TAMKANG JOURNAL OF MATHEMATICS
Volume vol, Number i, xxx-xxx, 20xx
DOI:10.5556/j.tkjm.56.2025.5752

A study on reaction-diffusion singular perturbation problems with non-classical conditions using collocation method

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Abstract. This article discuss about a numerical study to find the solution of second order reaction diffusion singular perturbation problem with non-local boundary conditions using cubic B-spline functions and collocation technique. Shishkin mesh is used to construct layer adapted meshes. The non-local boundary conditions are discretized using Trapezoidal rule. The study establishes that the discussed scheme's result is uniformly convergent up to second order in the supremum norm. To establish the efficiency of the discussed method, two numerical examples are presented along with their results in the form of tables and figures.

Keywords. Singularly perturbed problems, cubic B-spline collocation, non-classical condition, layer adaptive mesh

1 Introduction

When a small positive parameter μ appears in the highest derivative, differential equations are termed as singularly perturbed. Such problems arise in various fields of fluid dynamics, electrochemistry [8] and thermodynamics [9]. Typically, the solution to these problems exhibit boundary layers leading to challenges in both analytical and numerical approaches. There are several studies available in the literature for singular perturbation problems with and without turning points [3, 12] and for system of equations [23]. In the context of boundary conditions, one particular type of boundary condition that has received attention is the integral boundary condition (IBC). These conditions involve integrals over parts of the domain or its boundaries and their presence can significantly alter the mathematical structure and solution behavior of the differential equation. Singularly perturbed problems with IBC require the use of a closely spaced grid of mesh points in the layer regions. This requires parameter uniform convergence incorporating layer-adapted meshes and developing appropriate numerical methods adapted for singular perturbation problems (SPPs). In the solutions of singularly perturbed problems, one solution varies smoothly while the other varies rapidly illustrating multi-scale behavior. This complexity makes the numerical analysis of the problems considered more difficult than that of typical problems.

In [5, 10, 11], the researchers examined the finite difference technique for SPPs with IBC. Several authors, including Cen et al.[6], proposed a second-order adaptive grid technique for non

Received date: April 11, 2025; Published online: September 25, 2025.
2010 *Mathematics Subject Classification*. 65N06, 65N12 .
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linear SPPs with IBC. In [26, 27], Raja and Ayyadurai Tamilselvan adapted the finite difference technique on a uniform mesh during their analysis of singularly perturbed convection-diffusion equations with IBC. If we examine [25, 28, 32], researchers focused on singularly perturbed reaction-diffusion equations with IBC using non uniform meshes. Regardless of this, [7] focused on a synchronized difference approach for a reaction-diffusion problem that includes discontinuous data that are singularly perturbed in second order. In [4, 24, 31, 33], the authors concentrated on third-order SPPs using IBC. Moreover, several researchers [13, 19, 21, 22] concentrated on singularly perturbed convection-diffusion equations of second order with IBC.

In this work, the Cubic B-spline Collocation Method (CBSCM) is applied to resolve the singularly perturbed reaction-diffusion equations with IBC. The CBSCM provides enhanced flexibility and precision, mainly for irregular domains, by approximating solutions with smooth cubic functions.

The structure of the article is as follows: The problem description is given in Section 2. The results like maximum principle, stability and derivative estimates are stated in the form of results in Section 3. Section 4 explains the mesh discretization and mesh width along with the detailed derivation of the Cubic B-spline scheme using collocation technique. Error estimates are derived in Section 5. The numerical results are presented in Section 6 and the conclusion is found in Section 7.

2 Statement of the problem

Inspired by [34, 1, 5, 2, 16, 17, 15, 10, 11, 6], we consider the following SPP with IBC

$$L\vartheta = -\mu\vartheta''(t) + q(t)\vartheta(t) = r(t), \ 0 < t < 1, \tag{2.1}$$

$$\vartheta(0) = A_1, \tag{2.2}$$

$$L_1\vartheta(1) = \vartheta(1) - \mu \int_0^1 f(t)\vartheta(t)\, dt = A_2. \tag{2.3}$$

Here, $0 < \mu < 1$ is the perturbation parameter. A_1 and A_2 are constants. Further $q(t) \geq \alpha > 0, \ t \in [0, 1], \ f(t)$ is non-negative with $\int_0^1 f(t)dt < 1$ and $r(t), \ q(t)$ being sufficiently smooth functions on Ω. Boundary layers at $t = 0$ and $t = 1$ are found in the solution $\vartheta(t)$ of the problem (2.1) - (2.3). In this work, $\sqrt{\mu} \leq CN^{-1}$ is taken into consideration and C is a general positive constant. The convergence of the numerical solution to the exact solution of a singular perturbation problem is analyzed using the supremum norm $||\vartheta||_D = \sup\limits_{t\in D} |\vartheta(t)|$.

3 The continuous problem

The preliminary results which are derived in [25] are stated here.

Theorem 3.1. *(Maximum Principle) Let $\vartheta(t) \in C^2(\Omega)$ be any function that satisfies $\vartheta(0) \geq 0, \ L_1\vartheta(t) \geq 0, \ \forall t \in \Omega$. Then $\vartheta(t) \geq 0, \ \forall t \in \bar{\Omega}$.*

Lemma 3.1. *For $1 \leq m \leq 4$, let $\vartheta(t)$ be the solution of (2.1) - (2.3). Then*

$$\|\vartheta^{(m)}\|_\Omega \leq C(1 + \mu^{-\frac{m}{2}}). \tag{3.1}$$

To obtain error estimates, we separate the solution $\vartheta(t)$ into singular and smooth parts in the following manner

$$\vartheta(t) = u(t) + w(t).$$

The derivative estimates for smooth and layer parts are given in the following lemma:

Lemma 3.2. *For the problem (2.1) - (2.3), the singular components $v_L(t)$ and $v_R(t)$ and the regular component $u(t)$ of the solution $\vartheta(t)$ satisfy the following bounds:*

$$
\begin{aligned}
|u^{(m)}(t)| &\leq C(1+e^{-\frac{(k-2)}{2}}),\\
|w_L^{(m)}(t)| &\leq Ce^{-\frac{m}{2}}(\mu+e^{-t\sqrt{\frac{\alpha}{\mu}}}),\\
|w_R^{(m)}(t)| &\leq Ce^{-\frac{m}{2}}e^{(1-t)\sqrt{\frac{\alpha}{\mu}}},\ 0\leq m\leq 4,\ \forall\ t\in\Omega.
\end{aligned}
$$

4 Cubic B-spline Collocation Method (CBSCM)

For mesh construction, the widely used Shishkin mesh is employed. Take the boundary value problem (2.1) - (2.3) as an example. The boundary layers for the problem being examined are located at each end of the interval $[0,1]$. As a result, we split the interval into three smaller segments

$$[0,\nu],\ [\nu,1-\nu] \quad \text{and} \quad [1-\nu,1]$$

where $\nu=\min\left(\frac{1}{4},2\sqrt{\frac{\mu}{\alpha}}\ln N\right)$ which indicates the width of the boundary layer. In the aforementioned subdomains, each subinterval comprises n_1, n_2 and n_3 points, so that $n_1+n_2+n_3=N$. Define

$$
t_m=\begin{cases}
m\tilde{h}, & 0\leq m\leq N/4,\\
t_{N/4}+(m-N/4)\tilde{h}, & N/4+1\leq m\leq 3N/4,\\
t_{3N/4}+(m-3N/4)\tilde{h}, & 3N/4+1\leq m\leq N,
\end{cases}
$$

where

$$
\tilde{h}=\begin{cases}
\tilde{h}_1=4\nu/N, & 1\leq m\leq N/4,\\
\tilde{h}_2=2(1-2\nu)/N, & N/4+1\leq m\leq 3N/4,\\
\tilde{h}_3=4\nu/N, & 3N/4+1\leq m\leq N.
\end{cases}
$$

is the piecewise uniform mesh space and $\bar{\Omega}^N=\{t_m\}_{m=0}^N$.

4.1 Difference Scheme Derivation

This section employs the cubic B-spline collocation strategy to approximate the solution of the differential equations and the integral boundary conditions of the given problem (2.1)-(2.3).

Let $\bar{\Omega}$ be partitioned into $N+1$ number of mesh points such as $\psi=\{t_0,\ t_1,...,t_N\ \}$. The cubic B-splines $P_m(t)$ are defined [18, 20, 22, 29] as

$$
P_m(t)=\frac{1}{\tilde{h}^3}\begin{cases}
(t-t_{m-2})^3, & t_{m-2}\leq t\leq t_{m-1},\\
\tilde{h}^3+3\tilde{h}^2(t-t_{m-1})+3\tilde{h}(t-t_{m-1})^2-3(t-t_{m-1})^3, & t_{m-1}\leq t\leq t_m,\\
\tilde{h}^3+3\tilde{h}^2(t_{m+1}-t)+3\tilde{h}(t_{m+1}-t)^2-3(t_{m+1}-t)^3, & t_m\leq t\leq t_{m+1},\\
(t_{m+2}-t)^3, & t_{m+1}\leq t\leq t_{m+2},\\
0, & \text{otherwise},
\end{cases}
$$

where each $P_m(t)$ is continuously differentiable up to second order. By introducing three more fictitious points at both side of the end points t_0 and t_N such as $t_{-3} < t_{-2} < t_{-1} < t_0$ and $t_{N+3} > t_{N+2} > t_{N+1} > t_N$ in order to define $P_{-1}(t)$ and $P_{N+1}(t)$. Let $\Psi = \{P_{-1}, P_0, ..., P_{N+1}\}$ and let $Span(\Psi) = \phi_3(\psi)$. The function Ψ is linearly independent on $[0, 1]$ and hence $\phi_3(\psi)$ is $(N+3)$ dimensional. Let us define a linear operator $L : X \to X$, where X is the linear subspace of $L_2(\bar{\Omega})$, a vector space of all square integrable function. Define

$$v(t) = \sum_{m=-1}^{N+1} d_m P_m(t) \tag{4.1}$$

Here, d'_ms are real coefficients that need to be determined. The problems (2.1)-(2.3) takes the form.

$$Lv(t_m) = r(t_m), \quad 0 < m < N, \tag{4.2}$$

$$v(t_0) = A_1, \tag{4.3}$$

$$v(t_N) - \mu \sum_{m=1}^{N} \frac{f_{m-1}v_{m-1} + f_m v_m}{2} h_m = A_2. \tag{4.4}$$

After solving equations (4.2), we obtain a system of $N+1$ linear equations stated below in $N+3$ unknowns using the values of the B-spline functions P_m and their derivatives at the mesh points.

$$\left(\frac{-6\mu}{h_m^2} + q(t_m)\right) d_{m-1} + \left(\frac{12\mu}{h_m^2} + 4q(t_m)\right) d_m + \left(\frac{-6\mu}{h_m^2} + q(t_m)\right) d_{m+1} = r(t_m),\ 0 \le m \le N.$$

For simplicity, we represent the above system of equations in the form

$$E_m d_{m-1} + F_m d_m + G_m d_{m+1} = r_m, \qquad 0 \le m \le N, \tag{4.5}$$

where

$$E_m = \frac{-6\mu}{h_m^2} + q(t_m), \quad F_m = \frac{12\mu}{h_m^2} + 4q(t_m) \quad \text{and} \quad G_m = \frac{-6\mu}{h_m^2} + q(t_m).$$

From (4.3), we have

$$d_{-1} + 4d_0 + d_1 = A_1.$$

From the scheme (4.5), when $m = 0$, we have

$$E_0 d_{-1} + F_0 d_0 + G_0 d_1 = r_0.$$

After simplifying the above two equations, we get

$$d_0(-4E_0 + F_0) + d_1(-E_0 + G_0) = r_0 - A_1 E_0. \tag{4.6}$$

Similarly from Equation (4.4), we have

$$\begin{aligned} d_{N+1} = \gamma A_2 + \frac{\gamma\mu}{2} f_0 A_1 h_1 + \frac{\gamma\mu}{2}\{ & d_0(f_1h_1 + f_1h_2) + d_1(4f_1h_1 + 4f_1h_2 + f_2h_2 + f_2h_3) \\ & + d_2(f_1h_1 + f_1h_2 + 4f_2h_2 + 4f_2h_3 + f_3h_3 + f_3h_4) \\ & + \end{aligned}$$

$$\begin{aligned}
&\vdots\\
&+d_{N-2}(f_{N-3}h_{N-3}+f_{N-3}h_{N-2}+4f_{N-2}h_{N-2}+4f_{N-2}h_{N-1}+f_{N-1}h_{N-1}+f_{N-1}h_N)\}\\
&+\gamma d_{N-1}\left(\frac{\mu}{2}(f_{N-2}h_{N-2}+f_{N-2}h_{N-1}+4f_{N-1}h_{N-1}+4f_{N-1}h_N+f_Nh_N)-1\right)\\
&+\gamma d_N\left(\frac{\mu}{2}(f_{N-1}h_{N-1}+f_{N-1}h_N+4f_Nh_N)-4\right)
\end{aligned}$$

where $\gamma=\dfrac{2}{2-\mu f_Nh_N}$. Substituting the value of d_{N+1} in the Equation (4.5) when $m=N$, then the resulting equation follows

$$\begin{aligned}
&E_Nd_{N-1}+F_Nd_N+G_N[A_2\gamma+\frac{\gamma\mu}{2}[f_0A_1h_1+d_0(f_1h_1+f_2h_2)+d_1(4f_1h_1+4f_1h_2+f_2h_2+f_2h_3)\\
&+d_2(f_1h_1+f_1h_2+4f_2h_2+4f_2h_3+f_3h_3+f_3h_4)+d_3(f_2h_2+f_2h_3+4f_3h_3+4f_3h_4+f_4h_4+f_4h_5)\\
&+\ldots\\
&+d_{N-2}(f_{N-3}h_{N-3}+f_{N-3}h_{N-2}+4f_{N-2}h_{N-2}+4f_{N-2}h_{N-1}+f_{N-1}h_{N-1}+f_{N-1}h_N)]\\
&+d_{N-1}\left(\frac{\gamma\mu}{2}(f_{N-2}h_{N-2}+f_{N-2}h_{N-1}+4f_{N-1}h_{N-1}+4f_{N-1}h_N+f_Nh_N)-\gamma\right)\\
&+d_N\left(\frac{\gamma\mu}{2}(f_{N-1}h_{N-1}+f_{N-1}h_N+4f_Nh_N)-4\gamma\right)]=r_N.
\end{aligned}$$

On simplification, we get

$$\begin{aligned}
&d_0G_N\frac{\gamma\mu}{2}(f_1h_1+f_1h_2)+d_1G_N\frac{\gamma\mu}{2}(4f_1h_1+4f_1h_2+f_2h_2+f_2h_3)\\
&+d_2G_N\frac{\gamma\mu}{2}(f_1h_1+f_1h_2+4f_2h_2+4f_2h_3+f_3h_3+f_3h_4)\\
&+\\
&\vdots\\
&+d_{N-2}G_N\frac{\gamma\mu}{2}(f_{N-3}h_{N-3}+f_{N-3}h_{N-2}+4f_{N-2}h_{N-2}+4f_{N-2}h_{N-1}+f_{N-1}h_{N-1}+f_{N-1}h_N)\\
&+d_{N-1}(G_N\frac{\gamma\mu}{2}(f_{N-2}h_{N-2}+f_{N-2}h_{N-1}+4f_{N-1}h_{N-1}+4f_{N-1}h_N+f_Nh_N+f_{N-1}h_N)-\gamma G_N+E_N)\\
&+d_N(G_N\frac{\gamma\mu}{2}(f_{N-1}h_{N-1}+f_{N-1}h_N+4f_Nh_N)-4\gamma G_N+F_N)=r_N-G_NA_2\gamma-G_N\frac{\gamma\mu}{2}f_0A_1h_1.
\end{aligned}\tag{4.7}$$

Equations (4.5) along with (4.6) and (4.7) together form a system of equations $Ad=\delta$ where $d=(d_0,d_1,d_2,...,d_N)^T$ and $\delta=(r_0-A_1E_0,r_1,r_2,...,r_N-G_NA_2\gamma-G_N\gamma\mu f_0h_1/2)^T$. Further A is a diagonally dominant matrix and hence we can solve the system of equations. After solving, we can find the values of $d_0,d_1,d_2,...,d_N$ which are the solutions of the system of equations, from which we may calculate the values of d_{-1} and d_{N+1}. A unique solution $v(t)$ to the collocation technique with a basis of cubic B-splines applied to problems (2.1)-(2.3) is hence obtained from (4.1).

5 Error Estimate

Lemma 5.1. *The spline functions $P_{-1},P_0,P_1,...,P_{N+1}$ satisfy the inequality*

$$\sum_{m=-1}^{N+1}|P_m(t)|\le 10,\quad t\in\bar{\Omega}.$$

Theorem 5.1. *For the problem* (2.1) − (2.3) *under consideration, let* $v(t)$ *represent the collocation approximation from the cubic spline space* $\phi_3(\psi)$ *to the solution* $\vartheta(t)$*. If* $r \in C^2[0,1]$*, The parameter uniform error estimate is given by*

$$\sup_{0<\mu\leq 1}\max_{0\leq m\leq N}|\vartheta(t_m)-v(t_m)| \leq CN^{-2}(\ln N)^6$$

where C *is a positive constant independent of* μ *and* N*.*

Proof. $V(t)$ represents the unique spline interpolation from $\phi_3(\psi)$ to the BVP solution $\vartheta(t)$ of (2.1) − (2.3), as defined by

$$V(t) = \sum_{m=-1}^{N+1} \bar{d}_m P_m(t). \tag{5.1}$$

If $r(t) \in C^2[0,1]$, then $\vartheta(t) \in C^4[0,1]$ and according to de Boor and Hall error estimate [20] that

$$||D^j(\vartheta - V)||_\infty \leq C_j||\vartheta^{(4)}(t)||\bar{h}^{4-j}, \quad j = 0,1,2,3. \tag{5.2}$$

where $\bar{h} = \max\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\}$ and C_j are constants independent of μ and N.

$$\begin{aligned} |L\vartheta(t_m) - LV(t_m)| &= |-\mu\vartheta''(t_m) - q(t_m)\vartheta(t_m) - (-\mu V''(t_m) - q(t_m)V(t_m))|, \\ &\leq C(\mu\tilde{h}^2 + ||q||_\infty\bar{h}^4)||\vartheta^4(t_m)||, \\ &\leq C(\mu\bar{h}^2 + ||q||_\infty\bar{h}^4)(1 + \mu^{-4}\exp(-t_m\sqrt{\alpha/\mu}) + \mu^{-4}\exp(-(1-t_m)\sqrt{\alpha/\mu}). \end{aligned} \tag{5.3}$$

Two cases are now up for discussion: Case (i): Let $\nu = 1/4$. Then $1/4 \leq (2\sqrt{\mu/\alpha})\ln N$ which gives $\mu^{-1} \leq (C\ln N)^2$ and $\bar{h} = 1/N$. From (5.3), we obtain

$$|L\vartheta(t_m) - LV(t_m)| \leq CN^{-2}(\ln N)^6. \tag{5.4}$$

Case (ii): Let $\nu = 2\sqrt{\mu/\alpha}\ln N$. If m satisfies $N/4 \leq m \leq 3N/4$, we have Then (5.3) becomes

$$|L\vartheta(t_m) - LV(t_m)| \leq CN^{-2}. \tag{5.5}$$

Further, if m satisfies $1 \leq m \leq N/4$ and $3N/4 + 1 \leq m \leq N$, $\tilde{h} = 4\nu/N = CN^{-1}(\sqrt{\mu})\ln N$, and hence $\dfrac{\tilde{h}}{\sqrt{\mu}} = CN^{-1}\ln N$. subsequently by applying the lemma covered in [14], we can obtain from (5.3)

$$|L\vartheta(t_m) - LV(t_m)| \leq CN^{-2}(\ln N)^2. \tag{5.6}$$

From the above three equations, we get

$$|L\vartheta(t_m) - LV(t_m)| \leq CN^{-2}(\ln N)^6. \tag{5.7}$$

and hence we have

$$|Lv(t_m) - LV(t_m)| = |r(t_m) - LV(t_m)| = |L\vartheta(t_m) - LV(t_m)| \leq CN^{-2}(\ln N)^6. \tag{5.8}$$

Suppose that $LV(t_m) = \tilde{h}^2\bar{r}(t_m), \quad \forall\ 0 \leq m \leq N$ with the boundary conditions $V(t_0) = 0$, $V(t_N) = A_2 + \mu\int_0^1 f(t)\vartheta(t)dt$ leads to the linear system $A\bar{d} = \bar{\delta}$, which in turn results into

$$A(d-\bar{d}) = \delta - \bar{\delta}, \tag{5.9}$$

where $d-\bar{d} = (d_0 - \bar{d_0}, d_1 - \bar{d_1}, ..., d_N - \bar{d_N})^T$,

$$\delta - \bar{\delta} = (\tilde{h}^2(r(t_0) - \bar{r}(t_0)), \tilde{h}^2(r(t_1) - \bar{r}(t_1)), ..., \tilde{h}^2(r(t_N) - \bar{r}(t_N)))^T.$$

Using the equation (5.8), we get

$$||\delta - \bar{\delta}||_\infty \leq CN^{-2}(\ln N)^6. \tag{5.10}$$

Using (5.9) and (5.10) and using the result in [30], we have

$$|d_m - \bar{d_m}| \leq CN^{-2}(\ln N)^6,\ 0 \leq m \leq N.$$

Also from the boundary condition and from the above bounds, $|d_{-1} - \bar{d}_{-1}|$ and $|d_{N+1} - \bar{d}_{N+1}|$ provide the same estimates as above. Hence, we have

$$\max_{-1\leq m\leq N+1} |d_m - \bar{d_m}| \leq CN^{-2}(\ln N)^6.$$

From the above result and Lemma 5.1, we have

$$|v(t) - V(t)| \leq \max_{-1\leq m\leq N+1} |d_m - \bar{d}_m| \sum_{m=-1}^{N+1} |\mathscr{P}_m(t)| \leq CN^{-2}(\ln N)^6,$$

which gives

$$\max_{0\leq m\leq N} |v(t_m) - V(t_m)| \leq CN^{-2}(\ln N)^6.$$

Then, by the triangle inequality, we get

$$\sup_{0<\mu\leq 1}\max_{0\leq m\leq N} |\vartheta(t_m) - v(t_m)| \leq CN^{-2}(\ln N)^6.$$

□

6 Numerical Results

Example 1. [25]

$$-\mu\vartheta''(t) + \vartheta(t) = 1, \qquad 0 < t < 1,$$

with boundary conditions

$$\vartheta(0) = 0,\ \vartheta(1) - \mu\int_0^1 \frac{t}{2}\vartheta(t)dt = 0.$$

Its exact solution is provided by

$$\vartheta = \frac{(\mu - 2\mu^2 - 4 + 4e^{\frac{-1}{\sqrt{\mu}}}(1 + \frac{\mu^{3/2}}{2} + \frac{\mu^2}{2}))e^{\frac{t}{\sqrt{\mu}}}}{4e^{\frac{1}{\sqrt{\mu}}}(1 - \frac{\mu^{3/2}}{2} + \frac{\mu^2}{2}) - 4e^{\frac{-1}{\sqrt{\mu}}}(1 + \frac{\mu^{3/2}}{2} + \frac{\mu^2}{2})} + \frac{(2\mu^2 - \mu + 4 - 4e^{\frac{1}{\sqrt{\mu}}}(1 - \frac{\mu^{3/2}}{2} + \frac{\mu^2}{2}))e^{\frac{-t}{\sqrt{\mu}}}}{4e^{\frac{1}{\sqrt{\mu}}}(1 - \frac{\mu^{3/2}}{2} + \frac{\mu^2}{2}) - 4e^{\frac{-1}{\sqrt{\mu}}}(1 + \frac{\mu^{3/2}}{2} + \frac{\mu^2}{2})} + 1.$$

Given that an analytical solution exists for the problem, the maximum pointwise errors for each μ are approximated as follows:

$$E_\mu^N = \max_{t_i \in \bar{\Omega}} |\vartheta(t_m) - v^N(t_m)| \quad \text{and} \quad E^N = \max_{\mu} E_\mu^N$$

, where v^N represents the numerical solution. The parameter uniform convergence order is determined by

$$R_\mu^N = \log_2(\frac{E_\mu^N}{E_\mu^{2N}}) \ \text{ and } R^N = \log_2(\frac{E^N}{E^{2N}}).$$

Example 2. [25]

$$-\mu\vartheta''(t) + (5 + t)\vartheta(t) = 1, \qquad 0 < t < 1,$$

with boundary conditions

$$\vartheta(0) = 0, \ \vartheta(1) - \mu \int_0^1 \frac{t}{2}\vartheta(t)dt = 0.$$

For Example 2, the exact solution is not known and hence we employ the double mesh principle to estimate the error and find the experimental rate of convergence for the solution obtained. Let the double mesh difference be defined as $E_\mu^N = \max_{t_m \in \Omega^N} |v^N(t_m) - v^{2N}(t_m)|$ and let $E^N = \max_\mu D_\mu^N$ where $U^{2N}(t_m)$ represents the piecewise linear interpolate of the mesh function $U^{2N}(t_m)$ on the interval [0,1]. The order of convergence is determined from these quantities.

$$P_\mu^N = \log_2(\frac{E_\mu^N}{E_\mu^{2N}}) \text{ and } P^N = \log_2(\frac{E^N}{E^{2N}})$$

7 Discussion

The reaction-diffusion singular perturbation problem with non-local boundary conditions is studied using the CBSCM, resulting in a solution that shows boundary layers at $t = 0$ and $t = 1$. The layer adapted mesh namely Shishkin mesh is used to capture the layer behavior. The developed scheme resolves the problem arises due to the small perturbation parameter μ. The error of the scheme is determined through the discrete maximum norm. The numerical method is illustrated through two examples. Our numerical findings confirm the theoretical predictions. Tables 1,2 and 3 present the convergence rate and maximum pointwise errors for Examples 1 and 2 respectively. The error plots for Examples 1 and 2 are shown in Figure 3 and 4 to visually demonstrate how the maximum error E^N and D^N varies with the number of mesh points N for different values of μ. These plots confirm the theoretical convergence behaviour and enhance the clarity of results.Also the convergence rate of both examples are shown in Figure 5 .Based on the data, we can infer that this method is effective and indicates a superior rate of convergence.

Table 1: Numerical estimates for Example 1

	Number of mesh points N					
	64	128	256	512	1024	2048
2^{-1}	1.8987e-05	4.7465e-06	1.1866e-06	2.9666e-07	7.4164e-08	1.8541e-08
2^{-2}	4.6190e-05	1.1547e-05	2.8867e-06	7.2166e-07	1.8042e-07	4.5105e-08
2^{-3}	8.2934e-05	2.0729e-05	5.1819e-06	1.2955e-06	3.2386e-07	8.0966e-08
2^{-4}	1.3216e-04	3.3025e-05	8.2553e-06	2.0638e-06	5.1595e-07	1.2899e-07
2^{-5}	2.4327e-04	6.0752e-05	1.5185e-05	3.7964e-06	9.4907e-07	2.3727e-07
2^{-6}	4.7992e-04	1.1988e-04	2.9963e-05	7.4905e-06	1.8726e-06	4.6815e-07
2^{-7}	9.6355e-04	2.3985e-04	5.9898e-05	1.4971e-05	3.7424e-06	9.3558e-07
2^{-8}	1.9345e-03	4.7952e-04	1.1979e-04	2.9939e-05	7.4847e-06	1.8711e-06
2^{-9}	3.9220e-03	9.6354e-04	2.3985e-04	5.9898e-05	1.4970e-05	3.7423e-06
2^{-10}	4.2436e-03	1.4157e-03	4.6119e-04	1.1979e-04	2.9939e-05	7.4847e-06
2^{-11}	4.2436e-03	1.4157e-03	4.6119e-04	1.4572e-04	4.4947e-05	1.3597e-05
2^{-12}	4.2436e-03	1.4157e-03	4.6119e-04	1.4572e-04	4.4947e-05	1.3597e-05
2^{-13}	4.2436e-03	1.4157e-03	4.6119e-04	1.4572e-04	4.4947e-05	1.3597e-05
2^{-14}	4.2437e-03	1.4157e-03	4.6119e-04	1.4572e-04	4.4947e-05	1.3597e-05
2^{-15}	4.2437e-03	1.4157e-03	4.6119e-04	1.4572e-04	4.4947e-05	1.3597e-05
2^{-16}	4.2437e-03	1.4157e-03	4.6119e-04	1.4572e-04	4.4947e-05	1.3597e-05
2^{-17}	4.2437e-03	1.4157e-03	4.6119e-04	1.4572e-04	4.4947e-05	1.3597e-05
2^{-18}	4.2437e-03	1.4157e-03	4.6119e-04	1.4572e-04	4.4947e-05	1.3597e-05
E^N	1.8366e-02	5.7106e-03	1.8600e-03	5.8440e-04	1.7993e-04	5.4402e-05
R^N	1.6853	1.6183	1.6703	1.6995	1.7257	—

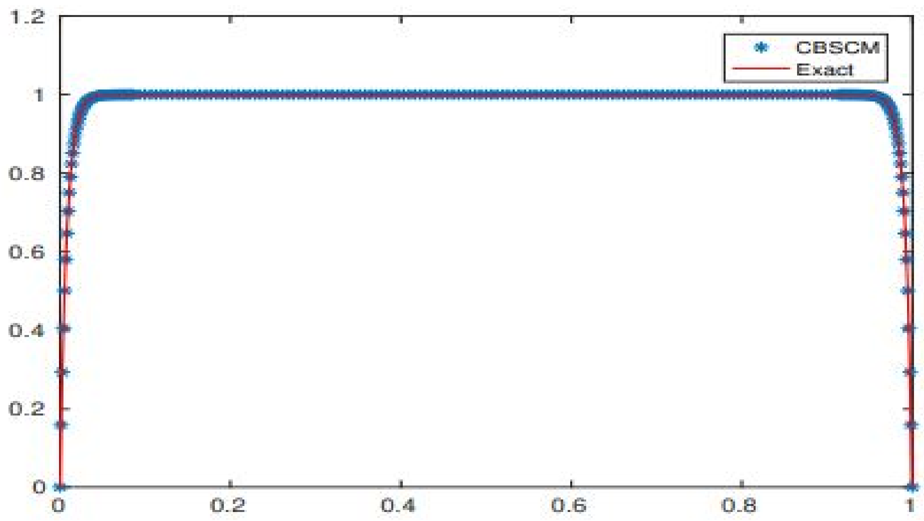

Figure 1: Solution graph of Example 1 using CBSCM for $\mu = 2^{-14}$ with $N = 256$.

Table 2: Numerical estimates for Example 2

	Number of mesh points N					
	64	128	256	512	1024	2048
2^{-1}	1.0202e-04	4.8916e-05	2.3936e-05	1.1837e-05	5.8860e-06	2.9348e-06
2^{-2}	1.6798e-04	8.0617e-05	3.9465e-05	1.9522e-05	9.7084e-06	4.8411e-06
2^{-3}	2.2522e-04	1.0874e-04	5.3404e-05	2.6461e-05	1.3171e-05	6.5703e-06
2^{-4}	2.5884e-04	1.2575e-04	6.2031e-05	3.0810e-05	1.5354e-05	7.6645e-06
2^{-5}	2.8524e-04	1.3518e-04	6.6482e-05	3.3017e-05	1.6458e-05	8.2165e-06
2^{-6}	3.4882e-04	1.4584e-04	6.9874e-05	3.4528e-05	1.7184e-05	8.5739e-06
2^{-7}	5.1288e-04	1.7586e-04	7.4226e-05	3.5866e-05	1.7780e-05	8.8607e-06
2^{-8}	8.8498e-04	2.5776e-04	8.8472e-05	3.7628e-05	1.8305e-05	9.0956e-06
2^{-9}	1.6518e-03	4.3737e-04	1.2895e-04	4.4454e-05	1.9014e-05	9.2988e-06
2^{-10}	3.4701e-03	8.1285e-04	2.1737e-04	6.4532e-05	2.2304e-05	9.5826e-06
2^{-11}	6.8747e-03	1.6054e-03	3.9889e-04	1.0852e-04	3.2302e-05	1.1181e-05
2^{-12}	1.3313e-02	3.4075e-03	7.7634e-04	1.9879e-04	5.4168e-05	1.6164e-05
2^{-13}	1.4743e-02	6.7181e-03	2.4590e-03	4.6303e-04	9.8741e-05	2.7103e-05
2^{-14}	1.4707e-02	6.7033e-03	2.4535e-03	8.5349e-04	2.9548e-04	9.6002e-05
2^{-15}	1.4683e-02	6.6928e-03	2.4497e-03	8.5096e-04	2.9465e-04	9.5616e-05
2^{-16}	1.4665e-02	6.6854e-03	2.4469e-03	8.4917e-04	2.9407e-04	9.5343e-05
2^{-17}	1.4653e-02	6.6801e-03	2.4450e-03	8.4790e-04	2.9365e-04	9.5150e-05
2^{-18}	1.4644e-02	6.6764e-03	2.4436e-03	8.4700e-04	2.9336e-04	9.5014e-05
D^N	1.4743e-02	6.7181e-03	2.4590e-03	8.5349e-04	2.9548e-04	9.6002e-05
P^N	1.1339	1.4500	1.5266	1.5303	1.6219	—

Table 3: Error estimates for Examples 1 and 2

	Number of mesh points N					
	64	128	256	512	1024	2048
E^N	1.8366e-02	5.7106e-03	1.8600e-03	5.8440e-04	1.7993e-04	5.4402e-05
R^N	1.6853	1.6183	1.6703	1.6995	1.7257	—
D^N	1.4743e-02	6.7181e-03	2.4590e-03	8.5349e-04	2.9548e-04	9.6002e-05
P^N	1.1339	1.4500	1.5266	1.5303	1.6219	—

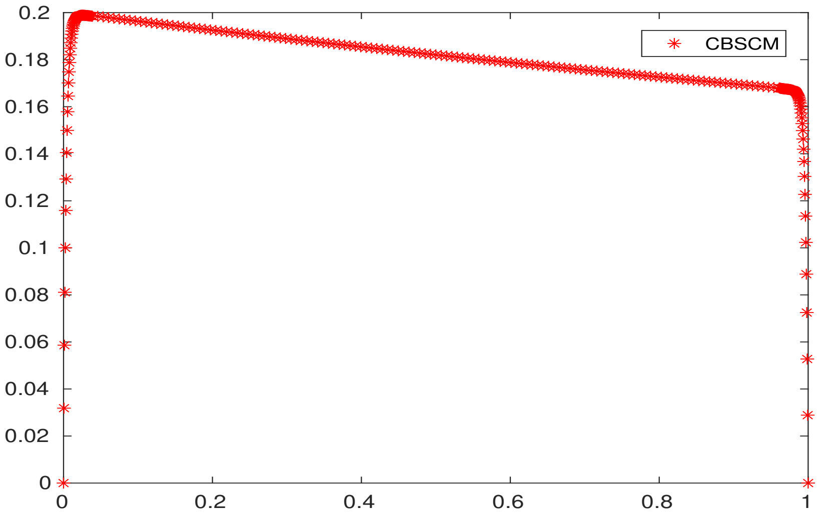

Figure 2: Solution graph of Example 2 using CBSCM for $\mu = 2^{-14}$ with $N = 256$.

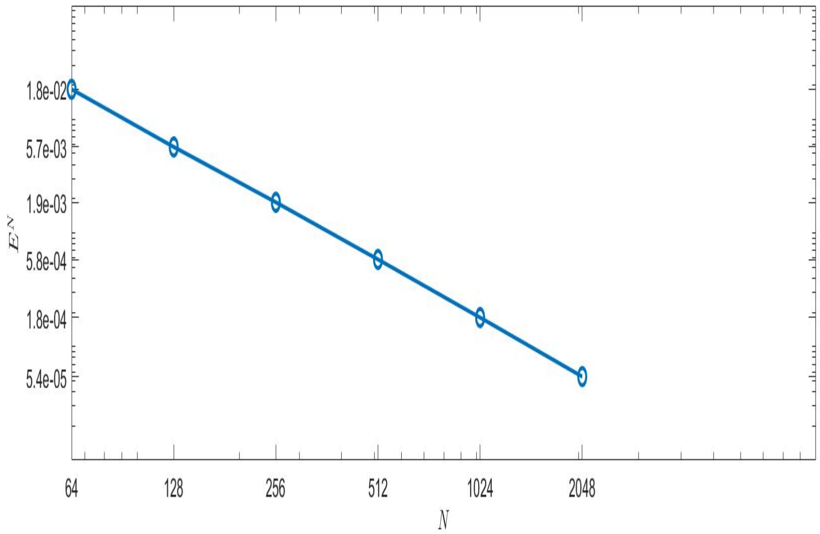

Figure 3: Error plot for Example 1

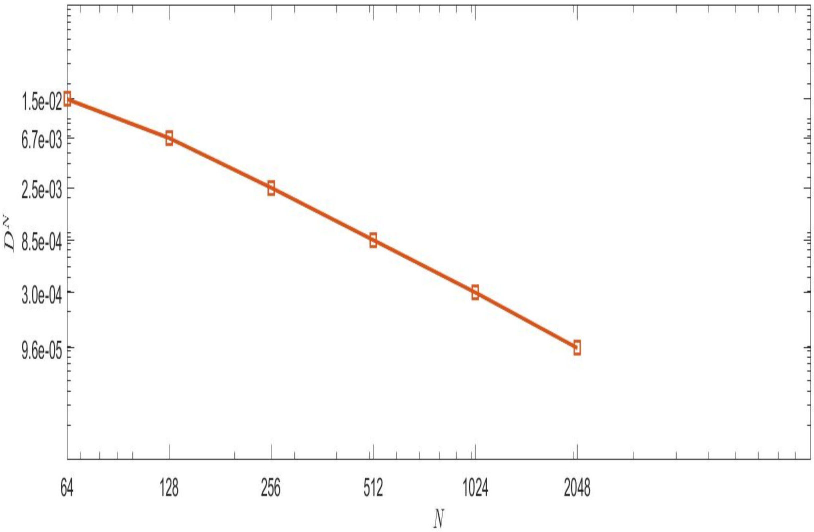

Figure 4: Error plot for Example 2

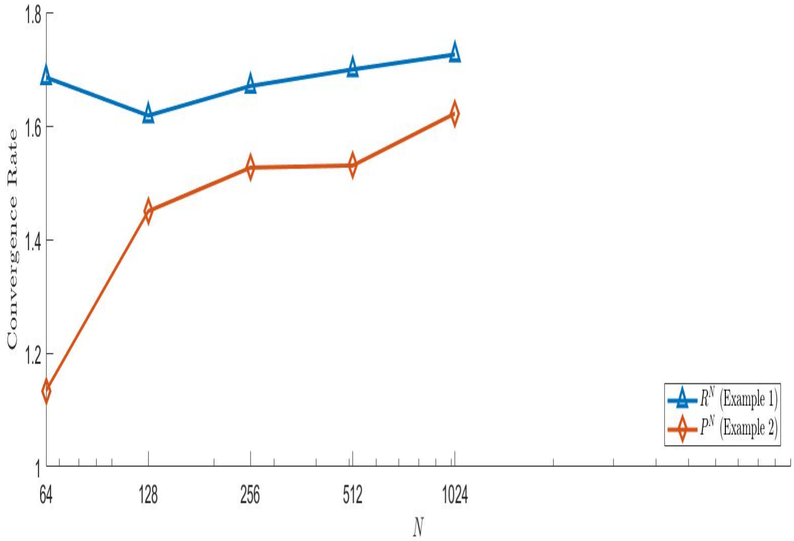

Figure 5: Convergence rate for Example 1 and 2

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