respectively given as

$$H_2 = diag\left(4\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right), 4\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right), \dots, 4\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right)\right),$$

and

$$H_3 = diag\left(\frac{4(m+\nu)}{m}, \frac{4(m+\nu)}{m}, \dots, \frac{4(m+\nu)}{m}\right).$$

As H_2 and H_3 are diagonal matrices, so their eigen values are well known. In order to find eigen values of H_1, we need to obatin the roots of characteristic equation

$$det|H_1 - \lambda I_m| = 0.$$

From foregoing equation and on simple computations, we get

$$\left(\frac{2(m+\nu)(m-1)+m\beta(\nu)}{m}\right)^{(m-2)} \begin{vmatrix} 2\left(\frac{\nu-m^2+2m}{m}+\frac{\beta(\nu)}{m-1}\right)-\lambda & -2(m-1) \\ -2 & \frac{2\nu}{m}-\lambda \end{vmatrix} = 0.$$

On further solvation the eigen values of matrix H_1 are given by

$$\lambda_{11} = 0, \quad \lambda_{22} = 2\left(\frac{2\nu-m^2+2m}{m}+\frac{\beta(\nu)}{m-1}\right),$$
$$\lambda_{33} = \cdots = \lambda_{mm} = 2\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right).$$

Consequetly we deduce that eigen values of $H(x)$ are

$$\begin{aligned} \lambda_{11} &= 0, \quad \lambda_{22} = 2\left(\frac{2\nu-m^2+2m}{m}+\frac{\beta(\nu)}{m-1}\right), \\ \lambda_{33} &= \cdots = \lambda_{mm} = 2\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right), \\ \lambda_{ij} &= 4\left(\frac{m+\nu}{m}+\frac{\beta(\nu)}{m-1}\right), \lambda_{im} = \frac{4(m+\nu)}{m}, \forall i,j \in \{1,2,\dots,m-1\}, i \neq j. \end{aligned}$$

Thus, $\mathcal{P}$ is parabolic and reaches at minimum $\mathcal{P}(\sigma^c) = 0$ for the solution σ^c of the system (3.12). Hence $\mathcal{P} \geq 0$ and hence

$$\begin{aligned} 2\tau &\leq \nu\mathcal{C} + \beta(\nu)\mathcal{C}(L) + m(m-1)\frac{(c+3\breve{\alpha})}{4} \\ &\quad + \frac{6(c-\breve{\alpha})}{4}[d_1 + d_2cos^2\theta_1 + d_3cos^2\theta_2]. \end{aligned}$$

From foregoing equation, we obtain

$$\begin{aligned} \rho &\leq \frac{\nu}{m(m-1)}\mathcal{C} + \frac{\beta(\nu)}{m(m-1)}\mathcal{C}(L) + \frac{(c+3\breve{\alpha})}{4} \\ &\quad + \frac{3(c-\breve{\alpha})}{2(m(m-1))}[d_1 + d_2cos^2\theta_1 + d_3cos^2\theta_2], \end{aligned}$$

for every tangent hyperplane L of N. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds if and only if

$$\sigma_{ij}^{\gamma} = 0, \ \forall\, i,j \in \{1,\ldots,m\},\ i \neq j \text{ and } \gamma \in \{m+1,\ldots,n\}, \tag{3.13}$$

and

$$\sigma_{mm}^{\gamma} = \frac{m(m-1)}{\nu}\sigma_{11}^{\gamma} = \cdots = \frac{m(m-1)}{\nu}\sigma_{m-1m-1}^{\gamma}, \forall\gamma \in \{m+1,\ldots,n\}. \tag{3.14}$$

From (3.13) and (3.14), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in N, such that the shape operator with respect to the orthonormal tangent and orthonormal normal frames takes the form (3.3). Inequality 3.2 can be proven in the same way. □

Theorem 3.2. *Let N be submanifold of generalized complex space forms then for generalised normalised δ-Casorati curvature, we have the following table where in each of the above inequalities*

M	N	Inequality
$M(c,\breve{\alpha})$	Bi-slant	(i) $\rho \leq \frac{\delta_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_2 cos^2\theta_1 + d_3 cos^2\theta_2]$. (ii) $\rho \leq \frac{\widehat{\delta}_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_2 cos^2\theta_1 + d_3 cos^2\theta_2]$.
$M(c,\breve{\alpha})$	Semi-slant	(i) $\rho \leq \frac{\delta_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_1 + d_2 cos^2\theta_1]$. (ii) $\rho \leq \frac{\widehat{\delta}_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_1 + d_2 cos^2\theta_1]$.

$0 < \nu < n(n-1)$ and $\nu > m(m-1)$ for (i) and (ii) respectively. The equality case holds in each of the above inequalities iff N is an invariantly quasi-umbilical submanifold with trivial normal connection in $M(c,\breve{\alpha})$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_m\}$ and normal orthonormal frame $\{e_{m+1},\ldots,e_n\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{m+1,\ldots,n\}$, take the form (3.3).

M	N	Inequality
$M(c,\breve{\alpha})$	Hemi-slant	$(i)\ \rho \leq \frac{\delta_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_2 cos^2\theta_1].$ $(ii)\ \rho \leq \frac{\widehat{\delta}_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_2 cos^2\theta_1].$
$M(c,\breve{\alpha})$	θ_1-slant	$(i)\ \rho \leq \frac{\delta_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)} cos^2\theta_1.$ $(ii)\ \rho \leq \frac{\widehat{\delta}_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)} cos^2\theta_1.$
$M(c,\breve{\alpha})$	θ_2-slant	$(i)\ \rho \leq \frac{\delta_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)} cos^2\theta_2.$ $(ii)\ \rho \leq \frac{\widehat{\delta}_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)} cos^2\theta_2.$
$M(c,\breve{\alpha})$	*Invariant*	$(i)\ \rho \leq \frac{\delta_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)}.$ $(ii)\ \rho \leq \frac{\widehat{\delta}_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)}.$
$M(c,\breve{\alpha})$	Anti-Invariant	$(i)\ \rho \leq \frac{\delta_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4}.$ $(ii)\ \rho \leq \frac{\widehat{\delta}_c(\nu;m-1)}{m(m-1)} + \frac{c+3\breve{\alpha}}{4}.$

Theorem 3.3. *Let N be an m-dimensional quasi bi-slant submanifold of a n-dimensional generalized complex space form $M(c,\breve{\alpha})$, then*
(i) The normalized δ-Casorati curvature $\delta_c(\nu;m-1)$ satisfies

$$\rho \leq \delta_c(\nu;m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}(d_1 + d_2 cos^2\theta_1 + d_3 cos^2\theta_2)$$

for any real number ν such that $0 < \nu < m(m-1)$.

Moreover , the equality holds iff N is an invariantly quasi-umbilical submanifold with trivial normal connection in $M(c,\breve{\alpha})$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_m\}$ and normal orthonormal frame $\{e_{m+1},\ldots,e_n\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{m+1,\ldots,n\}$, take the following form

$$A_{m+1} = \begin{bmatrix} \beta & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2\beta \end{bmatrix}, \qquad A_{m+2} = A_{m+3} = \cdots = A_n = 0. \tag{3.15}$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(r; m-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}(d_1 + d_2cos^2\theta_1 + d_3cos^2\theta_2)$$

for any real number $\nu > m(m-1)$. Moreover , the equality holds iff N is an invariantly quasi-umbilical submanifold with trivial normal connection in $M(c, \breve{\alpha})$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_m\}$ and normal orthonormal frame $\{e_{m+1}, \ldots, e_n\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{m+1, \ldots, n\}$, take the following form

$$A_{m+1} = \begin{bmatrix} 2\beta & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2\beta & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\beta & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta \end{bmatrix}, \qquad A_{m+2} = A_{m+3} = \cdots = A_n = 0. \tag{3.16}$$

Corollary 3.4. *Let N be a submanifold of generalized complex space forms then for normalised δ-Casorati curvature, we have the following table*

M	N	Inequality
$M(c, \breve{\alpha})$	Bi-slant	(i) $\rho \leq \delta_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_2cos^2\theta_1 + d_3cos^2\theta_2]$ (ii) $\rho \leq \widehat{\delta}_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_2cos^2\theta_1 + d_3cos^2\theta_2]$

M	N	Inequality
$M(c,\breve{\alpha})$	Semi-slant	$(i)\ \rho \leq \delta_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_1 + d_2 cos^2\theta_1]$. $(ii)\ \rho \leq \widehat{\delta}_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_1 + d_2 cos^2\theta_1]$.
$M(c,\breve{\alpha})$	Hemi-slant	$(i)\ \rho \leq \delta_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_2 cos^2\theta_1]$. $(ii)\ \rho \leq \widehat{\delta}_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{2m(m-1)}[d_2 cos^2\theta_1]$.
$M(c,\breve{\alpha})$	θ_1-slant	$(i)\ \rho \leq \delta_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)} cos^2\theta_1$. $(ii)\ \rho \leq \widehat{\delta}_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)} cos^2\theta_1$.
$M(c,\breve{\alpha})$	θ_2-slant	$(i)\ \rho \leq \delta_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)} cos^2\theta_2$. $(ii)\ \rho \leq \widehat{\delta}_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)} cos^2\theta_2$.
$M(c,\breve{\alpha})$	*Invariant*	$(i)\ \rho \leq \delta_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)}$. $(ii)\ \rho \leq \widehat{\delta}_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4} + \frac{3(c-\breve{\alpha})}{4(m-1)}$.
$M(c,\breve{\alpha})$	Anti-Invariant	$(i)\ \rho \leq \delta_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4}$. $(ii)\ \rho \leq \widehat{\delta}_c(\nu; m-1) + \frac{c+3\breve{\alpha}}{4}$.

Moreover, the equality for δ_c holds iff N is an invariantly quasi-umbilical submanifold with trivial normal connection in $M(c,\breve{\alpha})$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_m\}$ and normal orthonormal frame $\{e_{m+1},\ldots,e_n\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{m+1,\ldots,n\}$, take the form (3.15) and the equality for $\hat{\delta}$ holds iff N is an invariantly quasi-umbilical submanifold with trivial normal connection in $M(c,\breve{\alpha})$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_m\}$ and normal orthonormal frame $\{e_{m+1},\ldots,e_n\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{m+1,\ldots,n\}$, take the form (3.16).

4 Conclusion

This article established Casorati curvature inequalities for quasi bi-slant submanifolds within generalized complex space forms. These results offer significant insight into the intrinsic and extrinsic curvature properties of such submanifolds and elucidate the geometric constraints they satisfy in complex ambient spaces. To facilitate comparison and interpretation, we have summarized several particular cases in tabular form, thereby enhancing clarity and comprehension of the derived inequalities. Moreover, similar Casorati curvature inequalities can be formulated for quasi bi-slant submanifolds of contact manifolds by using the orthonormal frame structure introduced in this work.

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Conflict of interests
The authors declare that they have no conflict of interest, regarding the publication of this paper.

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