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Application of Lipschitz viscosity solutions for higher-order partial differential equations containing the special Lagrangian operator

S.M.E. Hosseini and S. Saiedinezhad

Abstract. Using the Lipschitz continuity of a class of viscosity solutions, we find a kind of viscosity solution for some higher-order partial differential equations containing the special Lagrangian operator. Additionally, we extend this analysis to equations that simultaneously contain the special Lagrangian and some other operators including Laplacian.

Keywords. Viscosity solutions to PDEs, Hamiltonian and Lagrangian structures, partial differential operators

1 Introduction

We can apply the theory of viscosity solutions to many partial differential equations (for example see [3], and [5]). Nevertheless, the theory of viscosity solutions is based on some versions of the maximum principle that have a second-order nature. As a result, the direct application of the theory is usually limited to the second-order partial differential equations (PDEs).

Recently, [2] introduced an approach to extend the application of the theory to higher-order PDEs. This approach can be expanded to cover more complex higher-order PDEs. Indeed, the main part of the approach uses a basic Holder continuity result (that can be found in earlier works like [6] or [7]). Here, we use a strong, recently proven, theorem from [8]. Also, by investigating a certain type of PDEs, we introduce a general method in the sense of change of variables, and we prove some theorems to expand the application of viscosity solutions to a new range of higher-order PDEs.

2 Preliminaries

For any natural number m, $\mathbb{S}^m$ is the set of symmetric $m\times m$ matrices. Also, for $w : \mathbb{R}^N \to \mathbb{R}$, the symbol $\bigtriangledown w$ denotes the gradient of w, and $D^2 w$ represents the Hessian matrix of w. In addition, for a function $u(\upsilon, \eta, \varrho, \omega) : \mathbb{R}^4 \to \mathbb{R}$, we use $\bigtriangledown_{(\upsilon,\eta,\varrho)} u$ for gradient vector, and $D^2{}_{(\upsilon,\eta,\varrho)} u$ for

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Corresponding author: Somayeh Saiedinezhad.

Hessian matrix of u with respect to the variables $(\upsilon, \eta, \varrho)$. In addition, $tanh(.)$, and $\triangle w$ represent the hyperbolic tangent and the Laplacian of a function w, respectively.

For a symmetric $n \times n$ matrix H with eigenvalues $\{\beta_j\}_{j=1}^n$, the Special Lagrangian is

$$L(H) = \sum_{j=1}^{n} \tan^{-1}(\beta_j) \quad . \tag{2.1}$$

We call a connected open set $U \subset \mathbb{R}^N$ a smooth domain if its boundary ∂U can be locally viewed as the graph of a smooth function. Also, for any $n \geq 1$, we use $B^n(R)$ as a ball of radius R in $\mathbb{R}^n$. Note that these balls may not centered at the origin. Finally, to get familiar with the theory of viscosity solutions, one can see [3], or [5].

3 main results

Here, by a same technique as the one introduced in [2], we expand the application of viscosity solutions to some important types of higher-order PDEs, and specially to those that contain Lagrangian, and Laplacian operators, simultaneously.

Indeed, our method for solving a higher-order PDE splits into two problems (in the sense of change of variables), so that one of them contains a PDE that we can find a viscosity solution for it (we call it the first problem), and the second one contains a PDE that admits a classical solution which can be considered as a viscosity solution (so that it can merge with the viscosity solution of the first problem). The outcome of this approach makes a generalized solution for the main higher-order PDE that we call it (inspired from Definition 1 in [2]) the viscosity solution of the higher-order PDE.

Definition 1. Let $G : \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$. The function υ is called a (generalized) viscosity solution of the PDE

$$Q\left(x, G(\triangledown \upsilon, D^2\upsilon), \triangledown G(\triangledown \upsilon, D^2\upsilon), D^2G(\triangledown \upsilon, D^2\upsilon)\right) = 0,$$

if and only if, there exists a function ϖ such that $G(\triangledown \upsilon, D^2\upsilon) = \varpi$, in classical sense, where ϖ is a viscosity solution for the equation $Q(x, \varpi, \triangledown\varpi, D^2(\varpi)) = 0$.

Now, to make a general formulation, we introduce a special type of functions.

Definition 2. Let $n \geq 1$, and $w : B^n(R) \to \mathbb{R}$ be Holder continuous (with an exponent $\gamma \leqslant 1$). The function $G : \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is in the set $\Upsilon(w, B^n(R))$ if and only if the following equation has a classical solution, in $B^n(R)$:

$$G(\triangledown \upsilon, D^2\upsilon){=}w(s) \quad , \ s \in B^n(R) \ . \tag{3.1}$$

Also, the notation $G \in \Upsilon$, means that for any Holder continuous function w, and any ball $B^n(R)$ we have $G \in \Upsilon(w, B^n(R))$.

Remark 1. As it is mentioned in Section 5 of [1] (Specifically, by Theorem 5.11 in [1]), when $G \in \Upsilon(w, B^n(R))$, the classical solution of (3.1) is also a viscosity solution (note that $G(s, \cdot)$ is non-increasing).

Example 1. Since For any Holder continuous function w, and any ball $B^n(R) \subset \mathbb{R}^n$, the equation $-\triangle\, \upsilon = tanh(w(s))$ has a classical solution (see Chapter 3 of [4]), thus $G_1(\triangledown \upsilon, D^2\upsilon) :=$

$-trace(D^2 v) = -\Delta v \in \Upsilon$. Similarly, since $tanh(.)$ is Lipschitz, and so Holder continuous, the equation $-\triangle v = tanh(w(s))$ has a classical solution, and therefore

$$G_2(\bigtriangledown v, D^2 v) := tanh^{-1}(-trace(D^2 v)) = tanh^{-1}(-\triangle v) \in \Upsilon.$$

Definition 3. The function $\alpha : \mathbb{R} \to \mathbb{R}$ is in the set $\Lambda^{(1)}$, if and only if, for every ball $B^1(R) \subset \mathbb{R}$, the following equation has a classical solution $\varphi : \mathbb{R} \to \mathbb{R}$:

$$\varphi^{'}(\omega) = (\alpha(\omega))^{-1}\omega \ .$$

Also, we say that the function α belongs to the set $\Lambda^{(2)}$, if and only if, the following equation has a classical solution $\varphi : \mathbb{R} \to \mathbb{R}$:

$$\varphi^{''}(\omega) = (\alpha(\omega))^{-1}\omega \ .$$

Remark 2. Some simple examples of the functions that satisfy the conditions of Definition 3 are exponential, polynomial, and constant functions that do not have any root.

Theorem 3.1. *Let $G \in \Upsilon$, and $\alpha \in \Lambda^{(1)}$. Also, for any $i \in \{1,2,3\}$, consider $b_i : \mathbb{R}^3 \to \mathbb{R}$ as a real valued function. Then, there exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon, \eta, \varrho, \omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that for*

$$K[u](\upsilon,\eta,\varrho,\omega) := \ (b_1(\upsilon,\eta,\varrho), b_2(\upsilon,\eta,\varrho), b_3(\upsilon,\eta,\varrho), \alpha(\omega)) \ . \bigtriangledown u \ ,$$

we have

$$(K[u])_\omega + L(D^2_{(\upsilon,\eta,\varrho)}(G(\bigtriangledown_{(\upsilon,\eta,\varrho)} u, D^2_{(\upsilon,\eta,\varrho)} u))) = 2\tau \ ,$$

in the viscosity sense in $B^4(R)$.

Proof. In [8], it is proved that there exist $\tau \in [0, \pi/2)$, and a smooth bounded domain $\Omega \subset \mathbb{R}^3$ such that the equation $L(D^2 v) = \tau$ has a Lipschitz viscosity solution v, in Ω. Now, let $R^{'} > 0$ such that $B^3(R^{'}) \subset \Omega$. Since $G \in \Upsilon$, $G(\bigtriangledown \tilde{u}, D^2\tilde{u}) = v$ admits a classical solution $\tilde{u}$, in $B^3(R^{'})$. Therefore, the function $\tilde{u}$ is a viscosity solution (based on our terminology) of the equation $L(D^2(G(\bigtriangledown \tilde{u}, D^2\tilde{u}))) = \tau$, in $B^3(R^{'}) \subset \mathbb{R}^3$.

Now, consider:

$$L(D^2_{(\upsilon,\eta,\varrho)}(G(\bigtriangledown_{(\upsilon,\eta,\varrho)} u, D^2_{(\upsilon,\eta,\varrho)} u))) = \tau \ ,$$

as an equation in $\mathbb{R}^3$. We have proved that we can find $\tau \in [0, \pi/2)$, $R^{'} > 0$ so that the above equation has a viscosity solution $\tilde{u}(\upsilon,\eta,\varrho)$, in a set $D := B^3(R^{'}) \subset \mathbb{R}^3$. Now, choose $R > 0$ such that there exists a ball $B^4(R) \subset D \times \mathbb{R}$. Note that for $\varphi \in \Lambda^{(1)}$, we have

$$\begin{aligned}
&\big(b_1(\upsilon,\eta,\varrho), b_2(\upsilon,\eta,\varrho), b_3(\upsilon,\eta,\varrho), \alpha(\omega)).(\frac{\partial \tilde{u}}{\partial \upsilon}, \frac{\partial \tilde{u}}{\partial \eta}, \frac{\partial \tilde{u}}{\partial \varrho}, \tau.\varphi^{'}(\omega))\big)_\omega \\
&= \tau.(\alpha(\omega).\varphi^{'}(\omega))^{'} = \tau \ .
\end{aligned}$$

Therefore, the function $u(\upsilon,\eta,\varrho,\omega) := \tilde{u}(\upsilon,\eta,\varrho) + \tau.\varphi(\omega)$, where $\varphi \in \Lambda^{(1)}$, is a viscosity solution of our main equation in the ball $B^4(R) \subset \mathbb{R}^4$. □

In addition, this method can be generalized to many PDEs, and specially, to those that contain Laplacian. In this regard, we prove the following theorem.

Theorem 3.2. *Let $G \in \Upsilon$, and $\alpha \in \Lambda^{(2)}$. Also, for any $i \in \{1,2,3\}$, suppose that $b_i : \mathbb{R}^3 \to \mathbb{R}$ is a real valued function. Then, there exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon, \eta, \varrho, \omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that for*

$$J[u](\upsilon,\eta,\varrho,\omega) := \ (b_1(\upsilon,\eta,\varrho), b_2(\upsilon,\eta,\varrho), b_3(\upsilon,\eta,\varrho), \alpha(\omega))\ .(u_{\upsilon\upsilon}, u_{\eta\eta}, u_{\varrho\varrho}, u_{\omega\omega})\ ,$$

we have

$$(J[u])_\omega + L(D^2_{(\upsilon,\eta,\varrho)}(G(\nabla_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u))) = 2\tau\ ,$$

in the viscosity sense, in $B^4(R)$.

Proof. First, consider:

$$L(D^2_{(\upsilon,\eta,\varrho)}(G(\nabla_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u))) = \tau\ .$$

as an equation in $\mathbb{R}^3$. We have discussed in the proof of Theorem 3.1 that we can find $\tau \in [0, \pi/2)$, and $R^{'} > 0$ so that the above equation has a viscosity solution $\hat{u}(\upsilon,\eta,\varrho)$ in $D := B^3(R^{'}) \subset \mathbb{R}^3$. Now, choose $R > 0$ such that there exists a ball $B^4(R) \subset D \times \mathbb{R}$. Note that for $\varphi \in \Lambda^{(2)}$, we have

$$\begin{aligned}&\big((b_1(\upsilon,\eta,\varrho), b_2(\upsilon,\eta,\varrho), b_3(\upsilon,\eta,\varrho), \alpha(\omega))\ .(\hat{u}_{\upsilon\upsilon}, \hat{u}_{\eta\eta}, \hat{u}_{\varrho\varrho}, \varphi^{''}(\omega))\big)_\omega\\ &= \tau.(\alpha(\omega)\varphi^{''}(\omega))^{'} = \tau\ .\end{aligned}$$

Therefore, the function $u(\upsilon,\eta,\varrho,\omega) := \hat{u}(\upsilon,\eta,\varrho) + \varphi(\omega)$, where $\varphi \in \Lambda^{(2)}$, is a viscosity solution of our main equation in the ball $B^4(R) \subset \mathbb{R}^4$. □

Now, we can prove many corollaries, and examples by the above theorems, and we provide some of them, here.

Corollary 3.3. *Let $G \in \Upsilon$. There exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon,\eta,\varrho,\omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that*

$$(\triangle u)_\omega + L(D^2_{(\upsilon,\eta,\varrho)}(G(\nabla_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u))) = 2\tau\ ,$$

in the viscosity sense, in $B^4(R)$.

Proof. By choosing $b_i(\upsilon,\eta,\varrho) = 1$ (for $i = 1,2,3$), and $\alpha(\omega) = 1$, Theorem 3.2 asserts that we can find $\tau \in [0, \pi/2)$, and $R > 0$ so that the above equation has a viscosity solution, in a ball $B^4(R)$. □

Corollary 3.4. *Let $G \in \Upsilon$. There exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon,\eta,\varrho,\omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that*

$$e^\omega.(u_\omega + u_{\omega\omega}) + L(D^2_{(\upsilon,\eta,\varrho)}(G(\nabla_{(\upsilon,\eta,\varrho)}u, D^2_{(\upsilon,\eta,\varrho)}u))) = 2\tau\ ,$$

in the viscosity sense, in $B^4(R)$.

Proof. By choosing $b_i(\upsilon,\eta,\varrho) = 0$ (for $i = 1,2,3$), and $\alpha(\omega) = e^\omega$, Theorem 3.1 asserts that we can find $\tau \in [0, \pi/2)$, and $R > 0$ so that the above equation has a viscosity solution, in a ball $B^4(R)$. □

Example 2. There exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon, \eta, \varrho, \omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that

$$e^{\omega}.(u_\omega + u_{\omega\omega}) + L(D^2_{(\upsilon,\eta,\varrho)}(-\triangle_{(\upsilon,\eta,\varrho)}(u))) = 2\tau \ ,$$

in the viscosity sense, in $B^4(R)$.

Proof. Choose $G(\bigtriangledown_{(\upsilon,\eta,\varrho)} u, D^2_{(\upsilon,\eta,\varrho)} u) := -\triangle_{(\upsilon,\eta,\varrho)}(u)$. Now, from Corollary 3.4, we can find $\tau \in [0, \pi/2)$, and $R > 0$ so that the above equation has a viscosity solution, in a ball $B^4(R)$. □

Example 3. There exist $\tau \in [0, \pi/2)$, $R > 0$, and a function $u(\upsilon, \eta, \varrho, \omega)$ defined on a ball $B^4(R) \subset \mathbb{R}^4$ such that

$$(\triangle u)_\omega + L(D^2_{(\upsilon,\eta,\varrho)}(tanh^{-1}(\triangle_{(\upsilon,\eta,\varrho)}(u))) = 2\tau \ ,$$

in the viscosity sense, in $B^4(R)$.

Proof. Choose $G(\bigtriangledown_{(\upsilon,\eta,\varrho)} u, D^2_{(\upsilon,\eta,\varrho)} u) := tanh^{-1}(\triangle_{(\upsilon,\eta,\varrho)}(u))$. Now, from Corollary 3.3, we can find $\tau \in [0, \pi/2)$, and $R > 0$ so that the above equation has a viscosity solution, in a ball $B^4(R)$. □

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SeyedMohammadErfan Hosseini School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16846-13114, Iran

E-mail: s_hosseini78@mathdep.iust.ac.ir

Somayeh Saiedinezhad School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16846-13114, Iran

E-mail: ssaiedinezhad@iust.ac.ir