TAMKANG JOURNAL OF MATHEMATICS
Volume 57, Number 1, 1-13, 2026
DOI:10.5556/j.tkjm.57.2026.5502

Weak solutions for the fractional Kirchhoff-type problem via Young measures

Ihya Talibi, Farah Balaadich, Brahim El Boukari and Jalila El Ghordaf

Abstract. The aim of this paper is to investigate the existence of weak solutions to the following Kirchhoff-type problem:

$$\begin{cases} M\left([u]_{sp}^{p}\right)(-\Delta)_p^s(u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $0 < s < 1 < p < \infty$, $[u]_{sp}$ is Gagliardo semi-norm, M is a continuous function with value in $\mathbb{R}^+$, f is a given function and $(-\Delta)_p^s$ is the fractional p-Laplacian operator. Under appropriate assumptions on the main functions, we obtain the existence results by applying the Galerkin method combined with the theory of Young measures.

Keywords. Kirchhoff equation, weak solution, fractional p-Laplacian system, Young measure, Galerkin method

1 Introduction

In this article, we are concerned with the existence of solutions to the following Kirchhoff-type problem:

$$\begin{cases} M\left([u]_{sp}^{p}\right)(-\Delta)_p^s(u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^m$, $m \in \{0, 1, 2...\}$ is a vector-valued function, $M : \mathbb{R}^+ \to \mathbb{R}^+$, $[u]_{sp}$ is Gagliardo semi-norm and f satisfies specific conditions that will be described later. Here $(-\Delta)_p^s u$ is the fractional p-Laplacian operator and will be introduced subsequently.

The Kirchhoff-type equation dates back to Kirchhoff's 1883 work [27], where he extended the classical D'Alembert wave equation by incorporating the effect of variations in string length during vibrations. Kirchhoff's model involved a nonlocal term dependent on the average kinetic energy. The mathematical literature on Kirchhoff-type problems has expanded considerably, especially with regard to the existence of solutions. For instance, the case of Kirchhoff problems

Received date: October 11, 2024; Published online: October 28, 2025.
2010 *Mathematics Subject Classification*. 35J05, 35J25, 35J60.
Corresponding author: Ihya Talibi.

involving the classical Laplace operator has been studied by [2, 21], while studies involving the p-Laplacian can be found in [4, 5]. The fractional Kirchhoff-type equation involving nonlocal operators was introduced in [23]. See also the application of these operators in various fields [3, 13, 14, 15, 16, 19, 25, 26, 28, 38].

Many authors have recently studied the existence results of the problem (1.1) by using different methods, for example, in [39], the authors showed the existence of nontrivial weak solutions by using the variational methods and the existence of two nontrivial weak solutions by applying the Mountain Pass Theorem. Based on the symmetric Mountain Pass Theorem and the Krasnoselskii genus theory, Xiang et al. in [30] proved the existence of nontrivial solutions. We also refer to [18, 34, 40, 41, 42] for more works solved by different methods. When $M \equiv 1$, problem (1.1) becomes the fractional p-Laplacian equation

$$\begin{cases} (-\Delta)_p^s(u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.2}$$

which is discussed by several researchers. For example, in [12], the authors proved the existence of weak solutions using the tool of Young measures. Qiu and Xiang in [32], used Leray-Schauder's nonlinear alternative to show the existence of nonnegative solutions. In [31], the authors proved the existence and multiplicity results via cohomological local splitting and critical groups. See also [17, 25, 33].

In the theory of nonlinear partial differential equations and the calculus of variations, Young measures have recently become an increasingly important tool to discuss the existence of solutions. For more details on this theory, we refer to see [24]. To the best of our knowledge, this is the first paper that treats a Kirchhoff problem involving fractional p-Laplacian by such a theory. The works studied in the literature are in the classical case; for example, Azroul and Balaadich in [7], proved the existence of solutions for a class of Kirchhoff-type problems

$$\begin{cases} -M\left(\int_\Omega A(x,\nabla u)dx\right) \operatorname{div} a(x,\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

where $a(.,\xi) = \nabla_\xi A(.,\xi)$, by using the theory of Young measures combined with the Galerkin method. We suggest to the readers to consult [8, 9, 10, 11, 12, 35, 36, 37] where this theory was applied to some quasi-linear elliptic systems.

In the present paper, motivated by the results in [7] and [12], we consider the problem (1.1) to study the existence results using the Galerkin method to build the approximation solutions and the theory of Young measures to pass to the limit.

There are three sections in this article. In section 2, we give some background information on fractional Sobolev spaces and some fundamental tools on Young measures. In section 3, we define weak solutions to problem (1.1), and we finish this section by proving the main results.

2 Preliminaries and notations

We recall some notations and definitions in this part, as well as some of the results that will be applied to this work.

Let $0 < s < 1 < p < \infty$ be real numbers, we define p_s^* the fractional critical exponent giving by:

$$p_s^* = \begin{cases} \infty & \text{if } ps \geq n, \\ np/(n-ps) & \text{if } ps < n. \end{cases}$$

The fractional p-Laplacian operator $(-\Delta)_p^s u$ is defined as follows:

$$(-\Delta)_p^s u(x) = P.V \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}} dy, \quad x \in \mathbb{R}^n$$

where $x \in \mathbb{R}^n$ and $P.V$, which stands for "in the principal value sense," is a frequently used abbreviation.

Let $\Omega \subset \mathbb{R}^n$, we denote $\Gamma = \mathbb{R}^{2n} \backslash \mathcal{J}$, where $\mathcal{J} = (\mathbb{R}^n \backslash \Omega) \times (\mathbb{R}^n \backslash \Omega) \subset \mathbb{R}^{2n}$. Based on W, which is a space of Lebesgue measurable and linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, and any function u in W its restriction belongs to $L^p(\Omega; \mathbb{R}^m)$ and

$$\iint_\Gamma \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \left(\iint_\Gamma \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}}.$$

The next closed linear space will be the space of work

$$W_0 = \{u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^n \backslash \Omega\},$$

equipped with the norm

$$\|u\|_{W_0} := [u]_{sp} = \left(\iint_\Gamma \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}}.$$

The dual space of $(W_0, \|\cdot\|_{W_0})$ is indecated by $\left(W_0^*, \|\cdot\|_{W_0^*}\right)$.

Lemma 2.1. *[39] $(W_0, \|\cdot\|_{W_0})$ is a uniformly convex Banach space.*

Remark 1. By the separability of W in [6] and Theorem 1.21 in [1], W_0 is separable.

Lemma 2.2. *[22] $\mathcal{C}_0^\infty(\Omega; \mathbb{R}^m)$ is a space of infinitely differentiable functions with compact support on Ω which is dense in W_0.*

Lemma 2.3. *[29] The following embedding $W_0 \hookrightarrow L^\lambda(\Omega; \mathbb{R}^m)$ is compact for all $1 \leq \lambda < p_s^*$, and continuous for all $1 \leq \lambda \leq p_s^*$.*

In the following, $\mathcal{C}_0(\mathbb{R}^m)$ stands for the space of continuous functions on $\mathbb{R}^m$ with compact support with regard to the $\|.\|_\infty$-norm. The space of signed Radon measures with finite mass is noted $\mathcal{M}(\mathbb{R}^m)$. The corresponding duality is given by

$$\langle \mu, \rho \rangle = \int_{\mathbb{R}^m} \rho(\lambda) d\mu(\lambda).$$

Definition 1. [12] Let a bounded sequence noted by $\{\tau_j\}_{j\geq 1}$ in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence $\{\tau_k\}\subset\{\tau_j\}$ and a Borel probability measure μ_x on $\mathbb{R}^m$ for almost every $x\in\Omega$, such that for a.e. $\rho\in\mathcal{C}(\mathbb{R}^m)$ we have $\rho(\tau_k)\rightharpoonup^* \bar{\rho}$ weakly in $L^\infty(\Omega)$, where

$$\bar{\rho}(x)=\langle\mu_x,\rho\rangle=\int_{\mathbb{R}^m}\rho(\lambda)d\mu_x(\lambda)$$

for a.e. $x\in\Omega$.

Lemma 2.4. *[20] Let $\Omega\subset\mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and τ_j from Ω to $\mathbb{R}^m$, with $j\in\mathbb{N}$, be a sequence of Lebesgue measurable functions. Then there exist a subsequence τ_k and a family $\{\mu_x\}_{x\in\Omega}$ of non-negative Radon measures on $\mathbb{R}^m$, such that*

(i) $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^m)}:=\int_{\mathbb{R}^m}d\mu_x(\lambda)\leq 1$ *for almost $x\in\Omega$.*

(ii) $\rho(\tau_k)\rightharpoonup^*\bar{\rho}$ *weakly in $L^\infty(\Omega)$ for all $\mathcal{C}_0(\mathbb{R}^m)$, where $\bar{\rho}=\langle\mu_x,\rho\rangle$.*

(iii) If for all $M>0$

$$\lim_{N\to\infty}\sup_{k\in\mathbb{N}}|\{x\in\Omega\cap B_M(0):|\tau_k(x)|\geq N\}|=0, \tag{2.1}$$

then $\|\mu_x\|=1$ for a.e. $x\in\Omega$, and for any measurable $\Omega'\subset\Omega$ we have $\rho(\tau_k)\rightharpoonup\bar{\rho}=\langle\mu_x,\rho\rangle$ weakly in $L^1(\Omega')$ for continuous function ρ provided the sequence $\rho(\tau_k)$ is weakly precompact in $L^1(\Omega')$.

In the sequel, let $ps<n$, $p'=\frac{p}{p-1}$, and C_i, $i=1,2,3...$ are positive constants, which vary from line to line and denote $U_k(x,y)=u_k(x)-u_k(y)$ and $U(x,y)=u(x)-u(y)$.

3 Existence of weak solutions

In this section, we will define weak solutions for the problem (1.1) and prove the main results. We start by citing the following statements:

(**F1**): The function f is a Carathéodory function from $\Omega\times\mathbb{R}^m$ to $\mathbb{R}$ satisfying:

$$\text{There exists } C_1>0,\quad 1<p<\frac{n}{s}\quad\text{such that}\quad |f(x,\zeta)|\leq\mathcal{E}(x)+C_1|\zeta|^{p-1},$$

for all $\zeta\in\mathbb{R}^m$ and almost every $x\in\Omega$, where $\mathcal{E}\in L^{p'}(\Omega)$, with $\mathcal{E}\geq 0$ a.e. in Ω.

(**F2**): M is a continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and satisfies

$$m_0 s^{\beta-1}\leq M(s)\leq m_1 s^{\beta-1},$$

for all $s>0$ and m_0,m_1 are real numbers such that $0<m_0\leq m_1$ and $\beta\geqslant 1$.

Definition 2. For the datum f, $u\in W_0$ is said to be a weak solution to the problem (1.1) if

$$M\left(||u||^p_{W_0}\right)\iint_\Gamma\frac{|U(x,y)|^{p-2}U(x,y)}{|x-y|^{n+ps}}(\phi(x)-\phi(y))dxdy=\int_\Omega f(x,u)\phi(x)dx$$

holds for any $\phi\in W_0$.

The aim of this paper is to prove Theorem 3.1 below, using the Galerkin method and fixed point theory to guarantee the approximate solution, and by passing through the limit, we obtain the existence of the weak solutions. We shall prove the following existence theorem:

Theorem 3.1. *Suppose that $(F_1)-(F_2)$ holds. Then, the problem (1.1) has a weak solution in the sense of Definition 2.*

Since W_0 is a uniformly convex Banach space and is separable, there is (V_k) a sequence of finite dimensional subspaces such that $V_1 \subset V_2 \subset \ldots \subset V_k \subset W_0$ with the property that $\cup_{k\geq 1} V_k$ is dense in W_0. We define the operator $\Psi(u) : W_0 \to W_0^*$ to build the approximating solutions:

$$\langle \Psi(u), \phi\rangle = M\left(||u||^p_{W_0}\right) \iint_\Gamma \frac{|U(x,y)|^{p-2}U(x,y)}{|x-y|^{n+ps}}(\phi(x)-\phi(y))dxdy - \int_\Omega f(x,u)\phi(x)dx, \tag{3.1}$$

with $\phi \in W_0$.

To prove the existence of the approximating solutions, we need the following assertions:

Assertion 1: The operator Ψ is well defined and bounded.

Indeed, by using the Hölder inequality and $(F1)-(F2)$, we obtain

$$\begin{aligned}
&|\langle \Psi(u), \phi\rangle| \\
&= \left| M\left(||u||^p_{W_0}\right) \iint_\Gamma \frac{|U(x,y)|^{p-2}U(x,y)}{|x-y|^{n+ps}}(\phi(x)-\phi(y))dxdy - \int_\Omega f(x,u)\phi(x)dx \right| \\
&\leq m_1\left(||u||^p_{W_0}\right)^{\beta-1} ||u||^{p-1}_{W_0}||\phi||_{W_0} + (||\mathcal{E}||_{p'} + C_1||u||^{p-1}_p)||\phi||_{W_0} \\
&\leq C_2||\phi||_{W_0}
\end{aligned}$$

for all $u, \phi \in W_0$.

Assertion 2: The restriction of Ψ to a finite linear subspace of W_0 is continuous.

Indeed, let $V_k = \text{span}\,\{e_1, \ldots, e_k\}$, where $\{e_i\}_{i=1}^k$ is a basis of V_k is a finite subspace of W_0. Let $\{u_k\} \subset W_0$ such that u_k converge to u in V_k. Then, for a subsequence still indexed by u_k, $u_k \to u$ a.e. and bounded in W_0. As a result for all $\phi \in W_0$ with $\|\phi\|_{W_0} \leq 1$,

$$\begin{aligned}
&\langle \Psi(u_k), \phi\rangle - \langle \Psi(u), \phi\rangle = M\left(||u_k||^p_{W_0}\right) \iint_\Gamma \frac{|U_k(x,y)|^{p-2}U_k(x,y)(\phi(x)-\phi(y))}{|x-y|^{n+ps}}dxdy \\
&- M\left(||u||^p_{W_0}\right) \iint_\Gamma \frac{|U(x,y)|^{p-2}U(x,y)(\phi(x)-\phi(y))}{|x-y|^{n+ps}}dxdy \\
&- \int_\Omega (f(x,u_k) - f(x,u))\phi(x)dx \\
&= \left[M\left(||u_k||^p_{W_0}\right) - M\left(||u||^p_{W_0}\right)\right] \iint_\Gamma \frac{|U_k(x,y)|^{p-2}U_k(x,y)}{|x-y|^{n+ps}}(\phi(x)-\phi(y))dxdy \\
&+ M\left(||u||^p_{W_0}\right) \iint_\Gamma \frac{|U_k(x,y)|^{p-2}U_k(x,y) - |U(x,y)|^{p-2}U(x,y)}{|x-y|^{\frac{n+ps}{p'}}} \frac{(\phi(x)-\phi(y))}{|x-y|^{\frac{n+ps}{p}}}dxdy
\end{aligned}$$

$$-\int_\Omega (f(x,u_k)-f(x,u))\phi(x)dx$$
$$=A_1+A_2-A_3,$$

where

$$A_1=\left[M\left(||u_k||^p_{W_0}\right)-M\left(||u||^p_{W_0}\right)\right]\iint_\Gamma \frac{|U_k(x,y)|^{p-2}U_k(x,y)}{|x-y|^{n+ps}}(\phi(x)-\phi(y))dxdy,$$
$$A_2=M\left(||u||^p_{W_0}\right)\iint_\Gamma \frac{|U_k(x,y)|^{p-2}U_k(x,y)-|U(x,y)|^{p-2}U(x,y)}{|x-y|^{\frac{n+ps}{p'}}}\frac{(\phi(x)-\phi(y))}{|x-y|^{\frac{n+ps}{p}}}dxdy$$
$$A_3=\int_\Omega \left(f(x,u_k)-f(x,u)\right)\phi(x)dx.$$

Due to the Hölder inequality, we have

$$A_1\le\left[M\left(||u_k||^p_{W_0}\right)-M\left(||u||^p_{W_0}\right)\right]||u_k||^{p-1}_{W_0},$$
$$A_2\le M\left(||u||^p_{W_0}\right)\left(\iint_\Gamma\left(\frac{|U_k(x,y)|^{p-2}U_k(x,y)-|U(x,y)|^{p-2}U(x,y)}{|x-y|^{\frac{p-1}{p}(n+ps)}}\right)^{\frac{p}{p-1}}dxdy\right)^{\frac{p-1}{p}}$$
$$A_3\le\left(\int_\Omega |f(x,u_k)-f(x,u)|^{p'}dx\right)^{\frac{1}{p'}}.$$

The continuity of M implies that

$$\lim_{k\to\infty}M\left(||u_k||^p_{W_0}\right)=M\left(||u||^p_{W_0}\right),$$

it follows that

$$A_1\le\left[M\left(||u_k||^p_{W_0}\right)-M\left(||u||^p_{W_0}\right)\right]||u_k||^{p-1}_{W_0}\longrightarrow 0\quad\text{as}\quad k\to\infty.$$

On the other hand, we have

$$A_2\le M\left(||u||^p_{W_0}\right)\|B_{a,k}-B_a\|_{L^{p'}(\Gamma,\mathbb{R}^m)},$$

where

$$B_{a,k}(x,y)=\frac{|U_k(x,y)|^{p-2}U_k(x,y)}{|x-y|^{\frac{p-1}{p}(n+ps)}}\in L^{p'}(\Gamma,\mathbb{R}^m),$$

and

$$B_a(x,y)=\frac{|U(x,y)|^{p-2}U(x,y)}{|x-y|^{\frac{p-1}{p}(n+ps)}}\in L^{p'}(\Gamma,\mathbb{R}^m).$$

Note that

$$L_{a,k}(x,y)=\frac{U_k(x,y)}{|x-y|^{\frac{(n+ps)}{p}}}\in L^p(\Gamma,\mathbb{R}^m),$$

and

$$L_a(x,y)=\frac{U(x,y)}{|x-y|^{\frac{(n+ps)}{p}}}\in L^p(\Gamma,\mathbb{R}^m).$$

Since $u_k \to u$ in W_0 then $L_{a,k}(x,y) \to L_a(x,y)$ in $L^p(\Gamma, \mathbb{R}^m)$, then there exists $g \in L^p(\Gamma, \mathbb{R}^m)$ such that $|L_{a,k}(x,y)| \le g(x,y)$, so we have $B_{a,k}(x,y) \to B_a(x,y)$ a.e in Γ. Then

$$|B_{a,k}(x,y)| = |L_{a,k}(x,y)|^{p-1} \le |g(x,y)|^{p-1}.$$

According to the dominant convergence theorem, we deduce that

$$B_{a,k}(x,y) \to B_a(x,y) \quad \text{in} \quad L^{p'}(\Gamma, \mathbb{R}^m) \text{ as } k \to \infty.$$

As a result

$$A_2 \le M\left(||u||^p_{W_0}\right) \|B_{a,k} - B_a\|_{L^{p'}(\Gamma, \mathbb{R}^m)} \to 0 \text{ as } k \to \infty.$$

However, we have

$$A_3 \le \left(\int_\Omega |f(x,u_k) - f(x,u)|^{p'} \, dx\right)^{\frac{1}{p'}}.$$

By using the boundedness of (u_k) in $L^p(\Omega; \mathbb{R}^m)$, we deduce from $(F1)$ that:

$$\int_\Omega |f(x,u_k)|^{p'} \, dx \le 2^{\frac{1}{p-1}} \int_\Omega \left(|\mathcal{E}(x)|^{p'} + C_1 |u_k|^p\right) dx \le C_3 \tag{3.2}$$

According to (3.2), the uniformly boundednes and equiintegrability of the sequence

$$\left\{|f(x,u_k) - f(x,u)|^{p'}\right\}$$

in $L^1(\Omega)$ can be easily deduced. We apply the Vitali Convergence Theorem, we find that

$$\lim_{k \to \infty} \int_\Omega |f(x,u_k) - f(x,u)|^{p'} \, dx = 0.$$

From the above discussion, we can infer that

$$\lim_{k \to \infty} |\langle \Psi(u_k), \phi\rangle - \langle \Psi(u), \phi\rangle| = 0.$$

Assertion 3: Ψ is coercive.
Indeed, as $\beta \geqslant 1$, we get

$$\begin{aligned}
\langle \Psi(u), u\rangle &= M\left(||u||^p_{W_0}\right) \iint_\Gamma \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy - \int_\Omega f(x,u)udx \\
&\geqslant m_0 \left(||u||^p_{W_0}\right)^{\beta-1} ||u||^p_{W_0} - C_4||\mathcal{E}||_{p'}||u||_{W_0} - C_1 C_5^p ||u||^p_{W_0} \\
&= m_0 \left(||u||^p_{W_0}\right)^{\beta} - C_4||\mathcal{E}||_{p'}||u||_{W_0} - C_1 C_5^p ||u||^p_{W_0} \\
&= m_0 ||u||^{p\beta}_{W_0} - C_4||\mathcal{E}||_{p'}||u||_{W_0} - C_1 C_5^p ||u||^p_{W_0}.
\end{aligned}$$

Under the assumption that $p > 1$ and with the chosen embedding constant $C_5 = \frac{1}{2}\left(\frac{m_0}{C_1}\right)^{\frac{1}{p}}$, we have

$$\lim_{||u||_{W_0} \to \infty} \frac{\langle \Psi(u), u\rangle}{||u||_{W_0}} = \infty. \qquad \square$$

From the instructions above, we can now construct the approximating solutions.

Lemma 3.1. *There exists $u_k \in V_k$, $\forall k \in \{0,1,2...\}$ such that*

$$\langle \Psi(u_k), \phi \rangle = 0 \quad \text{for all } \phi \in V_k. \tag{3.3}$$

Moreover, there is a constant $M > 0$ such that

$$\|u_k\|_{W_0} \leq M \quad \forall k \in \{0,1,2...\}. \tag{3.4}$$

Proof. On one hand, assume that $\dim V_k = r$ and fix k. For efficiency, we use the notation $\sum_{i=1}^{k} d^i e_i = d^i e_i$, where $(e_i)_{i=1}^r$ is a basis of V_k. We define the following map:

$$\mathcal{S} \quad : \quad \begin{array}{ccc} \mathbb{R}^r & \to & \mathbb{R}^r \\ (d^1, \dots, d^r) & \mapsto & \left(\left\langle \Psi\left(d^i e_i\right), e_j \right\rangle\right)_{j=1}^r. \end{array}$$

Let $u = d^i e_i$, via Assertion 1, the function $\mathcal{S}$ is continuous. We find that $\|d\|_{\mathbb{R}^r}$ tends to ∞ is equivalent to $\|u\|_{W_0}$ tends to ∞ and $\mathcal{S}(d)d = \langle \Psi(u), u \rangle$. Hence

$$\mathcal{S}(d)d \longrightarrow \infty \quad \text{as} \quad \|d\|_{\mathbb{R}^r} \longrightarrow \infty.$$

As a result, $M > 0$ exists such that for all $d \in \partial B_M(0) \subset \mathbb{R}^r$ we have $\mathcal{S}(d)d > 0$. There is $x \in B_M(0)$ solution of $\mathcal{S}(x) = 0$, this according to [28, Lemma 4.3]. Therefore, for all $k \in \{0,1,2...\}$ there exists $u_k \in V_k$ such that

$$\langle \Psi(u_k), \phi \rangle = 0 \quad \text{for all } \phi \in U_k.$$

On the other hand, observe that if $\|u_k\|_{W_0}$ tends to ∞, we get $\langle \Psi(u_k), u_k \rangle \to \infty$. There is a contradiction with (3.3). As a result, $\{u_k\}$ is uniformly bounded. □

As stated in the introduction, Young measures is the tool we use to prove the existence of a weak solution. To identify the weak limit, we consider the following lemma:

Lemma 3.2. *Assume that (3.4) holds. Then, the existence of a Young measure $\mu_{(x,y)}$ generated by $\frac{u_k(x)-u_k(y)}{|x-y|^{\frac{n+ps}{p}}} \in L^p(\Gamma; \mathbb{R}^m)$ has the following properties:*

1) $\left\|\mu_{(x,y)}\right\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for a.e. $(x,y) \in \Gamma$, i.e. $\mu_{(x,y)}$ is a probability measure.

2) $\left\langle \mu_{(x,y)}, id \right\rangle = \int_{\mathbb{R}^m} \lambda d\mu_{(x,y)}(\lambda)$ is the weak L^1-limit of u_k.

3) $\left\langle \mu_{(x,y)}, id \right\rangle = \frac{u(x)-u(y)}{|x-y|^{\frac{n+ps}{p}}}$ for a.e. $(x,y) \in \Gamma$.

Proof. 1) For simplicity reasons, we write

$$v_k(x,y) = \frac{u_k(x) - u_k(y)}{|x-y|^{\frac{n+ps}{p}}} \in L^p(\Gamma; \mathbb{R}^m). \tag{3.5}$$

We know that for any $M > 0$, $(\Omega \cap B_M)^2 \subseteq \Omega \times \Omega \subsetneqq \Gamma$, where B_M is the ball centered in 0 with radius M. Let $N \in \mathbb{R}$ such that $\Gamma_N \equiv \left\{(x,y) \in (\Omega \cap B_M)^2 : |v_k(x,y)| \geq N\right\}$. Moreover, by (3.4) we have

$$\|v_k\|_{L^p(\Gamma;\mathbb{R}^m)} = \left(\iint_\Gamma \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{\frac{1}{p}} = \|u_k\|_{W_0} \leq M. \tag{3.6}$$

Then, the sequence (v_k) is bounded in $L^p(\Gamma;\mathbb{R}^m)$. Consequently, there exists $C_6 \geq 0$ such that

$$C_6 \geq \iint_\Gamma |v_k(x,y)|^p \, dxdy \geq \iint_{\Gamma_N} |v_k(x,y)|^p \, dxdy \geq N^p |\Gamma_N|, \tag{3.7}$$

where $|\Gamma_N|$ is the Lebesgue measure of Γ_N. According to equation (3.7), the sequence (v_k) satisfies the equation (2.1). Hence, a Young measure noted by $\mu_{(x,y)}$ is generated by v_k such that $\left\|\mu_{(x,y)}\right\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for a.e. $(x,y) \in \Gamma$.

2) By (3.4), there exists a weak convergent subsequence still denoted by $(v_k)_k$ that convergences in $L^p(\Gamma;\mathbb{R}^m)$. As $L^p(\Gamma;\mathbb{R}^m)$ is reflexive $(p > 1)$, then v_k is weakly convergent in $L^1(\Gamma;\mathbb{R}^m)$. By the third assertion in Lemma 2.4, we replace the function ρ by the identity function, we then have

$$v_k \rightharpoonup \left\langle \mu_{(x,y)}, id \right\rangle = \int_{\mathbb{R}^m} \lambda d\mu_{(x,y)}(\lambda) \text{ weakly in } L^1(\Gamma;\mathbb{R}^m).$$

3) Using (3.4), we get $v_k \rightharpoonup v$ in $L^p(\Gamma;\mathbb{R}^m)$ (for a subsequence). Owing to the previous arguments, the uniqueness of limits implies that

$$\left\langle \mu_{(x,y)}, id \right\rangle = v(x,y) = \frac{u(x)-u(y)}{|x-y|^{\frac{n+ps}{p}}} \quad \text{for a.e. } (x,y) \in \Gamma.$$

□

Now, we are in a position to prove the main results.

Proof. (of Theorem 3.1) Let $\{v_k\}_k$ be the sequence given in (3.5). The weak convergence given in Lemma 3.2 shows that

$$\begin{aligned} |v_k|^{p-2} v_k \rightharpoonup & \int_{\mathbb{R}^m} |\lambda|^{p-2} \lambda d\mu_{(x,y)}(\lambda) \\ &= |v|^{p-2} v \\ &= \frac{|U(x,y)|^{p-2} U(x,y)}{|x-y|^{ps-s}} \end{aligned} \tag{3.8}$$

weakly in $L^1(\Gamma;\mathbb{R}^m)$. Since L^p is reflexive and $|v_k|^{p-2} v_k$ is bounded in $L^{p'}(\Gamma;\mathbb{R}^m)$, the sequence $|v_k|^{p-2} v_k$ converges in $L^{p'}(\Gamma;\mathbb{R}^m)$, and its weak $L^{p'}$ is also $|v|^{p-2} v$. Moreover, for any $\phi \in W_0$, we have

$$\frac{\phi(x)-\phi(y)}{|x-y|^{\frac{n+ps}{p}}} \in L^p(\Gamma;\mathbb{R}^m).$$

Finally, we deduce that

$$\begin{aligned} &\lim_{k\to\infty} \iint_\Gamma \frac{|U_k(x,y)|^{p-2} U_k(x,y)}{|x-y|^{n+ps}} (\phi(x)-\phi(y)) dxdy \\ &\qquad = \iint_\Gamma \frac{|U(x,y)|^{p-2} U(x,y)}{|x-y|^{n+ps}} (\phi(x)-\phi(y)) dxdy, \end{aligned} \tag{3.9}$$

for every $\phi \in W_0$.

On the other hand, according to (3.4), we have $u_k \to u$ in $L^p(\Gamma,\mathbb{R}^m)$ for a subsequence, we can deduce from the continuity of M that

$$M\left(\iint_\Gamma \frac{|U_k(x,y)|^p}{|x-y|^{n+ps}} dxdy\right) \to M\left(\iint_\Gamma \frac{|U(x,y)|^p}{|x-y|^{n+ps}} dxdy\right) \text{ a.e. for } k \to \infty, \tag{3.10}$$

From (3.9) and (3.10) it follows that

$$\begin{aligned}\lim_{k\to\infty} M\left(\iint_{\Gamma}\frac{|U_k(x,y)|^p}{|x-y|^{n+ps}}dxdy\right)\iint_{\Gamma}\frac{|U_k(x,y)|^{p-2}(u_k(x)-u_k(y))}{|x-y|^{n+ps}}(\phi(x)-\phi(y))dxdy \\ = M\left(\iint_{\Gamma}\frac{|U(x,y)|^p}{|x-y|^{n+ps}}dxdy\right)\iint_{\Gamma}\frac{|U(x,y)|^{p-2}U(x,y)}{|x-y|^{n+ps}}(\phi(x)-\phi(y))dxdy,\end{aligned} \tag{3.11}$$

for all $\phi \in W_0$.

On the other hand, by (3.4), we get

$$u_k \to u \text{ strongly in } L^p\left(\Omega;\mathbb{R}^m\right) \text{ and a.e in } \Omega \text{ (for a subsequence)}\,.$$

The property of the continuity in $(F1)$ leads to the conclusion that

$$f\left(x,u_k\right)\left(u_k-u\right)\to 0 \quad \text{a.e. in} \quad \Omega \quad \text{as} \quad k\to\infty.$$

According to the growth condition in $(F1)$, $\{f\left(x,u_k\right)\left(u_k-u\right)\}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. We apply the Vitali Convergence Theorem, then

$$\lim_{k\to\infty}\int_{\Omega} f\left(x,u_k\right)(u_k-u)dx = 0. \tag{3.12}$$

We conclude from (3.11), (3.12), Lemma 3.1, and density of $\cup_{k\geq 1}V_k$ in W_0, that $\langle \Psi\left(u\right),\phi\rangle = 0$ for all $\phi \in W_0$. □

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Ihya Talibi Laboratory LMACS, FST of Beni Mellal, Sultan Moulay Slimane University, Morocco

E-mail: ihya.talibi@usms.ma

Farah Balaadich Laboratory LMACS, FST of Beni Mellal, Sultan Moulay Slimane University, Morocco

E-mail: balaadich.edp@gmail.com

Brahim El Boukari Laboratory LMACS, Higher School of Technology of Beni Mellal, Sultan Moulay Slimane University, Morocco

E-mail: elboukaribrahim@yahoo.fr

Jalila El Ghordaf Laboratory LMACS, Sultan Moulay Slimane University, Morocco

E-mail: elg_jalila@yahoo.fr