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# Application of Lipschitz viscosity solutions for higher-order partial differential equations containing the special Lagrangian operator

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**Abstract**. Using the Lipschitz continuity of a class of viscosity solutions, we find a kind of viscosity solution for some higher-order partial differential equations containing the special Lagrangian operator. Additionally, we extend this analysis to equations that simultaneously contain the special Lagrangian and some other operators including Laplacian.

***Keywords.*** Viscosity solutions to PDEs, Hamiltonian and Lagrangian structures, partial differential operators

## 1 Introduction

We can apply the theory of viscosity solutions to many partial differential equations (for example see [3], and [5]). Nevertheless, the theory of viscosity solutions is based on some versions of the maximum principle that have a second-order nature. As a result, the direct application of the theory is usually limited to the second-order partial differential equations (PDEs).

Recently, [2] introduced an approach to extend the application of the theory to higher-order PDEs. This approach can be expanded to cover more complex higher-order PDEs. Indeed, the main part of the approach uses a basic Holder continuity result (that can be found in earlier works like [6] or [7]). Here, we use a strong, recently proven, theorem from [8]. Also, by investigating a certain type of PDEs, we introduce a general method in the sense of change of variables, and we prove some theorems to expand the application of viscosity solutions to a new range of higher-order PDEs.

## 2 Preliminaries

For any natural number $m$, $\mathbb{S}^m$ is the set of symmetric $m\times m$ matrices. Also, for $w:\mathbb{R}^N\to\mathbb{R}$, the symbol $\bigtriangledown w$ denotes the gradient of $w$, and $D^2w$ represents the Hessian matrix of $w$. In addition, for a function $u(\upsilon,\eta,\varrho,\omega):\mathbb{R}^4\to\mathbb{R}$, we use $\bigtriangledown_{(\upsilon,\eta,\varrho)}u$ for gradient vector, and ${D^2}_{(\upsilon,\eta,\varrho)}u$ for

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Hessian matrix of $u$ with respect to the variables $(\upsilon, \eta, \varrho)$. In addition, $tanh(.)$, and $\triangle w$ represent the hyperbolic tangent and the Laplacian of a function $w$, respectively.

For a symmetric $n \times n$ matrix $H$ with eigenvalues $\{\beta_j\}_{j=1}^n$, the Special Lagrangian is

$$L(H) = \sum_{j=1}^{n} \tan^{-1}(\beta_j) \quad . \tag{2.1}$$

We call a connected open set $U \subset \mathbb{R}^N$ a smooth domain if its boundary $\partial U$ can be locally viewed as the graph of a smooth function. Also, for any $n \geq 1$, we use $B^n(R)$ as a ball of radius $R$ in $\mathbb{R}^n$. Note that these balls may not centered at the origin. Finally, to get familiar with the theory of viscosity solutions, one can see [3], or [5].

# 3 main results

Here, by a same technique as the one introduced in [2], we expand the application of viscosity solutions to some important types of higher-order PDEs, and specially to those that contain Lagrangian, and Laplacian operators, simultaneously.

Indeed, our method for solving a higher-order PDE splits into two problems (in the sense of change of variables), so that one of them contains a PDE that we can find a viscosity solution for it (we call it the first problem), and the second one contains a PDE that admits a classical solution which can be considered as a viscosity solution (so that it can merge with the viscosity solution of the first problem). The outcome of this approach makes a generalized solution for the main higher-order PDE that we call it (inspired from Definition 1 in [2]) the viscosity solution of the higher-order PDE.

**Definition 1.** Let $G : \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$. The function $\upsilon$ is called a (generalized) viscosity solution of the PDE

$$Q\left(x, G(\bigtriangledown \upsilon, D^2\upsilon), \bigtriangledown G(\bigtriangledown \upsilon, D^2\upsilon), D^2G(\bigtriangledown \upsilon, D^2\upsilon)\right) = 0,$$

if and only if, there exists a function $\varpi$ such that $G(\bigtriangledown \upsilon, D^2\upsilon) = \varpi$, in classical sense, where $\varpi$ is a viscosity solution for the equation $Q(x, \varpi, \bigtriangledown \varpi, D^2(\varpi)) = 0$.

Now, to make a general formulation, we introduce a special type of functions.

**Definition 2.** Let $n \geq 1$, and $w : B^n(R) \to \mathbb{R}$ be Holder continuous (with an exponent $\gamma \leqslant 1$). The function $G : \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is in the set $\Upsilon(w, B^n(R))$ if and only if the following equation has a classical solution, in $B^n(R)$:

$$G(\bigtriangledown \upsilon, D^2\upsilon){=}w(s) \quad , \; s \in B^n(R) \; . \tag{3.1}$$

Also, the notation $G \in \Upsilon$, means that for any Holder continuous function $w$, and any ball $B^n(R)$ we have $G \in \Upsilon(w, B^n(R))$.

**Remark 1.** As it is mentioned in Section 5 of [1] (Specifically, by Theorem 5.11 in [1]), when $G \in \Upsilon(w, B^n(R))$, the classical solution of (3.1) is also a viscosity solution (note that $G(s, \cdot)$ is non-increasing).

**Example 1.** Since For any Holder continuous function $w$, and any ball $B^n(R) \subset \mathbb{R}^n$, the equation $-\triangle \upsilon = tanh(w(s))$ has a classical solution (see Chapter 3 of [4]), thus $G_1(\bigtriangledown \upsilon, D^2\upsilon) :=$

$-trace(D^2 v) = -\Delta v \in \Upsilon$. Similarly, since $tanh(.)$ is Lipschitz, and so Holder continuous, the equation $- \triangle v = tanh(w(s))$ has a classical solution, and therefore

$$G_2(\bigtriangledown v, D^2 v) := tanh^{-1}(-trace(D^2 v)) = tanh^{-1}(-\triangle v) \in \Upsilon.$$

**Definition 3.** The function $\alpha : \mathbb{R} \to \mathbb{R}$ is in the set $\Lambda^{(1)}$, if and only if, for every ball $B^1(R) \subset \mathbb{R}$, the following equation has a classical solution $\varphi : \mathbb{R} \to \mathbb{R}$ :

$$\varphi^{'}(\omega) = (\alpha(\omega))^{-1}\omega \ .$$

Also, we say that the function $\alpha$ belongs to the set $\Lambda^{(2)}$, if and only if, the following equation has a classical solution $\varphi : \mathbb{R} \to \mathbb{R}$ :

$$\varphi^{''}(\omega) = (\alpha(\omega))^{-1}\omega \ .$$

**Remark 2.** Some simple examples of the functions that satisfy the conditions of Definition 3 are exponential, polynomial, and constant functions that do not have any root.

**Theorem 3.1.** *Let $G \in \Upsilon$, and $\alpha \in \Lambda^{(1)}$. Also, for any $i \in \{1, 2, 3\}$, consider $b_i : \mathbb{R}^3 \to \mathbb{R}$ as a real valued function. Then, there exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon, \eta, \varrho, \omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that for*

$$K[u](\upsilon, \eta, \varrho, \omega) := \ (b_1(\upsilon, \eta, \varrho), b_2(\upsilon, \eta, \varrho), b_3(\upsilon, \eta, \varrho), \alpha(\omega)) \ . \bigtriangledown u \ ,$$

*we have*

$$(K[u])_\omega + L(D^2_{(\upsilon,\eta,\varrho)}(G(\bigtriangledown_{(\upsilon,\eta,\varrho)} u, D^2_{(\upsilon,\eta,\varrho)} u))) = 2\tau \ ,$$

*in the viscosity sense in $B^4(R)$.*

*Proof.* In [8], it is proved that there exist $\tau \in [0, \pi/2)$, and a smooth bounded domain $\Omega \subset \mathbb{R}^3$ such that the equation $L(D^2 v) = \tau$ has a Lipschitz viscosity solution $v$, in $\Omega$. Now, let $R^{'} > 0$ such that $B^3(R^{'}) \subset \Omega$. Since $G \in \Upsilon$, $G(\bigtriangledown \tilde{u}, D^2 \tilde{u}) = v$ admits a classical solution $\tilde{u}$, in $B^3(R^{'})$. Therefore, the function $\tilde{u}$ is a viscosity solution (based on our terminology) of the equation $L(D^2(G(\bigtriangledown \tilde{u}, D^2 \tilde{u}))) = \tau$, in $B^3(R^{'}) \subset \mathbb{R}^3$.

Now, consider:

$$L(D^2_{(\upsilon,\eta,\varrho)}(G(\bigtriangledown_{(\upsilon,\eta,\varrho)} u, D^2_{(\upsilon,\eta,\varrho)} u))) = \tau \ ,$$

as an equation in $\mathbb{R}^3$. We have proved that we can find $\tau \in [0, \pi/2)$, $R^{'} > 0$ so that the above equation has a viscosity solution $\tilde{u}(\upsilon, \eta, \varrho)$, in a set $D := B^3(R^{'}) \subset \mathbb{R}^3$. Now, choose $R > 0$ such that there exists a ball $B^4(R) \subset D \times \mathbb{R}$. Note that for $\varphi \in \Lambda^{(1)}$, we have

$$\begin{aligned}&\big(b_1(\upsilon, \eta, \varrho), b_2(\upsilon, \eta, \varrho), b_3(\upsilon, \eta, \varrho), \alpha(\omega)).(\frac{\partial \tilde{u}}{\partial \upsilon}, \frac{\partial \tilde{u}}{\partial \eta}, \frac{\partial \tilde{u}}{\partial \varrho}, \tau.\varphi^{'}(\omega))\big)_\omega \\ &= \tau.(\alpha(\omega).\varphi^{'}(\omega))^{'} = \tau \ .\end{aligned}$$

Therefore, the function $u(\upsilon, \eta, \varrho, \omega) := \tilde{u}(\upsilon, \eta, \varrho) + \tau.\varphi(\omega)$, where $\varphi \in \Lambda^{(1)}$, is a viscosity solution of our main equation in the ball $B^4(R) \subset \mathbb{R}^4$. □

In addition, this method can be generalized to many PDEs, and specially, to those that contain Laplacian. In this regard, we prove the following theorem.

**Theorem 3.2.** *Let $G \in \Upsilon$, and $\alpha \in \Lambda^{(2)}$. Also, for any $i \in \{1,2,3\}$, suppose that $b_i : \mathbb{R}^3 \to \mathbb{R}$ is a real valued function. Then, there exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon, \eta, \varrho, \omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that for*

$$J[u](\upsilon,\eta,\varrho,\omega) := \ (b_1(\upsilon,\eta,\varrho), b_2(\upsilon,\eta,\varrho), b_3(\upsilon,\eta,\varrho), \alpha(\omega)) \ .(u_{\upsilon\upsilon}, u_{\eta\eta}, u_{\varrho\varrho}, u_{\omega\omega}) \ ,$$

*we have*

$$(J[u])_\omega + L(D^2_{(\upsilon,\eta,\varrho)}(G(\bigtriangledown_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u))) = 2\tau \ ,$$

*in the viscosity sense, in $B^4(R)$.*

*Proof.* First, consider:

$$L(D^2_{(\upsilon,\eta,\varrho)}(G(\bigtriangledown_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u))) = \tau \ .$$

as an equation in $\mathbb{R}^3$. We have discussed in the proof of Theorem 3.1 that we can find $\tau \in [0, \pi/2)$, and $R^{'} > 0$ so that the above equation has a viscosity solution $\hat{u}(\upsilon,\eta,\varrho)$ in $D := B^3(R^{'}) \subset \mathbb{R}^3$. Now, choose $R > 0$ such that there exists a ball $B^4(R) \subset D \times \mathbb{R}$. Note that for $\varphi \in \Lambda^{(2)}$, we have

$$\begin{aligned}&\big((b_1(\upsilon,\eta,\varrho), b_2(\upsilon,\eta,\varrho), b_3(\upsilon,\eta,\varrho), \alpha(\omega)) \ .(\hat{u}_{\upsilon\upsilon}, \hat{u}_{\eta\eta}, \hat{u}_{\varrho\varrho}, \varphi^{''}(\omega))\big)_\omega \\ &= \tau.(\alpha(\omega)\varphi^{''}(\omega))^{'} = \tau \ .\end{aligned}$$

Therefore, the function $u(\upsilon,\eta,\varrho,\omega) := \hat{u}(\upsilon,\eta,\varrho) + \varphi(\omega)$, where $\varphi \in \Lambda^{(2)}$, is a viscosity solution of our main equation in the ball $B^4(R) \subset \mathbb{R}^4$. □

Now, we can prove many corollaries, and examples by the above theorems, and we provide some of them, here.

**Corollary 3.3.** *Let $G \in \Upsilon$. There exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon,\eta,\varrho,\omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that*

$$(\triangle u)_\omega + L(D^2_{(\upsilon,\eta,\varrho)}(G(\bigtriangledown_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u))) = 2\tau \ ,$$

*in the viscosity sense, in $B^4(R)$.*

*Proof.* By choosing $b_i(\upsilon,\eta,\varrho) = 1$ (for $i = 1,2,3$), and $\alpha(\omega) = 1$, Theorem 3.2 asserts that we can find $\tau \in [0, \pi/2)$, and $R > 0$ so that the above equation has a viscosity solution, in a ball $B^4(R)$. □

**Corollary 3.4.** *Let $G \in \Upsilon$. There exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon,\eta,\varrho,\omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that*

$$e^\omega.(u_\omega + u_{\omega\omega}) + L(D^2_{(\upsilon,\eta,\varrho)}(G(\bigtriangledown_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u))) = 2\tau \ ,$$

*in the viscosity sense, in $B^4(R)$.*

*Proof.* By choosing $b_i(\upsilon,\eta,\varrho) = 0$ (for $i = 1,2,3$), and $\alpha(\omega) = e^\omega$, Theorem 3.1 asserts that we can find $\tau \in [0, \pi/2)$, and $R > 0$ so that the above equation has a viscosity solution, in a ball $B^4(R)$. □

**Example 2.** There exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon, \eta, \varrho, \omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that

$$e^{\omega}.(u_\omega + u_{\omega\omega}) + L(D^2_{(\upsilon,\eta,\varrho)}(-\triangle_{(\upsilon,\eta,\varrho)}(u))) = 2\tau \ ,$$

in the viscosity sense, in $B^4(R)$.

*Proof.* Choose $G(\bigtriangledown_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u) := -\triangle_{(\upsilon,\eta,\varrho)}(u)$. Now, from Corollary 3.4, we can find $\tau \in [0, \pi/2)$, and $R > 0$ so that the above equation has a viscosity solution, in a ball $B^4(R)$. □

**Example 3.** There exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon, \eta, \varrho, \omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that

$$(\triangle u)_\omega + L(D^2_{(\upsilon,\eta,\varrho)}(tanh^{-1}(\triangle_{(\upsilon,\eta,\varrho)}(u))) = 2\tau \ ,$$

in the viscosity sense, in $B^4(R)$.

*Proof.* Choose $G(\bigtriangledown_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u) := tanh^{-1}(\triangle_{(\upsilon,\eta,\varrho)}(u))$. Now, from Corollary 3.3, we can find $\tau \in [0, \pi/2)$, and $R > 0$ so that the above equation has a viscosity solution, in a ball $B^4(R)$. □

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