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New studies on a family of q-weighted Bergman spaces on the unit disk and applications

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Abstract. In this paper, we give a family of q-weighted Bergman spaces $\{\mathcal{A}_{\alpha,n,q}\}_{n\in\mathbb{N}}$ which satisfies the continuous inclusion $\mathcal{A}_{\alpha,n,q} \subset ... \subset \mathcal{A}_{\alpha,1,q} \subset \mathcal{A}_{\alpha,0,q} = \mathcal{A}_{\alpha,q}$, where $\mathcal{A}_{\alpha,q}$ the q-weighted Bergman space. Moreover, a more general uncertainty inequality of the Heisenberg-type for the space $\mathcal{A}_{\alpha,n,q}$ is given by considering the operators $\nabla_{\alpha,n,q} := \nabla^n_{\alpha,q}$ and $L_{\alpha,n,q} := L^n_{\alpha,q}$. Also, we study on $\mathcal{A}_{\alpha,q}$ the q-Toeplitz operators, the q-Hankel operators and the q-Berezin operators. Finally, an application of the theory of extremal function and reproducing kernel of Hilbert space is given and we use it to establish the extremal function associated to an bounded linear operator $T : \mathcal{A}_{\alpha,q} \to H$, for any Hilbert space H. As application, we come up with some results regarding the extremal functions associated to the difference operator $Tf(z) := \frac{1}{z}(f(z) - f(0))$ and $Tf(z) := \frac{1}{1+q}(f(z) - f(-z))$.

Keywords. q-weighted Bergman spaces; uncertainty inequality; q-Toeplitz operators; q-Berezin operators; q-Hankel operators; extremal function

1 Introduction

Over the past decade, there has been considerable progress in understanding the behavior of operators on Bergman and weighted Bergman spaces. This area of research has drawn increasing attention due to its deep connections with complex analysis, functional analysis, and operator theory [6]. Various specialized techniques have been devised to investigate different types of operators. For example, Hankel operators have been thoroughly analyzed using function-theoretic and operator-theoretic approaches ([1], [12]); composition operators have been studied through dynamical and analytic techniques ([14]); and multiplier operators have been explored in the context of reproducing kernel Hilbert spaces and boundedness criteria. These developments have significantly enriched the theory and opened new directions for further investigation.

The aim of this paper is to deal with operators acting on a general q-weighted Bergman spaces $\{\mathcal{A}_{\alpha,n,q}\}_{n\in\mathbb{N}}$. We prove some properties concerning q-Toeplitz operators, q-Hankel operators and q-Berezin operators; we establish a more general Heisenberg-type uncertainty principle

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given in [16] for the space $\mathcal{A}_{\alpha,n,q}$ by considering the operators $\nabla_{\alpha,n,q} := \nabla^n_{\alpha,q}$ and $L_{\alpha,n,q} := L^n_{\alpha,q}$; we give an application of the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator T. Noting that, there exist many similar uncertainty principles, in physics [2, 4, 10], and mathematics [3, 19], that are based on position, momentum, energy, time, and so on.

In one complex variable, the weighted Bergman space is one of the complex analysis tools used in harmonic analysis [7]. Let $\mathbb{C}$ be the complex plane, $\mathbb{D} = \Big\{z \in \mathbb{C} : |z| < 1\Big\}$ the open unit disk and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. For any $\alpha > 0$,

$$d\nu_\alpha(z) := \frac{1}{\pi}\alpha(1-|z|^2)^{\alpha-1}\mathrm{d}x\mathrm{d}y$$

is the weighted Lebesgue measure on $\mathbb{D}$. The weighted Bergman space $\mathcal{A}_\alpha$ is the space $H(\mathbb{D})\bigcap L^2(\mathbb{D}, d\nu_\alpha)$. Noting that, it is an Hilbert when space equipped with the inner product

$$\langle f, g\rangle_{\mathcal{A}_\alpha} := \int_{\mathbf{D}} f(z)\overline{g(z)}d\nu_\alpha(z),$$

and the norm $\|f\|_{\mathcal{A}_\alpha} = \|f\|_{L^2_{\alpha,q}(\mathbb{D})}$, see [16, 8, 20] for more details on the theory of Bergman spaces. The contents of the paper are as follows. Section 2 reviewers from [16] the q-analogue of the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$ and we will introduce the q-analogue of q-weighted Bergman spaces $\{\mathcal{A}_{\alpha,n,q}\}_{n\in\mathbb{N}}$ which satisfies the continuous inclusion $\mathcal{A}_{\alpha,n,q} \subset ... \subset \mathcal{A}_{\alpha,1,q} \subset \mathcal{A}_{\alpha,0,q} = \mathcal{A}_{\alpha,q}$. In Sect. 3, we will study the q-derivative operator $\nabla_{\alpha,n,q}$ and its adjoint operator $L_{\alpha,n,q}$ on the q-weighted Bergman space $\mathcal{A}_{\alpha,n,q}$, we will prove some properties concerning q-Toeplitz operators, q-Hankel operators and q-Berezin operators and we will establish at the end of this section a general uncertainty inequality of Heisenberg type for the space $\mathcal{A}_{\alpha,n,q}$. In Sect. 4, we will give an application of the theory of extremal function and reproducing kernel of Hilbert space by establishing the extremal function associated to a bounded linear operator T; as application we come up with some results regarding the extremal functions associated to the difference operator $Tf(z) := \frac{1}{z}(f(z) - f(0))$ and $Tf(z) := \frac{1}{1+q}(f(z) - f(-z))$, respectively.

2 Preliminaries

In all the sequel, consider $0 < q < 1$ and $\alpha > 0$. We refer the reader to [9] and [13] for the definitions and notations of the basic hypergeometric series, the Jackson's q-derivative and q-integrals, q-Gamma and q-Beta functions. The reference [16] is devoted to the q-weighted Bergman space on the disk.

The standard Watson's notation for the q-shifted factorials are defined for any complex number a by

$$(a;q)_0 := 1,\ (a;q)_n := \prod_{k=0}^{n}(1-aq^{k-1}), \quad n = 1, 2, ..., \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^{k-1}),$$

and $[a]_q$ is standing for the number associated to a,

$$[a]_q := \frac{1-q^a}{1-q},\quad [a]_q! := \frac{(q;q)_n}{(1-q)^n},\ n \in \mathbb{N}.$$

For any complex z, $(a;q)_z$ is defined by

$$(a;q)_z := \frac{(a;q)_\infty}{(aq^z;q)_\infty}, \tag{2.1}$$

and the q-binomial theorem [9] is given by

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.2}$$

The q-analogue of the classical Euler Gamma and Beta functions defined by Jackson in [11] are

$$\Gamma_q(a) := \frac{(q;q)_\infty}{(q^a;q)_\infty}(1-q)^{1-a}, \quad \Re(a) > 0.$$

$$\beta_q(a,b) := \int_0^1 t^{a-1}(qt;q)_{b-1} d_q t = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}, \quad \Re(a), \Re(b) > 0. \tag{2.3}$$

The q-analogue exponential functions $e_q(z)$ and $E_q(z)$ [9] are given by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{(1-q)z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},$$

$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(1-q)z^n}{(q;q)_n} = (-z;q)_\infty.$$

The q-derivative [9] on a subset of $\mathbb{C}$ is defined by

$$D_{q,z} f(z) := \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0. \tag{2.4}$$

In all the sequel, we need the following spaces:

- $H(\mathbb{D})$ the space of all analytic functions on the unit open disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$.
- $L^2_{\alpha,q}(\mathbb{D}) := L^2_q(\mathbb{D}, d\nu_{\alpha,q})$ the space of measurable functions f on the unit disk $\mathbb{D}$ satisfying

$$\| f \|^2_{L^2_{\alpha,q}(\mathbb{D})} := \frac{[\alpha]_q}{2\pi} \int_0^1 \Big(\int_0^{2\pi} | f(re^{i\theta}) |^2 \, d\theta \Big) (qr^2;q)_{\alpha-1} d_q(r^2) := \int_{\mathbb{D}} | f(z) |^2 \, d\nu_{\alpha,q}(z)$$

is finite, where $d\nu_{\alpha,q}$ [5] the measure defined on the unit disk $\mathbb{D}$ for $\alpha > 0$ by

$$d\nu_{\alpha,q}(z) := \frac{[\alpha]_q}{2\pi}(qr^2;q)_{\alpha-1} d_q(r^2) d\theta; \quad z = re^{i\theta},$$

and $d\theta$ is the usual Lebesgue measure on $[0, 2\pi[$ and the integral with respect to $d_q(r^2)$ is related to the q-Jackson's integral over $[0,1]$ defined by:

$$\int_0^1 f(t) d_q t := (1-q) \sum_{n=0}^{\infty} f(q^n) q^n.$$

- $\mathcal{A}_{\alpha,q} := \mathcal{A}_{\alpha,q}(\mathbb{D})$ the q-weighted Bergman space of all functions in $H(\mathbb{D}) \bigcap L^2_{\alpha,q}(\mathbb{D})$. It is a Hilbert space when equipped with the inner product
$$\langle f, g\rangle_{\mathcal{A}_{\alpha,q}} = \int_{\mathbb{D}} f(z)\overline{g(z)}d\nu_{\alpha,q}(z).$$
and the norm
$$\|f\|_{\mathcal{A}_{\alpha,q}} = \Big(\int_{\mathbb{D}} |f(z)|^2 d\nu_{\alpha,q}(z)\Big)^{1/2}.$$
- $\mathcal{A}_{\alpha,n,q} := \mathcal{A}_{\alpha,n,q}(\mathbb{D})$, the Hilbert space of functions on $H(\mathbb{D})$, such that
$$\begin{aligned} \|f\|^2_{\mathcal{A}_{\alpha,n,q}} &:= |f(0)|^2 + \int_{\mathbb{D}} \mid N_q^n f(z) \mid^2 d\nu_{\alpha,q}, \quad n = 1, 2, ... \\ \|f\|^2_{\mathcal{A}_{\alpha,0,q}} &:= \|f\|^2_{\mathcal{A}_{\alpha,q}}, \end{aligned}$$
N_q is the q-multiplication operator on $\mathcal{A}_{\alpha,q}$ given by $N_q := zD_{q,z}$.

  Moreover, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ then
$$\|f\|^2_{\mathcal{A}_{\alpha,n,q}} = |a_0|^2 + \sum_{k=1}^{\infty} [k]_q^{2n} C_k(\alpha; q)|a_k|^2,$$
where
$$C_n(\alpha; q) := \frac{(q; q)_n}{(q^{\alpha+1}; q)_n}.$$

3 Uncertainty inequality on the q-weighted Bergman space $\mathcal{A}_{\alpha,n,q}$

In tyhis section, we will prove some properties concerning q-Toeplitz operators, q-Hankel operators and q-Berezin operators on the the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$ and we will establish at the end of this section a general uncertainty inequality of Heisenberg type for the space on the q-weighted Bergman space $\mathcal{A}_{\alpha,n,q}$.

3.1 Operators on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$

The q-operator $\nabla_{\alpha,q}$ and $L_{\alpha,q}$ [16] are the operators on $\mathcal{A}_{\alpha,q}$ defined by
$$\nabla_{\alpha,q} := q^{-\alpha-1} D_{q,z}, \qquad N_q := zD_{q,z} \qquad L_{\alpha,q} := z^2 D_{q,z} + [\alpha + 1]_q q^{-\alpha-1} z.$$
We recall the following q-commutation relation

Lemma 3.1. *[16]* $[\nabla_{\alpha,q}, L_{\alpha,q}]_q := \nabla_{\alpha,q} L_{\alpha,q} - L_{\alpha,q}\nabla_{\alpha,q} = q^{-\alpha-1}\Lambda_q\Big([\alpha+1]_q I + (1+q^{-1})q^{-\alpha-1}N_q\Big),$ *where I is the identity operator and Λ_q is the q-shift operator given by $\Lambda_q f(z) = f(qz)$.*

The domain of the operator $\nabla_{\alpha,q}$ denoted by $\mathrm{Dom}(\nabla_{\alpha,q})$ is defined by
$$\mathrm{Dom}(\nabla_{\alpha,q}) := \Big\{ f \in \mathcal{A}_{\alpha,q}(\mathbf{D}, d\nu_{\alpha,q});\ \nabla_{\alpha,q} f \in \mathcal{A}_{\alpha,q}(\mathbf{D}, d\nu_{\alpha,q}) \Big\},$$
and same for $\mathrm{Dom}_q(N_q)$ and $\mathrm{Dom}_q(L_{\alpha,q})$.

Lemma 3.2. *The operators $\nabla_{\alpha,q}$, N_q and $L_{\alpha,q}$ satisfies the following*

(i) $Dom(\nabla_{\alpha,q}) = Dom(L_{\alpha,q}) = Dom(N_q) = \mathcal{A}_{\alpha,1,q}$.

(ii) For any f, g in $\mathcal{A}_{\alpha,1,q}$ we have: $\langle \nabla_{\alpha,q} f, g\rangle_{\mathcal{A}_{\alpha,q}} = \langle f, L_{\alpha,q} g\rangle_{\mathcal{A}_{\alpha,q}}$.

(iii) For any f in $\mathcal{A}_{\alpha,1,q}$ we have

$$\| L_{\alpha,q} f \|^2_{\mathcal{A}_{\alpha,q}} = \| \nabla_{\alpha,q} f \|^2_{\mathcal{A}_{\alpha,q}} + q^{-\alpha-1}[\alpha+1]_q \| \Lambda_{q^{1/2}} f \|^2_{\mathcal{A}_{\alpha,q}} + q^{-\alpha-1}(1+q^{-1})\langle N_q \Lambda_{q^{1/2}} f, \Lambda_{q^{1/2}} f\rangle_{\mathcal{A}_{\alpha,q}}.$$

Proof. Let $f \in \mathcal{A}_{\alpha,1,q}$, with $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then using relation (2.4), we have respectively

$$\nabla_{\alpha,q} f(z) = \sum_{k=1}^{\infty} q^{-\alpha-1}[k]_q a_k z^{k-1} = \sum_{k=0}^{\infty} q^{-\alpha-1}[k+1]_q a_{k+1} z^k \tag{3.1}$$

and

$$L_{\alpha,q} f(z) = \sum_{k=0}^{\infty}([k]_q + q^{-\alpha-1}[\alpha+1]_q) a_k z^{k+1} = \sum_{k=1}^{\infty}([k-1]_q + q^{-\alpha-1}[\alpha+1]_q) a_{k-1} z^k. \tag{3.2}$$

Thus from the previous relation, we get

$$\| \nabla_{\alpha,q} f \|^2_{\mathcal{A}_{\alpha,q}} = \langle \nabla_{\alpha,q} f, \nabla_{\alpha,q} f\rangle = \langle f, L_{\alpha,q}\nabla_{\alpha,q} f\rangle = \sum_{k=1}^{\infty} q^{-\alpha-1}[k]_q \Big([k-1]_q + q^{-\alpha-1}[\alpha+1]_q\Big)|a_k|^2 C_k(\alpha;q), \tag{3.3}$$

$$\| L_{\alpha,q} f \|^2_{\mathcal{A}_{\alpha,q}} = \langle L_{\alpha,q} f, L_{\alpha,q} f\rangle = \langle f, \nabla_{\alpha,q} L_{\alpha,q} f\rangle = \sum_{k=1}^{\infty} q^{-\alpha-1}[k+1]_q \Big([k]_q + q^{-\alpha-1}[\alpha+1]_q\Big)|a_k|^2 C_k(\alpha;q), \tag{3.4}$$

and

$$\| N_q f \|^2_{\mathcal{A}_{\alpha,q}} = \langle N_q f, N_q f\rangle_{\mathcal{A}_{\alpha,q}} = \sum_{k=1}^{\infty} [k]_q^2 |a_k|^2 C_k(\alpha;q). \tag{3.5}$$

Therefore, from Proposition 3.1, (3.3), (3.4) and (3.5) we deduce easily

$$\begin{aligned}
\|f\|^2_{\mathcal{A}_{\alpha,1,q}} - |f(0)|^2 &\leq \|\nabla_{\alpha,q} f\|^2_{\mathcal{A}_{\alpha,q}} \leq (1 + q^{-\alpha-1}[\alpha+1]_q)\|f\|^2_{\mathcal{A}_{\alpha,1,q}} \\
\|f\|^2_{\mathcal{A}_{\alpha,1,q}} &\leq \|L_{\alpha,q} f\|^2_{\mathcal{A}_{\alpha,q}} \leq [2]_q(1 + q^{-\alpha-1}[\alpha+1]_q)\|f\|^2_{\mathcal{A}_{\alpha,1,q}} \\
\|f\|^2_{\mathcal{A}_{\alpha,1,q}} - |f(0)|^2 &\leq \|N_q f\|^2_{\mathcal{A}_{\alpha,q}} \leq \|f\|^2_{\mathcal{A}_{\alpha,1,q}}.
\end{aligned}$$

So, $\mathrm{Dom}(\nabla_{\alpha,q}) = \mathrm{Dom}(L_{\alpha,q}) = \mathrm{Dom}(N_q) = \mathcal{A}_{\alpha,1,q}$.

To prove (ii), let f, g in $\mathcal{A}_{\alpha,1,q}$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$. From Proposition 3.1, (3.1) and (3.2) we have

$$\begin{aligned}
\langle \nabla_{\alpha,q} f, g\rangle_{\mathcal{A}_{\alpha,q}} &= \sum_{k=0}^{\infty} q^{-\alpha-1}[k+1]_q a_{k+1}\bar{b}_k C_k(\alpha;q) \\
&= \sum_{k=0}^{\infty} a_{k+1}\bar{b}_k \frac{(q;q)_{k+1}}{(1-q)q^{\alpha+1}(q^{\alpha+1};q)_k} \\
&= \sum_{k=1}^{\infty} a_k \bar{b}_{k-1} \frac{(q;q)_k}{(1-q)q^{\alpha+1}(q^{\alpha+1};q)_{k-1}},
\end{aligned}$$

on the other hand

$$\begin{aligned}\langle f, L_{\alpha,q}g\rangle_{\mathcal{A}_{\alpha,q}} &= \sum_{k=0}^{\infty}([k-1]_q + q^{-\alpha-1}[\alpha+1]_q)[k+1]_q a_k\bar{b}_{k-1}C_k(\alpha;q)\\ &= \sum_{k=0}^{\infty}\frac{1-q^{\alpha+k}}{1-q}a_k\bar{b}_{k-1}C_k(\alpha;q)\\ &= \sum_{k=1}^{\infty}[k+\alpha]_q a_k\bar{b}_{k-1}\frac{(q;q)_k}{(q^{\alpha+1};q)_k}\\ &= \sum_{k=1}^{\infty}a_k\bar{b}_{k-1}\frac{(q;q)_k}{q^{\alpha+1}(1-q)(q^{\alpha+1};q)_{k-1}} = \langle\nabla_{\alpha,q}f, g\rangle_{\mathcal{A}_{\alpha,q}}.\end{aligned}$$

Finally, to prove (iii), using $[k+1]_q = [k]_q + q^k$ we deduce easily that

$$\begin{aligned}[k+1]_q\Big([k]_q + q^{-\alpha-1}[\alpha+1]_q\Big) &= \Big([k]_q + q^k\Big)\Big([k-1]_q + q^{k-1} + q^{-\alpha-1}[\alpha+1]_q\Big)\\ &= [k]_q\Big([k-1]_q + q^{-\alpha-1}[\alpha+1]_q\Big) + q^{k-\alpha-1}[\alpha+1]_q + \Big(1+q^{-1}\Big)q^k[k]_q.\end{aligned}$$

Which leads to the result using (3.3), (3.4), (3.5) and the fact that $\Lambda_q N_q = N_q\Lambda_q$. □

We derive the following results

Proposition 3.1. *Let $f, g \in \mathcal{A}_{\alpha,q}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, we have*

(i) $\langle f, g\rangle_{\mathcal{A}_{\alpha,q}} = \sum_{n=0}^{\infty} a_n\bar{b}_n\frac{(q;q)_n}{(q^{\alpha+1};q)_n} = \sum_{n=0}^{\infty} a_n\bar{b}_n\ C_n(\alpha;q).$

(ii) $||f||^2_{\mathcal{A}_{\alpha,q}} = \sum_{n=0}^{\infty}|a_n|^2\frac{(q;q)_n}{(q^{\alpha+1};q)_n} = \sum_{n=0}^{\infty}|a_n|^2\ C_n(\alpha;q).$

(iii) The set $\left\{\xi^{\alpha}_{n,q}(z) := \frac{z^n}{\sqrt{C_n(\alpha;q)}}\right\}_{n\geq 0}$*, forms a Hilbert's basis for the space $\mathcal{A}_{\alpha,q}$.*

Proof. Given $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, the result follows by using dominate convergence theorem and relation (4.6) in [5] we have

$$\langle f, g\rangle_{\mathcal{A}_{\alpha,q}} = \sum_{m,n=0}^{\infty} a_m\bar{b}_n\int_{\mathbb{D}} z^m\ \overline{z^n}\ d\nu_{\alpha,q}(z) = \sum_{n=0}^{\infty} a_m\bar{b}_n\frac{(q;q)_n}{(q^{\alpha+1};q)_n}. \tag{3.6}$$

The last assertion follows directly from Proposition 4.1 in [5]. □

Theorem 3.1. *The function $\mathcal{K}_{\alpha,q}$ given for $w, z \in \mathbb{D}$, by*

$$\mathcal{K}_{\alpha,q}(z,w) = \mathcal{K}_{\alpha,q}(z\overline{w}) = \frac{1}{(z\overline{w};q)_{\alpha+1}}, \tag{3.7}$$

is a reproducing kernel for the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$.
That is

(i) *for all* $w \in \mathbb{D}$, $z \longmapsto \mathcal{K}_{\alpha,q}(z,w)$ *belong to* $\mathcal{A}_{\alpha,q}$.

(ii) *for all* $w, z \in \mathbb{D}$ *and* $f \in \mathcal{A}_{\alpha,q}$, *we have* $\langle f, \mathcal{K}_{\alpha,q}(.,w)\rangle_{\mathcal{A}_{\alpha,q}} = f(w)$.

(iii) *For all* $f \in \mathcal{A}_{\alpha,q}$ *and* $z \in \mathbb{C}$, $\mid f(z) \mid \leq \left[e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)\right]^{1/2} \parallel f \parallel_{\mathcal{A}_{\alpha,q}}$.

(iv) *Let* $w \in \mathbb{D}$. *The function* $u(z) = \mathcal{K}_{\alpha,q}(z\overline{w})$ *is the unique analytic solution on* $\mathbb{D}$ *of the initial problem*

$$z\nabla_{\alpha,q}u(z) = wL_{\alpha,q}u(z), \quad u(0) = 1.$$

Proof. To prove the first assertion (i), we use Proposition 3.1 (iii) the function $\xi^{\alpha}_{n,q}(z)$ constitute an orthonormal basis of $\mathcal{A}_{\alpha,q}$. Therefore for any $z, w \in \mathbb{D}$, $\mathcal{K}_{\alpha,q}$ can be computed by evaluating the following sum

$$\mathcal{K}_{\alpha,q}(z,w) = \sum_{n=0}^{\infty} \xi^{\alpha}_{n,q}(z)\overline{\xi^{\alpha}_{n,q}(w)} = \sum_{n=0}^{\infty} \frac{1}{C_n(\alpha;q)} z^n\overline{w}^n.$$

Hence by (2.1) combined with (2.2) we deduce easily

$$\mathcal{K}_{\alpha,q}(z,w) = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1};q)_n}{(q;q)_n}(z\overline{w})^n = \frac{(q^{\alpha+1}z\overline{w};q)_\infty}{(z\overline{w};q)_\infty} = \frac{1}{(z\overline{w};q)_{\alpha+1}}.$$

To prove (ii), we use the same as in Proposition 4.2 in [5]. The last assertion follows by using (2.4). □

3.1.1 q-Toepliz Operator on $\mathcal{A}_{\alpha,q}$

Consider the orthogonal projection operator $P_{\alpha,q} : L^2_{\alpha,q}(\mathbb{D}) \to \mathcal{A}_{\alpha,q}$. Since $L^2_{\alpha,q}(\mathbb{D}) = \mathcal{A}_{\alpha,q} \oplus \mathcal{A}^{\perp}_{\alpha,q}$ then for any $f \in L^2_{\alpha,q}(\mathbb{D})$, we have $f = (f - f^{\perp}) + f^{\perp}$ where $f - f^{\perp} \in \mathcal{A}_{\alpha,q}$ and $f^{\perp} \in \mathcal{A}^{\perp}_{\alpha,q}$. Furthermore, for $z \in \mathbb{D}$,

$$P_{\alpha,q}f(z) = (f - f^{\perp})(z) = \langle (f - f^{\perp})(z), \mathcal{K}_{\alpha,q}(z,.)\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f(z), \mathcal{K}_{\alpha,q}(z,.)\rangle_{L^2_{\alpha,q}(\mathbb{D})},$$

where $\mathcal{K}_{\alpha,q}$ is the reproducing kernel given by (3.7). The following assertions then follow

Proposition 3.2. *For all* $f, g \in L^2_{\alpha,q}(\mathbb{D})$, *we have:*

(i) $P_{\alpha,q} \circ P_{\alpha,q}f = P_{\alpha,q}f$.

(ii) $\langle P_{\alpha,q}f, g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f, P_{\alpha,q}g\rangle_{L^2_{\alpha,q}(\mathbb{D})}$.

(iii) *The operator* $P_{\alpha,q}$ *is bounded with* $\parallel P_{\alpha,q} \parallel = 1$ *and* $\parallel I - P_{\alpha,q} \parallel \leq 1$.

Let $\phi \in L^\infty(\mathbb{D})$. The q-multiplication operators M_ϕ are the operators defined by

$$M_\phi : L^2_{\alpha,q}(\mathbb{D}) \to L^2_{\alpha,q}(\mathbb{D}), \quad M_\phi f(z) := \phi(z)f(z), \ z \in \mathbb{D}.$$

The q-Toepliz operators T_ϕ are the operators defined by

$$T_\phi : \mathcal{A}_{\alpha,q} \to \mathcal{A}_{\alpha,q}, \quad T_\phi f(z) := P_{\alpha,q}M_\phi(z)f(z), \ z \in \mathbb{D}.$$

Theorem 3.2. *Let* $\phi \in L^\infty(\mathbb{D})$.

(i) The operators T_ϕ are bounded and $\| T_\phi \| \leq \| \phi \|_\infty$.

(ii) For all $f, g \in \mathcal{A}_{\alpha,q}$, we have

$$\langle T_\phi f, g\rangle_{\mathcal{A}_{\alpha,q}} = \langle f, T_{\overline{\phi}} g\rangle_{\mathcal{A}_{\alpha,q}}.$$

Proof. Let $\phi \in L^\infty(\mathbb{D})$. To prove (i), let $f \in \mathcal{A}_{\alpha,q}$ then from Proposition 3.2 (iii) we have

$$\| T_\phi f \|_{\mathcal{A}_{\alpha,q}} = \| P_{\alpha,q} M_\phi f \|_{\mathcal{A}_{\alpha,q}} = \| P_{\alpha,q}(\phi f) \|_{\mathcal{A}_{\alpha,q}} \leq \| \phi f \|_{L^2_{\alpha,q}(\mathbb{D})} \leq \| \phi f \|_{L^\infty(\mathbb{D})} \| f \|_{\mathcal{A}_{\alpha,q}}.$$

Thus, $\| T_\phi \| \leq \| \phi \|$.
To prove the second assertion, we use the fact that for any $f, g \in \mathcal{A}_{\alpha,q}$, $P_{\alpha,q} f = f$ and $P_{\alpha,q} g = g$. From Proposition 3.2 (ii), we obtain

$$\langle T_\phi f, g\rangle_{\mathcal{A}_{\alpha,q}} = \langle \phi f, P_{\alpha,q} g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f, \overline{\phi} g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle P_{\alpha,q} f, T_{\overline{\phi}} g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f, T_{\overline{\phi}} g\rangle_{\mathcal{A}_{\alpha,q}}.$$

□

Theorem 3.3. *Let $\phi \in L^\infty(\mathbb{D})$ has compact support, then T_ϕ is a compact operator.*

Proof. Let $\phi \in L^\infty(\mathbb{D})$ and $n, m = 0, 1, 2, \ldots$. From Proposition 3.1, we have

$$T_\phi \xi^\alpha_{n,q}(z) = \sum_{m=0}^{\infty} \frac{\langle T_\phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{L^2_{\alpha,q}(\mathbb{D})}}{C_m(\alpha; q)} z^m.$$

So,

$$\langle T_\phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{\mathcal{A}_{\alpha,q}} = \langle \phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{L^2_{\alpha,q}(\mathbb{D})}$$

Since $\phi \in L^\infty(\mathbb{D})$ with compact support, there exist a positive constant a and K such that $| \phi(z) | \leq a$ and $\phi(z) = 0$, for any $| z | > a$. Then for all $n, m \in \mathbb{N}$, we get from (2.3) and Proposition 3.1 (i),

$$\langle \phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \frac{1}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}} \int_{|z| \leq a} \phi(z) z^n \overline{z^m} d\nu_{\alpha,q}(z)$$

Thus, we obtain

$$\begin{aligned} \left| \langle \phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{L^2_{\alpha,q}(\mathbb{D})} \right| &\leq \frac{K}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}} \int_{|z| \leq a} |z|^{n+m} d\nu_{\alpha,q}(z) \\ &\leq \frac{2K}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}} \int_0^a r^{n+m} d\nu_{\alpha,q}(z) \\ &\leq \frac{2K[\alpha]_q a^{n+m}}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}} \int_0^1 (qr^2; q)_{\alpha-1} d_q(r^2) \\ &\leq \frac{K a^{n+m}}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}}. \end{aligned}$$

Hence,

$$\sum_{n,m=0}^{\infty} \frac{\left| \langle T_\phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{\mathcal{A}_{\alpha,q}} \right|^2}{C_n(\alpha; q) C_m(\alpha; q)} \leq 4K^2 \left(\sum_{n=0}^{\infty} \frac{a^{2n}}{C_n(\alpha; q)} \right)^2 \leq 4K^2 e_q(a^2) (q^{\alpha+1}; q)^2_\infty < \infty$$

Then T_ϕ is an Hilbert-Schmidt operator, and consequently it is compact. □

3.1.2 q-Hankel Operator on $\mathcal{A}_{\alpha,q}$

Let $\phi \in L^\infty(\mathbb{D})$. The q-Hankel operators H_ϕ are the operators defined by

$$H_\phi : \mathcal{A}_{\alpha,q} \to \mathcal{A}_{\alpha,q}, \quad H_\phi := (I - P_{\alpha,q})M_\phi.$$

Theorem 3.4. *Let $\phi, \psi \in L^\infty(\mathbb{D})$.*

(i.) The operators H_ϕ are bounded and $\| H_\phi \| \leq \| \phi \|_\infty$.

(ii) For all $f \in \mathcal{A}_{\alpha,q}$ and $g \in L^2_{\alpha,q}(\mathbb{D})$, we have

$$\langle H_\phi f, g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f, H^*_\phi g\rangle_{\mathcal{A}_{\alpha,q}}, \quad H^*_\phi = P_{\alpha,q}M_{\overline{\phi}}(I - P_{\alpha,q}).$$

(iii) $T_{\phi\psi} - T_\phi T_\psi = H^*_{\overline{\phi}} H_\phi$.

Proof. Let $\phi, \psi \in L^\infty(\mathbb{D})$. To prove (i), from Proposition 3.2 (iii) for any $f \in \mathcal{A}_{\alpha,q}$

$$\| H_\phi \|_{L^2_{\alpha,q}(\mathbb{D})} = \| (I - P_{\alpha,q}) \|_{L^2_{\alpha,q}(\mathbb{D})} \leq \| \phi f \|_{L^2_{\alpha,q}(\mathbb{D})} \leq \| \phi \|_{L^\infty_{\alpha,q}(\mathbb{D})} \| \phi f \|_{L^2_{\alpha,q}(\mathbb{D})} .$$

So, $\| H_\phi \| \leq \| \phi \|_{L^\infty(\mathbb{D})}$.
(ii) Let $f \in \mathcal{A}_{\alpha,q}$ and $g \in L^2_{\alpha,q}(\mathbb{D})$. From Proposition 3.2 (ii) and the fact that $P_{\alpha,q} f = f$ we obtain

$$\begin{aligned}
\langle H_\phi f, g\rangle_{L^2_{\alpha,q}(\mathbb{D})} &= \langle \phi f, g\rangle_{L^2_{\alpha,q}(\mathbb{D})} - \langle \phi f, P_{\alpha,q} g\rangle_{L^2_{\alpha,q}(\mathbb{D})} \\
&= \langle f, \overline{\phi}(I - P_{\alpha,q})g\rangle_{L^2_{\alpha,q}(\mathbb{D})} \\
&= \langle P_{\alpha,q} f, \overline{\phi}(I - P_{\alpha,q})g\rangle_{L^2_{\alpha,q}(\mathbb{D})} \\
&= \langle f, P_{\alpha,q} M_{\overline{\phi}}(I - P_{\alpha,q})g\rangle_{\mathcal{A}_{\alpha,q}}.
\end{aligned}$$

(iii) Let $\phi, \psi \in L^\infty(\mathbb{D})$. Then

$$H^*_{\overline{\phi}} H_\psi = P_{\alpha,q} M_\phi (I - P_{\alpha,q})^2 M_\psi = P_{\alpha,q} M_\phi (I - P_{\alpha,q}) M_\psi = P_{\alpha,q} M_{\phi\psi} - P_{\alpha,q} M_\phi P_{\alpha,q} M_\psi = T_{\phi\psi} - T_\phi T_\psi.$$

□

3.1.3 q-Berezin Operators on $\mathcal{A}_{\alpha,q}$

The q-normalized Fock kernel denoted by $\mathcal{S}^{\alpha,q}_z$ is given by for any $z, w \in \mathbb{D}$

$$\mathcal{S}^{\alpha,q}_z(w) := \frac{\mathcal{K}_{\alpha,q}(z,w)}{\| \mathcal{K}_{\alpha,q}(z,.) \|_{\mathcal{A}_{\alpha,q}}} := \frac{\sqrt{(|z|^2; q)_{\alpha+1}}}{(\overline{z}w; q)_{\alpha+1}}.$$

The q-Berezin transform of a function $f \in L^\infty(\mathbb{D})$ is defined by

$$\mathcal{B}er_{\alpha,q}(f)(z) := \langle T_f \mathcal{S}^{\alpha,q}_z, \mathcal{S}^{\alpha,q}_z\rangle_{\mathcal{A}_{\alpha,q}}.$$

Thus the following theorem follow immediately

Theorem 3.5. *Let $f \in L^\infty(\mathbb{D})$. Then $\| \mathcal{B}er_{\alpha,q}(f) \|_{L^\infty(\mathbb{D})} \leq \| f \|_{L^\infty(\mathbb{D})}$ and*

$$\mathcal{B}er_{\alpha,q}(f)(z) = \int_{\mathbb{D}} f(w) \mid \mathcal{S}^{\alpha,q}_z(w) \mid^2 d\nu_{\alpha,q}(w).$$

3.2 Uncertainty inequality on the q-weighted Bergman space $\mathcal{A}_{\alpha,n,q}$

In this subsection we will give a more general uncertainty inequality of the Heisenberg-type for the space $\mathcal{A}_{\alpha,n,q}$, by the virtue of the following lemma and theoorem:

Lemma 3.3. *[6] Let X and Y be self-adjoint operators on Hilbert space H (i.e $X^* = X$ and $Y^* = Y$). Then*

$$\| (X-a)f \|_H \| (Y-b)f \|_H \geq \frac{1}{2} \mid \langle [X,Y]f, f\rangle_H \mid,$$

for all f in $Dom(XY) \cap Dom(YX)$ and $a, b \in \mathbb{R}$

Theorem 3.6. *[16] Let $f \in \mathcal{A}_{\alpha,2,q}$. For all $a, b \in \mathbb{R}$, we have*

$$\begin{aligned} \| (\nabla_{\alpha,q} + L_{\alpha,q} - a)f \|_{\mathcal{A}_{\alpha,q}} \| (\nabla_{\alpha,q} - L_{\alpha,q} + ib)f \|_{\mathcal{A}_{\alpha,q}} &\geq q^{-\alpha-1}[\alpha+1]_q \| \Lambda_{q^{1/2}} f \|^2_{\mathcal{A}_{\alpha,q}} \\ &+ q^{-\alpha-1}(1+q^{-1})\langle N_q \Lambda_{q^{1/2}} f, \Lambda_{q^{1/2}} f\rangle_{\mathcal{A}_{\alpha,q}}. \end{aligned}$$

Consider the operators

$$\nabla_{\alpha,n,q} f(z) := \nabla^n_{\alpha,q} f(z) = q^{-n(\alpha+1)} D^n_{q,z} f(z), \qquad L_{\alpha,n,q} f(z) := L^n_{\alpha,q} f(z)$$

and the commutator operator

$$\Delta_{\alpha,n,q} f(z) := \big[\nabla_{\alpha,n,q} - L_{\alpha,n,q}\big]_q f(z) := \nabla_{\alpha,n,q} L_{\alpha,n,q} f(z) - L_{\alpha,n,q} \nabla_{\alpha,n,q} f(z).$$

The domain of the operator $\nabla_{\alpha,n,q}$ denoted by $\mathrm{Dom}(\nabla_{\alpha,n,q})$ defined by

$$\mathrm{Dom}(\nabla_{\alpha,n,q}) := \Big\{ f \in \mathcal{A}_{\alpha,q};\ \nabla_{\alpha,n,q} f \in \mathcal{A}_{\alpha,q} \Big\},$$

and same for $\mathrm{Dom}_q(L_{\alpha,n,q})$. So, we have the following q-commutation relation

Lemma 3.4. *$Dom(\nabla_{\alpha,n,q}) = Dom(L_{\alpha,n,q}) = \mathcal{A}_{\alpha,n,q}$.*

Proof. Let $f \in \mathcal{A}_{\alpha,q}$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then from relations (3.1), (3.2) and by induction on $n \in \mathbb{N}$ we obtain respectively

$$\nabla_{\alpha,n,q} f(z) = \sum_{k=n}^{\infty} q^{-n(\alpha+1)} \frac{[k]_q!}{[k-n]_q!} a_k z^{k-n} = \sum_{k=0}^{\infty} q^{-n(\alpha+1)} \frac{[k+n]_q!}{[k]_q!} a_{k+n} z^k,$$

$$L_{\alpha,n,q} f(z) = \sum_{k=0}^{\infty} \Big[\prod_{i=1}^{n} ([n+k-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big] a_k z^{n+k} = \sum_{k=n}^{\infty} \Big[\prod_{i=1}^{n} ([k-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big] a_{k-n} z^k.$$

Thus

$$\| L_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} = \langle L_{\alpha,n,q} f, L_{\alpha,n,q} f\rangle = \sum_{k=0}^{\infty} q^{-n(\alpha+1)} \frac{[k+n]_q!}{[k]_q!} \Big[\prod_{i=1}^{n} ([k+n-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big] |a_k|^2 C_k(\alpha;q), \tag{3.8}$$

and

$$\| \nabla_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} = \langle \nabla_{\alpha,n,q} f, \nabla_{\alpha,n,q} f\rangle = \sum_{k=n}^{\infty} q^{-n(\alpha+1)} \frac{[k]_q!}{[k-n]_q!} \Big[\prod_{i=1}^{n} ([k-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big] |a_k|^2 C_k(\alpha;q). \tag{3.9}$$

Now, using the fact that for any $n,k\in\mathbb{N}$

$$\begin{aligned}\frac{[k+n]_q!}{[k]_q!}\prod_{i=1}^{n}([k+n-i]_q+q^{-(\alpha+1)}[\alpha+1]_q) &\leq q^{-n(\alpha+1)}(1-q)^n[\alpha+1]_q^n\prod_{i=1}^{n}(1+q^{\alpha+1}\frac{[k+n-i]_q}{[\alpha+1]_q})\\ &\leq q^{-n(\alpha+1)}(q^{\alpha+1};q)_n\prod_{i=1}^{n}(1+\frac{q^{\alpha+1}}{1-q^{\alpha+1}})\\ &\leq q^{-n(\alpha+1)}\frac{(q^{\alpha+1};q)_n}{(1-q^{\alpha+1})^n}[k+n]_q^{2n},\end{aligned}$$

then we obtain respectively

$$\begin{aligned}\parallel \nabla_{\alpha,n,q}f\parallel^2_{\mathcal{A}_{\alpha,q}} &\leq \frac{(q^{\alpha+1};q)_n}{[q^{\alpha+1}(1-q^{\alpha+1})]^n}\sum_{k=0}^{\infty}[k+n]_q^{2n}|a_{k+n}|^2C_{k+n}(\alpha;q)\\ &= \frac{(q^{\alpha+1};q)_n}{[q^{\alpha+1}(1-q^{\alpha+1})]^n}\sum_{k=n}^{\infty}[k]_q^{2n}|a_k|^2C_k(\alpha;q),\end{aligned}$$

and

$$\parallel L_{\alpha,n,q}f\parallel^2_{\mathcal{A}_{\alpha,q}} \leq \frac{(q^{\alpha+1};q)_n}{[q^{\alpha+1}(1-q^{\alpha+1})]^n}\sum_{k=0}^{\infty}[k+n]_q^{2n}|a_k|^2C_k(\alpha;q).$$

Therefore,

$$\parallel f\parallel^2_{\mathcal{A}_{\alpha,n,q}}-c_{\alpha,n,q}(f)\leq [n]_q^{2n}\parallel \nabla_{\alpha,n,q}f\parallel^2_{\mathcal{A}_{\alpha,q}}\leq [n]_q^{2n}\frac{(q^{\alpha+1};q)_n}{[q^{\alpha+1}(1-q^{\alpha+1})]^n}\parallel f\parallel^2_{\mathcal{A}_{\alpha,n,q}},$$

where

$$c_{\alpha,n,q}(f)=|a_0|^2+\sum_{k=1}^{n-1}[k]_q^{2n}|a_k|^2C_k(\alpha;q)$$

and

$$\parallel f\parallel^2_{\mathcal{A}_{\alpha,n,q}}\leq\parallel L_{\alpha,n,q}f\parallel^2_{\mathcal{A}_{\alpha,q}}\leq [n+1]_q^{2n}\frac{(q^{\alpha+1};q)_n}{[q^{\alpha+1}(1-q^{\alpha+1})]^n}\parallel f\parallel^2_{\mathcal{A}_{\alpha,n,q}},$$

which leads to the result. □

Lemma 3.5.

(i) For any $f\in\mathcal{A}_{\alpha,q}$, $\quad Dom(\nabla_{\alpha,n,q}L_{\alpha,n,q})=Dom(L_{\alpha,n,q}\nabla_{\alpha,n,q})=\mathcal{A}_{\alpha,2n,q}$.

(ii) For any $f\in\mathcal{A}_{\alpha,2n,q}$, let $\beta_k(\alpha,n;q)=q^{-n(\alpha+1)}[k]_q!/[k-n]_q!\Big[\prod_{i=1}^{n}([k-i]_q+q^{-(\alpha+1)}[\alpha+1]_q)\Big]$. Then

$$\Delta_{\alpha,n,q}f(z)=\big[\nabla_{\alpha,n,q}-L_{\alpha,n,q}\big]_qf(z)=\sum_{k=0}^{\infty}r_k(\alpha,n;q)a_kz^k,\quad r_k(\alpha,n;q)\geq 0,$$

where $\quad r_k(\alpha,n;q)=\begin{cases}\beta_{k+n}(\alpha,n;q), & 0\leq k\leq n-1,\\ \beta_{k+n}(\alpha,n;q)-\beta_k(\alpha,n;q), & k\geq n\end{cases}.$

Proof. (i) Let $f \in \mathcal{A}_{\alpha,q}$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then from relations (3.8) and (3.9) we have

$$\nabla_{\alpha,n,q} L_{\alpha,n,q} f(z) = \sum_{k=0}^{\infty} q^{-n(\alpha+1)} \frac{[k+n]_q!}{[k]_q!} \Big[\prod_{i=1}^{n} ([k+n-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big] a_k z^k, \tag{3.10}$$

and

$$L_{\alpha,n,q} \nabla_{\alpha,n,q} f(z) = \sum_{k=n}^{\infty} q^{-n(\alpha+1)} \frac{[k]_q!}{[k-n]_q!} \Big[\prod_{i=1}^{n} ([k-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big] a_k z^k. \tag{3.11}$$

So

$$\begin{aligned} \| \nabla_{\alpha,n,q} L_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} &= \sum_{k=0}^{\infty} q^{-2n(\alpha+1)} \frac{[k+n]_q^2!}{[k]_q^2!} \Big[\prod_{i=1}^{n} ([k+n-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big]^2 |a_k|^2 C_k(\alpha;q) \\ &\leq \frac{(q^{\alpha+1};q)_n^2}{[q^{\alpha+1}(1-q^{\alpha+1})]^{2n}} \sum_{k=0}^{\infty} [k+n]_q^{4n} |a_k|^2 C_k(\alpha;q), \end{aligned}$$

and

$$\begin{aligned} \| L_{\alpha,n,q} \nabla_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} &= \sum_{k=n}^{\infty} q^{-2n(\alpha+1)} \frac{[k]_q^2!}{[k-n]_q^2!} \Big[\prod_{i=1}^{n} ([k-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big]^2 |a_k|^2 C_k(\alpha;q) \\ &\leq \frac{(q^{\alpha+1};q)_n^2}{[q^{\alpha+1}(1-q^{\alpha+1})]^{2n}} \sum_{k=n}^{\infty} [k]_q^{4n} |a_k|^2 C_k(\alpha;q). \end{aligned}$$

Therefore

$$\| f \|^2_{\mathcal{A}_{\alpha,2n,q}} \leq \| \nabla_{\alpha,n,q} L_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} \leq [n+1]_q^{4n} \frac{(q^{\alpha+1};q)_n^2}{[q^{\alpha+1}(1-q^{\alpha+1})]^{2n}} \| f \|^2_{\mathcal{A}_{\alpha,2n,q}},$$

and

$$\| f \|^2_{\mathcal{A}_{\alpha,2n,q}} - d_{\alpha,n,q}(f) \leq [n]_q^{4n} \| L_{\alpha,n,q} \nabla_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} \leq [n]_q^{4n} \frac{(q^{\alpha+1};q)_n^2}{[q^{\alpha+1}(1-q^{\alpha+1})]^{2n}} \| f \|^2_{\mathcal{A}_{\alpha,2n,q}},$$

where

$$d_{\alpha,n,q}(f) = |a_0|^2 + \sum_{k=1}^{n-1} [k]_q^{4n} |a_k|^2 C_k(\alpha;q).$$

(ii) The result follows from relations (3.10), (3.11) and the fact that for any $k \geq n$, $[k+n]_q![k-n]_q! \geq [k]_q^2!$ and $[k]_q![k-n]_q! \leq [k]_q^2!$. So

$$\begin{aligned} r_k(\alpha,n;q) &\geq \Big[\prod_{i=1}^{n} ([k+n-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big] - \Big[\prod_{i=1}^{n} ([k-i]_q + q^{-(\alpha+1)}[\alpha+1]_q) \Big] \\ &\geq \big([k]_q + q^{-(\alpha+1)}[\alpha+1]_q \big)^n - \big([k-1]_q + q^{-(\alpha+1)}[\alpha+1]_q \big)^n \\ &\geq 0. \end{aligned}$$

□

By (3.8) and (3.9), one has

$$\| L_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} = \| \Delta_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} + \langle \Delta_{\alpha,n,q} f, f \rangle_{\mathcal{A}_{\alpha,q}}.$$

Then using the previous lemma and if we proceed in the same way of Theorem 3.6, we deduce the following result.

Theorem 3.7. *Let $f \in \mathcal{A}_{\alpha,2n,q}$. For all $a, b \in \mathbb{R}$, we have*

(i) $\| (\nabla_{\alpha,q} + L_{\alpha,q} - a) f \|_{\mathcal{A}_{\alpha,q}} \| (\nabla_{\alpha,n,q} - L_{\alpha,n,q} + ib) f \|_{\mathcal{A}_{\alpha,q}} \geq \langle \Delta_{\alpha,n,q} f, f \rangle_{\mathcal{A}_{\alpha,q}}$.

(ii) $\| (\nabla_{\alpha,q} + L_{\alpha,q} - a) f \|_{\mathcal{A}_{\alpha,q}} \| (\nabla_{\alpha,n,q} - L_{\alpha,n,q} + ib) f \|_{\mathcal{A}_{\alpha,q}} \geq \| L_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}} - \| \Delta_{\alpha,n,q} f \|^2_{\mathcal{A}_{\alpha,q}}$.

Note that by Lemma 3.4 this uncertainty inequality can be extended to the space $\mathcal{A}_{\alpha,n,q}$.

4 Extremal Function on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$

Let $\eta > 0$ and $T : \mathcal{A}_{\alpha,q} \to H$ be a bounded operator from $\mathcal{A}_{\alpha,q}$ into a Hilbert space H. We denote by $\langle .,. \rangle_{T,\eta,q}$ the inner product defined on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$ by

$$\langle f, g \rangle_{T,\eta,q} := \eta \langle f, g \rangle_{\mathcal{A}_{\alpha,q}} + \langle Tf, Tg \rangle_H,$$

and $\| f \|_{T,\eta,q} := \sqrt{\langle f, f \rangle_{T,\eta,q}}$.

By the virtue of the theory of reproducing kernels of Hilbert space, we study the extremal function associated to the operator T on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$.

Theorem 4.1. *Let $\eta > 0$. The space $(\mathcal{A}_{\alpha,q}, \langle .,. \rangle_{T,\eta,q})$ possesses a reproducing kernel $\mathcal{K}_{T,\eta,q}(z,w); z, w \in \mathbb{D}$ which satisfies the equation $(\eta I + T^*T)\mathcal{K}_{T,\eta,q}(z,.) = \mathcal{K}_{\alpha,q}(z,.)$, where $\mathcal{K}_{\alpha,q}$ is the kernel given by (3.7). Moreover, the kernel $\mathcal{K}_{T,\eta,q}$ satisfies the following properties*

(i) $\| \mathcal{K}_{T,\eta,q}(z,.) \|_{\mathcal{A}_{\alpha,q}} \leq \frac{1}{\eta} \sqrt{e_q(|z|^2) E_q(q^{\alpha+1}|z|^2)}$.

(ii) $\| T\mathcal{K}_{T,\eta,q}(z,.) \|_H \leq \sqrt{\frac{e_q(|z|^2) E_q(q^{\alpha+1}|z|^2)}{2\eta}}$.

(iii) $\| T^*T\mathcal{K}_{T,\eta,q}(z,.) \|_{\mathcal{A}_{\alpha,q}} \leq \sqrt{e_q(|z|^2) E_q(q^{\alpha+1}|z|^2)}$,

Proof. Let $f \in \mathcal{A}_{\alpha,q}$. Using Theorem 3.1 (iii), the map $f \mapsto f(z)$ is a continuous linear functional on $(\mathcal{A}_{\alpha,q}, \langle .,. \rangle_{T,\eta,q})$. Thus, $(\mathcal{A}_{\alpha,q}, \langle .,. \langle_{T,\eta,q})$ has a reproducing kernel denoted $\mathcal{K}_{T,\eta,q}$. Now using the fact that

$$f(z) = \eta \langle f, \mathcal{K}_{T,\eta,q}(z,.) \rangle_{\mathcal{A}_{\alpha,q}} + \langle Tf, T\mathcal{K}_{T,\eta,q}(z,.) \rangle_H = \langle f, (\eta I + T^*T)\mathcal{K}_{T,\eta,q}(z,.) \rangle_{\mathcal{A}_{\alpha,q}},$$

we deduce easily that $(\eta I + T^*T)\mathcal{K}_{T,\eta,q}(z,.) = \mathcal{K}_{\alpha,q}(z,.)$. So the previous relation implies that

$$\eta^2 \| \mathcal{K}_{T,\eta,q}(z,.) \|^2_{\mathcal{A}_{\alpha,q}} + 2\eta \| T\mathcal{K}_{T,\eta,q}(z,.) \|^2_H + \| T^*T\mathcal{K}_{T,\eta,q}(z,.) \|^2_{\mathcal{A}_{\alpha,q}} = \| \mathcal{K}_{\alpha,q}(z,.) \|^2_{\mathcal{A}_{\alpha,q}} .$$

So we obtain the properties (i), (ii) and (iii) by using relation (3.7). □

Since relations (3.3), (3.4) and (3.5), we get

Example 1. For any $w, z \in \mathbb{D}$, let $H = \mathcal{A}_{\alpha,q}$.

(a) If $T = \nabla_{\alpha,q}$, then

$$\mathcal{K}_{T,\eta,q}(z,w) = \frac{1}{\eta C_0(\alpha;q)} + \sum_{n=1}^{\infty} \frac{(z\overline{w})^n}{\Big(\eta + q^{-\alpha-1}[n]_q([n-1]_q + q^{-\alpha-1}[\alpha+1]_q)\Big) C_n(\alpha;q)}.$$

(b) If $T = L_{\alpha,q}$, then

$$\mathcal{K}_{T,\eta,q}(z,w) = \frac{1}{\eta C_0(\alpha;q)} + \sum_{n=1}^{\infty} \frac{(z\overline{w})^n}{\Big(\eta + q^{-\alpha-1}[n+1]_q([n]_q + q^{-\alpha-1}[\alpha+1]_q)\Big) C_n(\alpha;q)}.$$

(c) If $T = N_q$, then

$$\mathcal{K}_{T,\eta,q}(z,w) = \frac{1}{\eta C_0(\alpha;q)} + \sum_{n=1}^{\infty} \frac{(z\overline{w})^n}{\Big(\eta + [n]_q^2\Big) C_n(\alpha;q)}.$$

We can state now the main result of this section.

Theorem 4.2. *For any $h \in H$ and $\eta > 0$, there exists a unique function $f^*_{\eta,h}$, where the infimum*

$$\inf_{f \in \mathcal{A}_{\alpha,q}} \Big\{ \eta \parallel f \parallel^2_{\mathcal{A}_{\alpha,q}} + \parallel h - Tf \parallel^2_H \Big\} \tag{4.1}$$

is attained. Moreover, the extremal function $f^*_{\eta,h}$ is given by

$$f^*_{\eta,h}(z) = \langle h, T\mathcal{K}_{T,\eta,q}(z,.)\rangle_H, \tag{4.2}$$

and satisfies the following $\quad \mid f^*_{\eta,h}(z) \mid \leq \sqrt{\dfrac{e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)}{2\eta}} \parallel h \parallel_H .$

Proof. The existence and unicity of the extremal function $f^*_{\eta,h}$ satisfying (4.1) is obtained in [15, 17]. In particular, $f^*_{\eta,h}$ is given by the reproducing kernel of $\mathcal{A}_{\alpha,q}$ with $\parallel . \parallel_{T,\eta,q}$ norm as $f^*_{\eta,h}(z) = \langle h, T\mathcal{K}_{T,\eta,q}(z,.)\rangle_H$. This yields the result, by using relation (4.2), Theorem 4.1 (ii) and the fact that

$$\mid f^*_{\eta,h}(z) \mid \leq \parallel h \parallel_H \parallel T\mathcal{K}_{T,\eta,q}(z,.) \parallel_H \leq \sqrt{\frac{e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)}{2\eta}} \parallel h \parallel_H,$$

which completes the proof of the theorem. □

Theorem 4.3. *(**Calderón's formula.**) Let $\eta > 0$ and $f \in \mathcal{A}_{\alpha,q}$. The extremal function $f^*_{\eta,h}$ is given by*

$$f^*_{\eta,h}(z) = \langle Tf, T\mathcal{K}_{T,\eta,q}(z,.)\rangle_{\mathcal{A}_{\alpha,q}},$$

satisfies

$$f(z) = \lim_{\eta \to 0^+} f^*_{\eta,h}(z) = \lim_{\eta \to 0^+} \langle h, T\mathcal{K}_{T,\eta,q}(z,.)\rangle_{\mathcal{A}_{\alpha,q}}.$$

Proof. Let $f \in \mathcal{A}_{\alpha,q}$, $h = Tf$ and $f^*_\eta = f^*_{\eta,Tf}$. Then $f^*_\eta(z) = \langle f, T^*T\mathcal{K}_{T,\eta,q}(z,.)\rangle_{\mathcal{A}_{\alpha,q}}$. Moreover, from Theorem 3.7 we deduce easily that

$$\lim_{\eta\to 0^+}(\eta I + T^*T)\mathcal{K}_{T,\eta,q}(z,.) = \lim_{\eta\to 0^+} T^*T\mathcal{K}_{T,\eta,q}(z,.) = \mathcal{K}_{\alpha,q}(z,.).$$

Thus

$$\lim_{\eta\to 0^+} f^*_\eta = \langle f, \mathcal{K}_{\alpha,q}(z,.)\rangle_{\mathcal{A}_{\alpha,q}} = f(z),$$

which leads to the result. □

4.1 Applications.

Let H be the prehilbertian space of analytic functions on the disk $\mathbb{D}$ equipped with the inner product

$$\langle f, g\rangle_H := \int_{\mathbb{D}} f(z)\overline{g(z)}|z|^2 d\nu_{\alpha,q}(z).$$

For any $f, g \in H$ with $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ we have from Proposition 3.1 and relation (3.6)

$$\langle f, g\rangle_H = \sum_{n\geq 0} a_n\overline{b_n}C_{n+1}(\alpha;q), \qquad \| f \|_H = \sum_{n\geq 0} | a_n |^2\, C_{n+1}(\alpha;q).$$

The space H is a Hilbert space with Hilberts basis $\left\{\dfrac{z^n}{\sqrt{C_{n+1}(\alpha;q)}}\right\}_{n\geq 0}$ and reproducing kernel

$$\mathcal{S}_{\alpha,q}(z,w) = \sum_{n=0}^{\infty}\frac{(z\overline{w})^n}{C_{n+1}(\alpha;q)} = \frac{\mathcal{K}_{\alpha,q}(z\overline{w}) - 1}{z\overline{w}}. \tag{4.3}$$

4.1.1 Application 1.

Let T be the q-difference operator defined on $\mathcal{A}_{\alpha,q}$ by

$$Tf(z) := \frac{1}{z}(f(z) - f(0)).$$

The operator T maps continuously from $\mathcal{A}_{\alpha,q}$ into H and $\| Tf \|_H \leq \| f \|_{\mathcal{A}_{\alpha,q}}$. So, if $f, g \in \mathcal{A}_{\alpha,q}$ with $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ we can deduce easily that

$$\langle f, g\rangle_{T,\eta} = \eta a_0\overline{b_0} + (\eta + 1)\sum_{n=1}^{\infty} a_n\overline{b_n}C_n(\alpha;q).$$

Thus, for $z, w \in \mathbb{D}$ we have

$$\begin{aligned}\mathcal{K}_{T,\eta,q}(z,w) &= \frac{1}{\eta} + \frac{1}{\eta+1}(\mathcal{K}_{\alpha,q}(z\overline{w}) - 1),\\ T\mathcal{K}_{T,\eta,q}(z,.)(w) &= \frac{1}{\eta+1}\frac{\mathcal{K}_{\alpha,q}(z\overline{w}) - 1}{\overline{w}},\end{aligned}$$

hence for all $h \in H$ we deduce that

$$f^*_{\eta,h}(z) = \frac{1}{\eta+1} z h(z).$$

4.1.2 Application 2.

Let T be the q-difference operator defined on $\mathcal{A}_{\alpha,q}$ by

$$Tf(z) := \frac{1}{1+q}(f(z) - f(-z)).$$

Same, the operator T maps continuously from $\mathcal{A}_{\alpha,q}$ into H and $\parallel Tf \parallel_H \leq \parallel f \parallel_{\mathcal{A}_{\alpha,q}}$. So, if $f, g \in \mathcal{A}_{\alpha,q}$ with $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ we can deduce easily that

$$\langle f, g\rangle_{T,\eta} = \eta a_0 \overline{b_0} + \sum_{n=1}^{\infty} \Big[\eta + \frac{1}{1+q}(1 - (-1)^n)\Big] a_n \overline{b_n} C_n(\alpha; q).$$

Therefore, from equation (4.3) for $z, w \in \mathbb{D}$ we have

$$\begin{aligned} \mathcal{K}_{T,\eta,q}(z, w) &= \frac{1}{\eta}\sum_{n=0}^{\infty} \frac{(z\overline{w})^{2n}}{C_{2n}(\alpha; q)} + \frac{1}{\eta+1}\sum_{n=0}^{\infty} \frac{(z\overline{w})^{2n+1}}{C_{2n+1}(\alpha; q)}, \\ T\mathcal{K}_{T,\eta,q}(z, .)(w) &= \frac{1}{\eta+1}\sum_{n=0}^{\infty} \frac{z^{2n+1}\overline{w}^{2n}}{C_{2n+1}(\alpha; q)}, \end{aligned}$$

and for all $h \in H$ we deduce that

$$f^*_{\eta,h}(z) = \frac{1}{(1+q)(\eta+1)} z\Big[h(z) + h(-z)\Big].$$

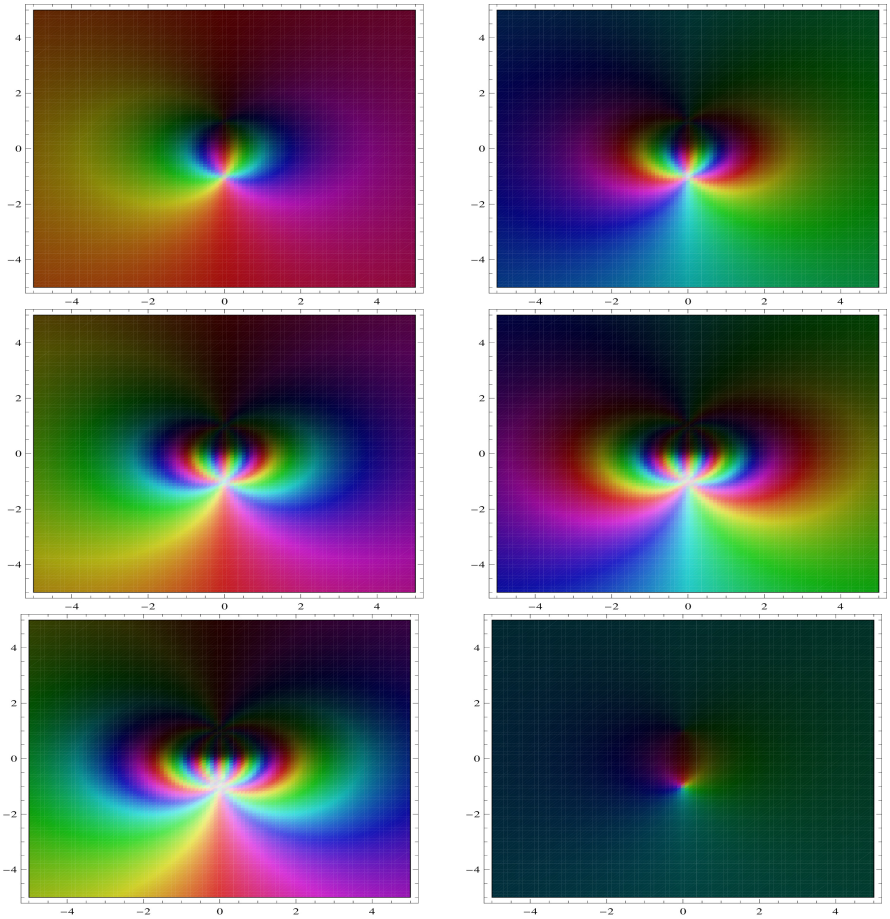

Figure 1: The following is the color function of $f^*_{\eta,h}(z)$ associated to the q-difference operator $Tf(z) := \frac{1}{z}(f(z) - f(0))$ for $\lambda = 10$, $z = x + iy$, $(x, y) \in [-5, 5] \times [-5, 5]$ and respectively $h(z) = 1, z, z^2, z^3, z^4, z^5$. The argument of a complex value is encoded by the hue of a color (red = positive real, and then counterclockwise through yellow, green, cyan, blue and purple; cyan stands for negative real). Strong colors denote points close to the origin, black = 0, weak colors denote points with large absolute value, white = ∞.

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