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The Split feasibility problems with multiple output sets in real Banach spaces

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Abstract. This paper aims to present a new iterative method for solving the split feasibility problem with multiple output sets for generalized demimetric mappings and solution sets of a generalized equilibrium problem in the real Banach spaces. The strong convergence of the sequence generated by our method is proven under certain conditions. We also give a numerical example to support our main result. The results of this paper extend and improve some recent corresponding results announced by many other authors.

Keywords. Split feasibility problem with multiple output set , generalized demimetric mappings, fixed point problem, generalized equilibrium problem

1 Introduction

Suppose that C is a closed and convex subset of a real Banach space E, and let E^* denote its dual space. Consider two bifunctions $f : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$, and a nonlinear mapping $B : C \to 2^{E^*}$. We focus on the generalized equilibrium problem (GEP), which consists of finding a point $u \in C$ such that

$$f(u, y) + \langle x^*, y - u\rangle + \phi(u, y) - \phi(u, u) \geq 0, \quad \forall y \in C, \ \forall x^* \in Bu. \tag{1.1}$$

The solution set of (1.1) is denoted by $GEP(f, B, \phi)$. If $\phi \equiv 0$, problem (1.1) reduces to finding $u \in C$ such that

$$f(u, y) + \langle x^*, y - u\rangle \geq 0, \quad \forall y \in C, \ \forall x^* \in Bu, \tag{1.2}$$

whose solution set is written as $GEP(f, B)$. When $f \equiv 0$ and $\phi \equiv 0$, problem (1.1) becomes the problem of finding $u \in C$ such that

$$\langle x^*, y - u\rangle \geq 0, \quad \forall y \in C, \ \forall x^* \in Bu,$$

which was introduced by Hartman and Stampacchia [10].

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Suppose C and Q are nonempty, closed, and convex subsets of real Banach spaces E_1 and E_2, respectively, and let E_1^* and E_2^* denote their dual spaces. Let $A : E_1 \to E_2$ be a bounded linear operator. The split feasibility problem (SFP) [6] consists of finding a point

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{1.3}$$

To solve the SFP, Schöpher et al. [23] proposed the following algorithm in the setting of p-uniformly convex and uniformly smooth real Banach spaces: for $x_1 \in E_1$, define

$$x_{n+1} = \Pi_C J^q_{E_1^*}\left[J^p_{E_1} x_n - \gamma_n A^* J^p_{E_2}(Ax_n - P_Q(x_n))\right], \quad n \geq 1, \tag{1.4}$$

where Π_C denotes the Bregman projection of E_1 onto C, and J^p_E represents the duality mapping. Under the assumption that E_1 is p-uniformly convex, uniformly smooth, and the duality mapping of E_1 is sequentially weak-to-weak continuous, the authors showed that algorithm (1.4) converges weakly. Given two nonlinear mappings $T : E_1 \to E_1$ and $S : E_2 \to E_2$, and a nonzero bounded linear operator $A : E_1 \to E_2$, the split common fixed point problem (SCFP) [8] is formulated as follows:

$$\text{Find } x^* \in F(S) \text{ such that } Ax^* \in F(T). \tag{1.5}$$

We denote by Ω the solution set of this SCFPP for the mappings T and S; That is, $\Omega = \{x^* \in F(S) : Ax^* \in F(T)\}$.

In the literature, there are several extensions and generalizations of the SFP, and one notable example is the split feasibility problem with multiple output sets (SFPMOS) [20, 21, 22], which is defined as follows:

$$\text{Find } x^* \in \Omega = C \cap \left(\cap_{i=1}^N A_i^{-1}(Q_i)\right), \tag{1.6}$$

where for each $i = 1, 2, ..., N$, Q_i is a closed and convex subset of a real Hilbert space H_i, and $A_i : H \to H_i$ is a bounded linear operator. Recently, Wang [26] studied the following generalization of SFPMOS in a Hilbert space:

$$\text{Find } x^* \in \Omega = F(T_0) \cap \left(\cap_{i=1}^N A_i^{-1}(F(T_i))\right), \tag{1.7}$$

where, for each $i = 0, 1, ..., N$, T_i is a nonlinear mapping from H_i into itself. To solve problem (1.7), Wang [26] introduced a new iterative method and established its weak and strong convergence under suitable conditions.

Consider a Banach space E with a subset $C \subseteq E$. A nonlinear mapping $B : C \to 2^{E^*}$ is said to be:

(i) Monotone, if $\langle x - y, x^* - y^* \rangle \geq 0, \ \forall x, y \in C, \ \forall x^* \in Bx, \ \forall y^* \in By.$

(ii) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x - y, x^* - y^* \rangle \geq \alpha \|x^* - y^*\|^2, \quad \forall x^* \in Bx, \quad \forall y^* \in By.$$

Motivated by the aforementioned research, this paper focuses on the following problem within the framework of p-uniformly convex Banach spaces:

$$\text{Find } x^* \in \Omega = GEP(B, f, \phi) \cap \Gamma, \tag{1.8}$$

where Γ is the solution set of the following generalization SFPMOS: Find x^* so that

$$x^* \in \cap_{j=1}^M F(S_j) \text{ and } x^* \in \cap_{i=1}^N A_i^{-1}(F(T_i)), \tag{1.9}$$

with the following specifications:

- For each $i = 1, 2, \dots, N$, $T_i : E_i \to E_i$ is a finite family of ξ_i-generalized demimetric mappings;
- For each $j = 1, 2, \dots, M$, $S_j : E \to E$ is a finite family of μ_j-generalized demimetric mappings;
- $B : C \to E^*$ is an α-inverse strongly monotone mapping.

We propose a self-adaptive algorithm to solve problem (1.8) and establish the strong convergence of the generated sequences under appropriate conditions on the parameters.

2 Preliminaries

Let E be a real Banach space and let E^* be its dual. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. We will show the strong convergence of $\{x_n\}$ to x by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

Let $1 < q \leq 2 \leq p$ and $\frac{1}{p} + \frac{1}{q} = 1$. A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U_E = \{x \in E : \|x\| = 1\}$ with $x \neq y$. The modulus of convexity, denoted as δ, is defined for a Banach space E by:

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\},$$

for every $\epsilon \in (0, 2]$.

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for every $\epsilon > 0$, and p-uniformly convex if there is a $c_p > 0$ so that $\delta_E(\epsilon) \geq c_p \epsilon^p$ for any $\epsilon \in (0, 2]$. The modulus of smoothness $\rho_E : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_E(\tau) = \sup\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1; \|x\| = \|y\| = 1\}.$$

A Banach space E is called uniformly smooth if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$, and E is called q-uniformly smooth if there exists a $c_q > 0$ so that $\rho_E(\tau) \leq c_q \tau^q$ for all $\tau > 0$. It is known that every p-uniformly convex space is also a uniformly convex space. Moreover, a Banach space E is p-uniformly convex if and only if its dual space E^* is q-uniformly smooth [2]. For instance, L_p spaces are $\min\{p, 2\}$-uniformly smooth for each $p > 1$. Further details can be found in [27].

For $p > 1$, the definition of the generalized duality mapping $J_E^p : E \to 2^{E^*}$ is given by

$$J_E^p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}, \qquad \forall x \in E.$$

In particular, $J_E^2 = J$ is called the normalized duality mapping. In this case, we assume that E is a p-uniformly convex and uniformly smooth, which implies that its dual space E^* is q-uniformly smooth and uniformly convex. In this setting, the duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$ where $J_{E^*}^q$ is the generalized duality mapping of E^* (see [2]).

Lemma 2.1. *[27] A uniformly smooth Banach space E has a modulus of smoothness of power type $q > 1$ if and only if there exists a constant $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q x \rangle + C_q \|y\|^q,$$

for all $x, y \in E$.

Lemma 2.2. *[27] Let $p > 1$ and $r > 0$ be two fixed real numbers, and let E be a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$, $g(0) = 0$ such that for all $x, y \in B_r = \{x \in E : \|x\| \leq r\}$ and $0 \leq \alpha \leq 1$,*

$$\|\alpha x + (1-\alpha)y\|^p \leq \alpha\|x\|^p + (1-\alpha)\|y\|^p - W_p(\alpha)g(\|x-y\|),$$

where $W_p(\alpha) := \alpha^p(1-\alpha) + \alpha(1-\alpha)^p$.

Let $1 < p, q < \infty$ be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f_p : E \to \mathbb{R}$ be a function given by $f_p(x) = \frac{1}{p}\|x\|^p$. The Bregman distance with respect to f_p is defined as follows:

$$\Delta_p(x, y) = \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p, \qquad \forall x, y \in E. \tag{2.1}$$

In the case of Hilbert spaces, where the duality mapping J is the identity operator, we have $\Delta_2(x, y) = \|x - y\|^2$.

According to [3, 7], we utilize the function $V_p : E \times E^* \to [0, \infty)$ associated with f_p, which is defined as follows:

$$V_p(x, \tilde{x}) = \Delta_p(x, J_{E^*}^q(\tilde{x})), \tag{2.2}$$

for all $x \in E$ and $\tilde{x} \in E^*$. In addition, using the subdifferential inequality, we have

$$V_p(x, \tilde{x}) + \langle (J_{E^*}^q)(\tilde{x}) - x, \tilde{y}\rangle \leq V_p(x, \tilde{x} + \tilde{y}), \tag{2.3}$$

for all $x \in E$ and $\tilde{x}, \tilde{y} \in E^*$ [14]. In addition, the function V_p is convex in the second variable. Thus, for all $z \in E$,

$$\Delta_p\big(z, J_{E^*}^q(\sum_{i=1}^N t_i J_E^p(x_i))\big) \leq \sum_{i=1}^N t_i \Delta_p(z, x_i), \tag{2.4}$$

where $\{x_i\}_{i=1}^N \subset E$, and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.3. *[19] Let E be a uniformly convex and uniformly smooth Banach space. If $x_1 \in E$ and the sequence $\{\Delta_p(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.4. *[11] Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\Delta_p(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.5. *[23] Let E be a p-uniformly convex Banach space. Then for $x, y \in E$, there exists a fixed constant $\tau_p > 0$ such that*

$$\tau_p\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p x - J_E^p y\rangle.$$

Let C be a nonempty, closed, and convex subset of E. The metric projection

$$P_C x := \arg\min_{y \in C} \|x - y\|, \quad x \in E,$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0, \quad \forall z \in C.$$

The Bregman projection, introduced by Bregman [5], is a mapping Π_C from E onto C is defined by

$$\Delta_p(\Pi_C(x), x) = \min_{y \in C} \Delta_p(y, x).$$

In Hilbert spaces, the Bregman projection Π_C reduces to the metric projection.

For any $x \in E$, then Bregman projections can be characterized by a variational inequality [19]:

$$\langle y - \Pi_C x, J_E^p x - J_E^p(\Pi_C x)\rangle \le 0, \quad \forall y \in C. \tag{2.5}$$

Moreover, this variational inequality is equivalent to the following inequality

$$\Delta_p(y, \Pi_C x) + \Delta_p(\Pi_C x, x) \le \Delta_p(y, x), \quad \forall y \in C.$$

Let E be a smooth Banach space and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : E \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [25] if, for any $x \in E$ and $p \in F(U)$,

$$\langle x - p, J_E(x - Ux)\rangle \ge \frac{1-\eta}{2}\|x - Ux\|^2$$

where J_E is the duality mapping on E. We have from [25] that $F(U)$ is nonempty, closed, and convex.

Recently, Kawasaki and Takahashi [12] generalized the concept of demimetric mappings as follows: Let E be a smooth, strictly convex, and reflexive Banach space. Let ξ be a real number with $\xi \neq 0$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called ξ-generalized demimetric if the following inequality holds:

$$\xi\langle x - q, J_E(x - Ux)\rangle \ge \|x - Ux\|^2,$$

for all $x \in C$ and $q \in F(U)$.

Let E be a p-uniformly convex and uniformly smooth Banach space. Shahzad et al., [24] introduced the following modifications of demimetric mappings: A mapping $U : E \to E$ with $F(U) \neq \emptyset$ is called η-demimetric if, for any $x \in E$ and $p \in F(U)$,

$$\langle x - p, J_E^p(x - Ux)\rangle \ge \frac{1-\eta}{2}\|x - Ux\|^p$$

In this paper, we adopt the following concept of demimetric mappings in a p-uniformly convex Banach:

Let E be a p-uniformly convex and uniformly smooth, and let $\xi \neq 0$ be a real number. A mapping $U : E \to E$ with $F(U) \neq \emptyset$ is called ξ-generalized demimetric if the following inequality holds:

$$\xi\langle x - q, J_E^p(x - Ux)\rangle \ge \|x - Ux\|^p,$$

for all $x \in C$ and $q \in F(U)$ (see [9]).

Example 1. Let E be a p-uniformly convex Banach space which is uniformly smooth and B be a maximal monotone mapping with $B^{-1}(0) \neq 0$. Then, for any $\lambda > 0$, the metric resolvent $J_\lambda = (J_E + \lambda B)^{-1} J_E$ is a 1-generalized demimetric mapping [25].

Example 2. Let E be a p-uniformly convex Banach space and uniformly smooth and let C be a nonempty, closed and convex subset of E. Then, P_C is a 1-generalized demimetric mapping.

Lemma 2.6. *[18] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_k}\}_{k\geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_k} \leq \Gamma_{n_k+1}$ for all $k \geq 0$. Also, consider a sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) := \max\{k \leq n \mid \Gamma_{n_k} \leq \Gamma_{n_k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence satisfying $\lim_{n\to\infty} \tau(n) = \infty$. Moreover, if $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for all $n \geq n_0$, then we have

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$$

Lemma 2.7. *[17] Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\delta_n)a_n + b_n + c_n, \quad \forall n \geq 0,$$

where $\{\delta_n\}$ is a sequence in $(0,1)$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:
(a) If there exists $M > 0$ such that $\frac{b_n}{\delta_n} < M$ for all n, then $\{a_n\}$ is bounded.
(b) If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Suppose that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following assumptions:
(i) $f(x, x) = 0$, for all $x \in C$;
(ii) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;
(iii) $\limsup_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y)$, for all $x, y, z \in C$;
(iv) For each $x \in C$, $y \to f(x, y)$ is convex and lower semicontinuous. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:
(i) ϕ is skew-symmetric, i.e., $\phi(x, x) - \phi(x, y) - \phi(y, x) + \phi(y, y) \geq 0$, for all $x, y \in C$;
(ii) ϕ is convex in the second argument;
(iii) ϕ is continuous.

Lemma 2.8. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E. Let $f : C \times C \to \mathbb{R}$ satisfy Assumption 2 and $\phi : C \times C \to \mathbb{R}$ satisfy Assumption 2. Assume that $B : C \to E^*$ is an α-inverse-strongly monotone mapping. Define the resolvent $K^{f,\phi}$ associated with f and ϕ by*

$$K^{f,\phi}(x) = \{z \in C : f(z, y) + \langle Bz, y - z\rangle + \langle y - z, J_E^p z - J_E^p x\rangle + \phi(z, y) - \phi(z, z) \geq 0\},$$

for all $x \in E$. Then, the following properties hold:
(a) $K^{f,\phi}$ is single-valued.
(b) $K^{f,\phi}$ is a Bregman firmly nonexpansive type mapping, i.e., for all $x, y \in E$,

$$\langle K^{f,\phi}x - K^{f,\phi}y, J_E^p(K^{f,\phi}x) - J_E^p(K^{f,\phi}y)\rangle \leq \langle K^{f,\phi}x - K^{f,\phi}y, J_E^p(x) - J_E^p(y)\rangle.$$

(c) $F(K^{f,\phi}) = GEP(f, B, \phi)$ is closed and convex;
(d) $\Delta_p(q, K^{f,\phi}x) + \Delta_p(K^{f,\phi}x, x) \leq \Delta_p(q, x)$, for all $q \in F(K^{f,\phi})$, and $x \in E$;
(e) $K^{f,\phi}$ is Bregman quasi-nonexpansive.

Proof. Define a bifunction $F : C \times C \to \mathbb{R}$ by

$$F(x, y) = f(x, y) + \langle Bx, y - x\rangle, \quad \forall x, y \in C.$$

It is straightforward to verify that F satisfies Assumption 2. Therefore, the results can be deduced immediately from Lemma 2.2 in [13]. □

3 Main result

In this section, we propose a self-adaptive algorithm to solve the SFPMOS in Banach spaces and establish a strong convergence theorem for the sequences generated by the proposed method.

(i) Let E and E_i $(i = 0, 1, 2, ..., N)$, be p-uniformly convex and uniformly smooth Banach and let C and C_i be nonempty, closed, and convex subsets of E and E_i respectively.

(ii) Let $A_i : E \to E_i$, $i = 1, 2, ..., N$ be nonzero bounded linear operators and A_i^* be the adjoint of A_i.

(iii) For each $j \in \{1, 2, ..., M\}$ with $\mu_j \neq 0$, let $S_j : C \to C$ be a finite family of μ_j-generalized demimetric and demiclosed mappings. Similarly, for each $i \in \{1, 2, ..., N\}$ with $\xi_i \neq 0$, let $T_i : C_i \to C_i$ be a finite family of ξ_i-generalized demimetric and demiclosed mappings.

(iv) Let $f : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ be bifunctions satisfying Assumptions 2 and 2 respectively, and let $B : C \to E^*$ be an α-inverse-strongly monotone mapping.

(v) $\Omega = \Gamma \cap GEP(B, f, \phi) \neq \emptyset$, where Γ is the solution set of the split common fixed point problem.

Theorem 3.1. *Let $\{v_n\}$ be a sequence in C such that $v_n \to v$. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} u_n \in C \quad s.t. \quad f(u_n, y) + \langle Bu_n, y - u_n\rangle + \langle y - u_n, J_E^p u_n - J_E^p x_n\rangle \\ + \phi(u_n, y) - \phi(u_n, u_n) \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J_{E^*}^q \big[\sum_{j=1}^{M} b_{n,j}\big(J_E^p u_n - \sigma_{n,j} r_j J_E^p(u_n - S_j u_n)\big)\big], \\ z_n = \Pi_C J_{E^*}^q \big[\sum_{i=1}^{N} a_{n,i}\big(J_E^p y_n - \tau_{n,i} l_i A_i^* J_{E_i}^p(A_i y_n - T_i A_i y_n)\big)\big], \\ x_{n+1} = \Pi_C J_{E^*}^q \big[\beta_n J_E^p x_n + (1 - \beta_n)(\alpha_n J_E^p v_n + (1 - \alpha_n) J_E^p z_n)\big], \end{cases} \tag{3.1}$$

where $l_i = \frac{\xi_i}{|\xi_i|}$, $r_j = \frac{\mu_j}{|\mu_j|}$, and the stepsizes are chosen as follows:

$$\tau_{n,i} = \frac{\rho_{n,i}\|A_i y_n - T_i A_i y_n\|^{p(p-1)}}{\overline{\tau}_{n,i}^p}, \quad for \quad i = 1, 2, ..., N,$$

with

$$\overline{\tau}_{n,i} := \max\{\|A_i^* J_{E_i}^p (A_i y_n - T_i A_i y_n)\|, \gamma\}$$

where $\gamma > 0$, $\{\rho_{n,i}\} \subset (0, (\frac{q}{|\xi_i|C_q})^{\frac{1}{q-1}})$, and $\{\sigma_{n,j}\} \subset (0, (\frac{q}{|\mu_j|C_q})^{\frac{1}{q-1}})$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$,$\{a_{n,i}\}$, and $\{b_{n,j}\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subseteq (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\lim_{n\to\infty} a_{n,i}\rho_{n,i}(\frac{1}{|\xi_i|} - \frac{C_q \rho_{n,i}^{q-1}}{q}) > 0$, *and* $\lim_{n\to\infty} b_{n,j}\sigma_{n,j}(\frac{1}{|\mu_j|} - \frac{C_q \sigma_{n,j}^{q-1}}{q}) > 0$.

(iii) $\{a_{n,i}\}_{i=1}^N, \{b_{n,j}\}_{j=1}^M \subset (0, 1]$, $\sum_{i=1}^N a_{n,i} = \sum_{j=1}^M b_{n,j} = 1$;

(iv) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, where $x^* = \Pi_\Omega v$.

Proof. Let $x^* \in \Omega$. Then, using (3.1), (2.1), and Lemma 2.1, we obtain

$$\begin{aligned}
\Delta_p(x^*, z_n) &\leq \Delta_p\Big(x^*, J^q_{E^*}\big[\sum_{i=1}^N a_{n,i}\big(J^p_E y_n - \tau_{n,i} l_i A_i^* J^p_{E_i}(A_i y_n - T_i A_i y_n)\big)\big]\Big) \\
&\leq \frac{1}{p}\|x^*\|^p - \langle x^*, J^p_E y_n\rangle + \sum_{i=1}^N \tau_{n,i} a_{n,i} l_i \langle A_i x^*, J^p_{E_i}(A_i y_n - T_i A_i y_n)\rangle \\
&+ \frac{1}{q}\|\sum_{i=1}^N a_{n,i}(J^p_{E_i} y_n - \tau_{n,i} l_i A_i^* J^p_{E_i}(A_i y_n - T_i A_i y_n))\|^q \\
&\leq \frac{1}{p}\|x^*\|^p - \langle x^*, J^p_E y_n\rangle + \sum_{i=1}^N \tau_{n,i} a_{n,i} l_i \langle A_i x^*, J^p_{E_i}(A_i y_n - T_i A_i y_n)\rangle \\
&+ \frac{1}{q}\sum_{i=1}^N a_{n,i}\|J^p_{E_i} y_n - \tau_{n,i} l_i A_i^* J^p_{E_i}(A_i y_n - T_i A_i y_n)\|^q \\
&\leq \frac{1}{p}\|x^*\|^p - \langle x^*, J^p_E y_n\rangle + \frac{\|y_n\|^p}{q} - \sum_{i=1}^N \tau_{n,i} a_{n,i} l_i \langle A_i y_n - A_i x^*, J^p_{E_i}(A_i y_n - T_i A_i y_n)\rangle \\
&+ \frac{C_q}{q}\sum_{i=1}^N \tau^q_{n,i} a_{n,i}\|A_i^* J^p_{E_i}(A_i y_n - T_i A_i y_n)\|^q \\
&\leq \Delta_p(x^*, y_n) - \sum_{i=1}^N \tau_{n,i} a_{n,i} l_i \frac{1}{|\xi_i|}\|A_i y_n - T_i A_i y_n\|^p \\
&+ \frac{C_q}{q}\sum_{i=1}^N \tau^q_{n,i} a_{n,i}\|A_i^* J^p_{E_i}(A_i y_n - T_i A_i y_n)\|^q \\
&\leq \Delta_p(x^*, y_n) - \sum_{i=1}^N \frac{a_{n,i}}{|\xi_i|}\frac{\rho_{n,i}\|A_i y_n - T_i A_i y_n\|^{p(p-1)}}{\overline{\tau}^p_{n,i}}\|A_i y_n - T_i A_i y_n\|^p \\
&+ \frac{C_q}{q}\sum_{i=1}^N a_{n,i}\frac{\rho^q_{n,i}\|A_i y_n - T_i A_i y_n\|^{pq(p-1)}}{\overline{\tau}^{pq}_{n,i}}\overline{\tau}^q_{n,i} \\
&\leq \Delta_p(x^*, y_n) - \sum_{i=1}^N a_{n,i}\rho_{n,i}\Big(\frac{1}{|\xi_i|} - \frac{C_q \rho^{q-1}_{n,i}}{q}\Big)\frac{\|A_i y_n - T_i A_i y_n\|^{p^2}}{\overline{\tau}^p_{n,i}}.
\end{aligned} \tag{3.2}$$

In the same way, we deduce that

$$\Delta_p(x^*, y_n) \leq \Delta_p(x^*, u_n) - \sum_{j=1}^M b_{n,j}\sigma_{n,j}\Big(\frac{1}{|\mu_j|} - \frac{C_q \sigma^{q-1}_{n,j}}{q}\Big)\|u_n - S_j u_n\|^p. \tag{3.3}$$

From (3.1) and condition (d) of Lemma 2.8, we obtain

$$\Delta_p(x^*, u_n) \leq \Delta_p(x^*, K^{f,\phi} x_n) \leq \Delta_p(x^*, x_n). \tag{3.4}$$

By substituting (3.4) and (3.3) into (3.2), we get

$$\Delta_p(x^*, z_n) \leq \Delta_p(x^*, x_n) - \sum_{j=1}^{M} b_{n,j}\sigma_{n,j}(\frac{1}{|\mu_j|} - \frac{C_q\sigma_{n,j}^{q-1}}{q})\|u_n - S_ju_n\|^p$$
$$- \sum_{i=1}^{N} a_{n,i}\rho_{n,i}(\frac{1}{|\xi_i|} - \frac{C_q\rho_{n,i}^{q-1}}{q})\frac{\|A_iy_n - T_iA_iy_n\|^{p^2}}{\overline{\tau}_{n,i}^p}. \quad (3.5)$$

Since $\{\sigma_{n,j}\} \subset (0, (\frac{q}{|\mu_j|C_q})^{\frac{1}{q-1}})$ and $\{\rho_{n,i}\} \subset (0, (\frac{q}{|\xi_i|C_q})^{\frac{1}{q-1}})$, we have

$$\Delta_p(x^*, z_n) \leq \Delta_p(x^*, x_n). \quad (3.6)$$

Let $t_n = J_{E^*}^q\left[\alpha_n J_E^p v_n + (1-\alpha_n)J_E^p z_n\right]$, then

$$\begin{aligned}\Delta_p(x^*, t_n) &\leq \Delta_p(x^*, J_{E^*}^q\left[\alpha_n J_E^p v_n + (1-\alpha_n)J_E^p z_n\right]) \\ &\leq \alpha_n\Delta_p(x^*, v_n) + (1-\alpha_n)\Delta_p(x^*, z_n) \\ &\leq \alpha_n\Delta_p(x^*, v_n) + (1-\alpha_n)\Delta_p(x^*, x_n)\end{aligned}$$

It follows that

$$\begin{aligned}\Delta_p(x^*, x_{n+1}) &\leq \Delta_p(x^*, J_{E^*}^q\left[\beta_n J_E^p x_n + (1-\beta_n)J_E^p t_n\right]) \\ &\leq \beta_n\Delta_p(x^*, x_n) + (1-\beta_n)\Delta_p(x^*, t_n) \\ &\leq \beta_n\Delta_p(x^*, x_n) + (1-\beta_n)\alpha_n\Delta_p(x^*, v_n) + (1-\alpha_n)(1-\beta_n)\Delta_p(x^*, x_n) \\ &\leq (1-\alpha_n(1-\beta_n))\Delta_p(x^*, x_n) + \alpha_n(1-\beta_n)\Delta_p(x^*, v_n).\end{aligned}$$

Since $\{v_n\}$ is bounded, $\{\Delta_p(x^*, v_n)\}$ is also bounded. Hence, by Lemma 2.7, $\{x_n\}$ is bounded. Consequently, $\{z_n\}$, $\{y_n\}$, and $\{u_n\}$ are bounded.

We now prove that the sequence $\{x_n\}$ converges strongly to an element of the solution set. From (2.2) and (2.3), we have

$$\begin{aligned}\Delta_p(x^*, t_n) &= \Delta_p(x^*, J_{E^*}^q[\alpha_n J_E^p v_n + (1-\alpha_n)J_E^p z_n]) \\ &= V_p(x^*, \alpha_n J_E^p v_n + (1-\alpha_n)J_E^p z_n) \\ &\leq V_p(x^*, \alpha_n J_E^p v_n + (1-\alpha_n)J_E^p z_n - \alpha_n(J_E^p v_n - J_E^p x^*)) \\ &+ \langle t_n - x^*, \alpha_n(J_E^p v_n - J_E^p x^*)\rangle \\ &= \alpha_n\Delta_p(x^*, x^*) + (1-\alpha_n)\Delta_p(x^*, z_n) + \alpha_n\langle t_n - x^*, J_E^p v_n - J_E^p x^*\rangle \\ &= (1-\alpha_n)\Delta_p(x^*, x_n) - (1-\alpha_n)\sum_{j=1}^{M} b_{n,j}\sigma_{n,j}(\frac{1}{|\mu_j|} - \frac{C_q\sigma_{n,j}^{q-1}}{q})\|u_n - S_ju_n\|^p \\ &- (1-\alpha_n)\sum_{i=1}^{N} a_{n,i}\rho_{n,i}(\frac{1}{|\xi_i|} - \frac{C_q\rho_{n,i}^{q-1}}{q})\frac{\|A_iy_n - T_iA_iy_n\|^{p^2}}{\overline{\tau}_{n,i}^p} \\ &+ \alpha_n\langle t_n - x^*, J_E^p v_n - J_E^p x^*\rangle.\end{aligned} \quad (3.7)$$

Consequently,

$$\Delta_p(x^*, x_{n+1}) \leq \beta_n\Delta_p(x^*, x_n) + (1-\alpha_n)\Delta_p(x^*, t_n)$$

$$\begin{aligned}
&\leq (1-\alpha_n(1-\beta_n))\Delta_p(x^*,x_n)\\
&-(1-\alpha_n)(1-\beta_n)\sum_{j=1}^{M} b_{n,j}\sigma_{n,j}\Big(\frac{1}{|\mu_j|}-\frac{C_q\sigma_{n,j}^{q-1}}{q}\Big)\|u_n-S_ju_n\|^p\\
&-(1-\alpha_n)(1-\beta_n)\sum_{i=1}^{N} a_{n,i}\rho_{n,i}\Big(\frac{1}{|\xi_i|}-\frac{C_q\rho_{n,i}^{q-1}}{q}\Big)\frac{\|A_iy_n-T_iA_iy_n\|^{p^2}}{\overline{\tau}_{n,i}^{p}}.\\
&+\alpha_n(1-\beta_n)\langle t_n-x^*, J_E^p v_n - J_E^p v\rangle\\
&+\alpha_n(1-\beta_n)\langle t_n-x^*, J_E^p v - J_E^p x^*\rangle. \qquad (3.8)
\end{aligned}$$

Let $\Gamma_n=\Delta_p(x^*,x_n)$ for all $n\geq 1$. From (3.8), we have

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\sum_{j=1}^{M} b_{n,j}\sigma_{n,j}\Big(\frac{1}{|\mu_j|}-\frac{C_q\sigma_{n,j}^{q-1}}{q}\Big)\|u_n-S_ju_n\|^p\\
&+(1-\alpha_n)(1-\beta_n)\sum_{i=1}^{N} a_{n,i}\rho_{n,i}\Big(\frac{1}{|\xi_i|}-\frac{C_q\rho_{n,i}^{q-1}}{q}\Big)\frac{\|A_iy_n-T_iA_iy_n\|^{p^2}}{\overline{\tau}_{n,i}^{p}}\\
&\leq \Gamma_n-\Gamma_{n+1}\\
&+\alpha_n(1-\beta_n)\langle t_n-x^*, J_E^p v_n - J_E^p v\rangle\\
&+\alpha_n(1-\beta_n)\langle t_n-x^*, J_E^p v - J_E^p x^*\rangle.
\end{aligned}$$

Case 1: Suppose that the sequence $\{\Gamma_n\}_{n=n_0}^{\infty}$ is nonincreasing for some $n_0\in\mathbb{N}$. Then $\{\Gamma_n\}$ is convergent. Hence,

$$\Gamma_n-\Gamma_{n+1}\to 0$$

Now, by conditions (i) and (iv), we obtain

$$\lim_{n\to\infty}\|u_n-S_ju_n\|=0, \quad j=1,2,...,M, \tag{3.9}$$

and

$$\lim_{n\to\infty}\|A_iy_n-T_iA_iy_n\|=0, \quad i=1,2,...,N. \tag{3.10}$$

Using (3.9), we have

$$\|J_E^p(y_n)-J_E^p(u_n)\|\leq \sum_{j=1}^{M} b_{n,j}\sigma_{n,j}\|J_E^p(u_n-S_ju_n)\|\to 0, \tag{3.11}$$

By the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^* and (3.11), we get

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.12}$$

Moreover, (3.1) and (3.10) yield

$$\|J_E^pz_n-J_E^py_n\|\leq \sum_{i=1}^{N} a_{n,i}\tau_{n,i}\|A_i\|\|J_{E_i}^p(A_iy_n-T_iA_iy_n)\|\to 0, \quad \text{as} \quad n\to\infty. \tag{3.13}$$

By the uniform continuity of $J^q_{E^*}$ on bounded subsets of E^* and (3.13), we obtain

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.14}$$

Let $\eta = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|t_n\|\}$. Then, by Lemma 2.2, there exists a continuous, strictly increasing, and convex function $g : [0, 2\eta] \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$\begin{aligned}
\Delta_p(x^*, x_{n+1}) &= \Delta_p(x^*, J^q_{E^*}\left[\beta_n J^p_E x_n + (1-\beta_n)J^p_E t_n\right]) \\
&= \frac{1}{p}\|x^*\|^p - \langle x^*, \beta_n J^p_E x_n + (1-\beta_n)J^p_E t_n\rangle \\
&+ \frac{1}{q}\|\beta_n J^p_E x_n + (1-\beta_n)J^p_E t_n\|^q \\
&= \frac{1}{p}\|x^*\|^p - \beta_n\langle x^*, J^p_E x_n\rangle + (1-\beta_n)\langle x^*, J^p_E t_n\rangle \\
&+ \frac{1}{q}\beta_n\|x_n\|^p + \frac{1}{q}(1-\beta_n)\|t_n\|^p \\
&- \frac{1}{q}(\beta_n^q(1-\beta_n) + \beta_n(1-\beta_n)^q)g(\|J^p_E x_n - J^p_E t_n\|) \\
&= \beta_n\Delta_p(x^*, x_n) + (1-\beta_n)\Delta_p(x^*, t_n) \\
&- \frac{1}{q}(\beta_n^q(1-\beta_n) + \beta_n(1-\beta_n)^q)g(\|J^p_E x_n - J^p_E t_n\|) \\
&\le (1-\alpha_n(1-\beta_n))\Delta_p(x^*, x_n) + \alpha_n(1-\beta_n)\Delta_p(x^*, v_n) \\
&- \frac{1}{q}(\beta_n^q(1-\beta_n) + \beta_n(1-\beta_n)^q)g(\|J^p_E x_n - J^p_E t_n\|).
\end{aligned} \tag{3.15}$$

Therefore,

$$\begin{aligned}
&\frac{1}{q}(\beta_n^q(1-\beta_n) + \beta_n(1-\beta_n)^q)g(\|J^p_E x_n - J^p_E t_n\|) \le \alpha_n\Delta_p(x^*, v_n) \\
&+ \Delta_p(x^*, x_n) - \Delta_p(x^*, x_{n+1}).
\end{aligned}$$

Using conditions (i) and (iv), we conclude that $\lim_{n\to\infty} g(\|J^p_E x_n - J^p_E t_n\|) = 0$. Therefore,

$$\lim_{n\to\infty} \|J^p_E x_n - J^p_E t_n\| = 0.$$

By the uniform continuity of $J^q_{E^*}$ on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{3.16}$$

This, together with Lemma 2.5 implies that

$$\lim_{n\to\infty} \Delta_p(x_n, t_n) = 0.$$

It follows that

$$\Delta_p(z_n, t_n) \le \alpha_n\Delta_p(z_n, v_n) + (1-\alpha_n)\Delta_p(z_n, z_n) \to 0, \text{ as } n \to \infty.$$

Hence, by Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|z_n - t_n\| = 0.$$

Consequently,

$$\|z_n - x_n\| \le \|z_n - t_n\| + \|t_n - x_n\| \to 0, \text{ as } n \to \infty. \tag{3.17}$$

Now, using (3.12), (3.14), and (3.17), we get

$$\|u_n - x_n\| \le \|u_n - y_n\| + \|y_n - z_n\| + \|z_n - x_n\| \to 0. \tag{3.18}$$

Now, since E is a reflexive Banach space and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$. From (3.18), there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup z$. Then, by (3.9) and the demiclosedness of S_j, $j = 1, 2, ..., M$, we have $z \in \cap_{j=1}^{M} F(S_j)$. Moreover, from (3.17), we have $z_{n_k} \rightharpoonup z$. By the continuity of A_i, we obtain $A_i y_{n_k} \to A_i z$. Hence, using (3.11) and the demiclosedness of $I - T_i$, we conclude that $A_i z \in \cap_{i=1}^{N} F(T_i)$.

Next, we show that $z \in GEP(f, B, \phi)$. Using (3.18) and uniformly norm-to-norm continuity of J_E^p on bounded sets, we obtain

$$\lim_{n\to\infty} \|J_E^p u_n - J_E^p x_n\| = 0.$$

According to the definition of $u_n = K^{F,\phi} x_n$, we get

$$F(u_n, y) + \langle y - u_n, J_E^p u_n - J_E^p x_n\rangle + \phi(u_n, y) - \phi(u_n, u_n) \ge 0, \quad \forall y \in C, \tag{3.19}$$

where $F(u_n, y) = f(u_n, y) + \langle Bu_n, y - u_n\rangle$. It is clear that $y \to f(x, y) + \langle Bx, y - x\rangle$ is lower semicontinuous and convex. Hence, the bifunction $F : C \times C \to \mathbb{R}$ satisfies Assumption 2. It follows from (3.19) that

$$\begin{aligned}\langle y - u_{n_k}, J_E^p u_{n_k} - J_E^p x_{n_k}\rangle &\ge -F(u_{n_k}, y) - \phi(u_{n_k}, y) + \phi(u_{n_k}, u_{n_k}) \\ &\ge F(y, u_{n_k}) - \phi(u_{n_k}, y) + \phi(u_{n_k}, u_{n_k})\end{aligned}$$

for all $y \in C$. Taking the limit inferior on both sides of the last inequality and using the continuity of ϕ, we deduce that

$$F(y, z) - \phi(z, y) + \phi(z, z) \le 0, \quad \forall y \in C.$$

Let $y_t = ty + (1 - t)z$ for all $y \in C$ and all $0 < t < 1$, by the convexity of C, we have $y_t \in C$ and hence

$$F(y_t, z) - \phi(z, y_t) + \phi(z, z) \le 0.$$

Now,

$$\begin{aligned}0 = F(y_t, y_t) &\le tF(y_t, y) + (1 - t)F(y_t, z) \\ &\le tF(y_t, y) + (1 - t)[\phi(z, y_t) - \phi(z, z)] \\ &\le tF(y_t, y) + (1 - t)t[\phi(z, y) - \phi(z, z)].\end{aligned}$$

As $t > 0$, we have

$$F(z, y) + \phi(z, y) - \phi(z, z) \ge 0, \quad \forall y \in C.$$

Therefore,

$$f(z, y) + \langle Bz, y - z\rangle + \phi(z, y) - \phi(z, z) \ge 0, \quad \forall y \in C.$$

So $z \in GEP(f, B, \phi)$. Hence $z \in \Omega$. Next, we show that

$$\limsup_{n\to\infty} \langle t_n - x^*, J_E^p v - J_E^p x^* \rangle \leq 0$$

To derive this inequality, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle t_n - x^*, J_E^p v - J_E^p x^* \rangle = \lim_{k\to\infty} \langle t_{n_k} - x^*, J_E^p v - J_E^p x^* \rangle.$$

Since $x_{n_k} \rightharpoonup z$ and by (3.16), we also have $t_{n_k} \rightharpoonup z$. Hence,

$$\limsup_{n\to\infty} \langle t_n - x^*, J_E^p v - J_E^p x^* \rangle = \langle z - x^*, J_E^p v - J_E^p x^* \rangle \leq 0. \tag{3.20}$$

Since $v_n \to v$, it follows that $\lim_{n\to\infty} \langle t_n - x^*, J_E^p v_n - J_E^p v \rangle = 0$. Combining this with (3.8) and (3.20), we deduce by Lemma 2.7 that $\Gamma_n \to 0$ as $n \to \infty$. Hence, $x_n \to x^*$.

Case 2: Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined by $\tau(n) := \max\{k \in \mathbb{N} : k \leq n,\ \Gamma_k \leq \Gamma_{k+1}\}$ for all $n \geq n_0$. Then, by Lemma 2.6 we have $\tau(n) \to \infty$ and $0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for all $n \geq n_0$. Following the same argument as in case 1, we have

$$\lim_{n\to\infty} \|A_i y_{\tau(n)} - T_i A_i y_{\tau(n)}\| = \lim_{n\to\infty} \|u_{\tau(n)} - S_j u_{\tau(n)}\| = 0.$$

Moreover, we can show that

$$\limsup_{n\to\infty} \langle t_{\tau(n)} - x^*, J_E^p v - J_E^p x^* \rangle \leq 0.$$

Also, from (3.8) and $\leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have

$$\Gamma_{\tau(n)} \leq \langle t_{\tau(n)} - x^*, J_E^p v_{\tau(n)} - J_E^p x^* \rangle + \langle t_{\tau(n)} - x^*, J_E^p v - J_E^p x^* \rangle. \tag{3.21}$$

Therefore, $\limsup_{n\to\infty} \Gamma_{\tau(n)} \leq 0$ and hence $\lim_{n\to\infty} \Gamma_{\tau(n)} = 0$. Again, from (3.8), we have

$$\begin{aligned} \Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} &\leq \alpha_{\tau(n)}(1 - \beta_{\tau(n)}) \langle t_{\tau(n)} - x^*, J_E^p v_{\tau(n)} - J_E^p x^* \rangle \\ &\quad + \alpha_{\tau(n)}(1 - \beta_{\tau(n)}) \langle t_{\tau(n)} - x^*, J_E^p v - J_E^p x^* \rangle \\ &\quad \to 0. \end{aligned}$$

It follows from (3.21) that $\lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$. Hence,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1} \to 0.$$

Thus, $\lim_{n\to\infty} \Delta_p(x^*, x_n) = 0$, and therefore, $x_n \to x^*$. This completes the proof. □

Remark 1. It is worth comparing the structure of our proposed algorithm (15) with that of recent inertial CQ algorithms, such as the one introduced by Jolaoso and Shehu [16]. Although both algorithms are iterative and use self-adaptive step sizes, their structures differ in several important aspects.

1 The algorithm in [16] is designed for Hilbert spaces, while our algorithm operates in the more general framework of p-uniformly convex and uniformly smooth Banach spaces. This structural difference allows our method to address problems that cannot be formulated in the Hilbert space setting.

2 The method in [16] solves the classical split feasibility problem (SFP) with a single output set, incorporating inertial and correction terms for acceleration. Our algorithm addresses the more general split feasibility problem with multiple output sets (SFPMOS). Its structure includes a resolvent operator $K^{f,\phi}$ for the generalized equilibrium problem and two families of ξ_i-generalized demimetric mappings $\{T_i\}$ and η_j-generalized demimetric mappings $\{S_j\}$ for multiple fixed point constraints.

3 The convergence analysis in [16] relies on firmly nonexpansive mappings in Hilbert spaces, while our proof depends on Banach space tools such as Bregman distance inequalities and properties of generalized duality mappings.

Theorem 3.2. *Let $\{v_n\}$ be a sequence in C such that $v_n \to v$. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} u_n \in C \quad s.t. \quad \langle Bu_n, y-u_n\rangle + \langle y-u_n, J_E^p u_n - J_E^p x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J_{E^*}^q \big[\sum_{j=1}^{M} b_{n,j}\big(J_E^p u_n - \sigma_{n,j} r_j J_E^p(u_n - S_j u_n)\big)\big], \\ z_n = \Pi_C J_{E^*}^q \big[\sum_{i=1}^{N} a_{n,i}\big(J_E^p y_n - \tau_{n,i} l_i A_i^* J_{E_i}^p(A_i y_n - T_i A_i y_n)\big)\big], \\ x_{n+1} = \Pi_C J_{E^*}^q \big[\beta_n J_E^p x_n + (1-\beta_n)(\alpha_n J_p^E v_n + (1-\alpha_n) J_E^p z_n)\big], \end{cases} \tag{3.22}$$

where $l_i = \frac{\xi_i}{|\xi_i|}$, $r_j = \frac{\mu_j}{|\mu_j|}$, and the step-sizes are chosen as follows:

$$\tau_{n,i} = \frac{\rho_{n,i}\|A_i y_n - T_i A_i y_n\|^{p(p-1)}}{\overline{\tau}_{n,i}^p}, \ \text{for} \ \ i = 1,2,...,N,$$

with

$$\overline{\tau}_{n,i} := \max\{\|A_i^* J_{E_i}^p(A_i y_n - T_i A_i y_n)\|, \gamma\}$$

where $\gamma > 0$, $\{\rho_{n,i}\} \subset (0, (\frac{q}{|\xi_i| C_q})^{\frac{1}{q-1}})$, and $\{\sigma_{n,j}\} \subset (0, (\frac{q}{|\mu_j| C_q})^{\frac{1}{q-1}})$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$,$\{a_{n,i}\}$, and $\{b_{n,j}\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subseteq (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\lim_{n\to\infty} a_{n,i}\rho_{n,i}(\frac{1}{|\xi_i|} - \frac{C_q \rho_{n,i}^{q-1}}{q}) > 0$, *and* $\lim_{n\to\infty} b_{n,j}\sigma_{n,j}(\frac{1}{|\mu_j|} - \frac{C_q \sigma_{n,j}^{q-1}}{q}) > 0$.

(iii) $\{a_{n,i}\}_{i=1}^N, \{b_{n,j}\}_{j=1}^M \subset (0,1]$, $\sum_{i=1}^N a_{n,i} = \sum_{j=1}^M b_{n,j} = 1$;

(iv) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, where $x^* = \Pi_\Omega v$.

Proof. Letting $f = 0$ and $\phi = 0$ in Theorem 3.1, we obtain the desired result. □

4 Applications

In this section, we present some applications of our main result.
Let $\{C_j\}_{j=1}^M$, and $\{Q_i\}_{i=1}^N$ be nonempty, closed, and convex subsets of E and E_i, respectively. Consider the following problem:

$$\Omega = \Gamma \cap \mathrm{Sol}(B, f, \phi), \tag{4.1}$$

where Γ denotes the solution set of the following generalized SFPMOS: fined x^* so that

$$x^* \in \cap_{j=1}^{M} C_j \quad \text{and} \quad x^* \in \cap_{i=1}^{N} A_i^{-1} Q_i.$$

Let $S_j = P_{C_j}$ for all $j = 1,..,M$ and $T_i = P_{Q_i}$ for all $i = 1,...,N$. Then T_j and S_i are 1-generalized demimetric mappings (see [9, 24] for details). Now, if the solution set Ω of problem (4.1) is nonempty, then by applying Theorem 3.1 we obtain the following theorem

Theorem 4.1. *Let $\{v_n\}$ be a sequence in C such that $v_n \to v$. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} u_n \in C \quad s.t. \quad f(u_n, y) + \langle Bu_n, y - u_n\rangle + \langle y - u_n, J_E^p u_n - J_E^p x_n\rangle \\ + \phi(u_n, y) - \phi(u_n, u_n) \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J_{E^*}^q \big[\sum_{j=1}^{M} b_{n,j}\big(J_E^p u_n - \sigma_{n,j} r_j J_E^p(u_n - P_{C_j} u_n)\big)\big], \\ z_n = \Pi_C J_{E^*}^q \big[\sum_{i=1}^{N} a_{n,i}\big(J_E^p y_n - \tau_{n,i} l_i A_i^* J_{E_i}^p(A_i y_n - P_{Q_i} A_i y_n)\big)\big], \\ x_{n+1} = \Pi_C J_{E^*}^q \big[\beta_n J_E^p x_n + (1-\beta_n)(\alpha_n J_p^E v_n + (1-\alpha_n) J_E^p z_n)\big], \end{cases} \tag{4.2}$$

where the stepsizes are chosen as follows:

$$\tau_{n,i} = \frac{\rho_{n,i}\|A_i y_n - P_{Q_i} A_i y_n\|^{p(p-1)}}{\overline{\tau}_{n,i}^{p}}, \ \ \textit{for} \ \ i = 1, 2, ..., N,$$

with

$$\overline{\tau}_{n,i} := \max\{\|A_i^* J_{E_i}^p(A_i y_n - P_{Q_i} A_i y_n)\|, \gamma\}$$

where $\gamma > 0$, $\{\rho_{n,i}\} \subset (0, (\frac{q}{C_q})^{\frac{1}{q-1}})$, and $\{\sigma_{n,j}\} \subset (0, (\frac{q}{C_q})^{\frac{1}{q-1}})$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$,$\{a_{n,i}\}$, and $\{b_{n,j}\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subseteq (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\lim_{n\to\infty} a_{n,i}\rho_{n,i}(1 - \frac{C_q \rho_{n,j}^{q-1}}{q}) > 0$, *and* $\lim_{n\to\infty} b_{n,j}\sigma_{n,j}(1 - \frac{C_q \sigma_{n,j}^{q-1}}{q}) > 0$.

(iii) $\{a_{n,i}\}_{i=1}^{N}, \{b_{n,j}\}_{j=1}^{M} \subset (0, 1]$, $\sum_{i=1}^{N} a_{n,i} = \sum_{j=1}^{M} b_{n,j} = 1$;

(iv) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, where $x^* = \Pi_\Omega v$.

Let $B : E \to 2^E$ be a set-valued mapping. The effective domain of B is denoted by $D(B)$, i.e., $D(B) = \{x \in E : Bx \neq \emptyset\}$. The mapping B on E is said to be monotone if

$$\langle u - v, x - y\rangle \geq 0 \quad \text{for all } x, y \in D(B),\ u \in Bx,\ v \in By.$$

A monotone mapping B on E is called maximal if its graph is not properly contained in the graph of any other monotone mapping on E.

Now, let E be a p-uniformly convex and uniformly smooth Banach space, and let $B : E \to E^*$ be a maximal monotone mapping. For each $r > 0$, the resolvent of B is defined by

$$J_r^B(x) = (J_p^E + rB)^{-1} J_P^E x, \quad \text{for each} \quad x \in E.$$

It is easy to verify that $F(J_r^B) = B^{-1}(0)$, where $B^{-1}(0) := \{x \in E : 0 \in Bx\}$ (see [15]).
For each $x \in E$ and $\mu > 0$, we define the metric resolvent of B by

$$Q_\mu^B(x) = (I + \mu(J_E^p)^{-1}B)^{-1}(x), \quad \text{for all} \quad x \in E,$$

(see [4]).

Let $M_j : E \to E^*$, $1 \le j \le M$ and $N_i : E_i \to E_i^*$, $1 \le i \le N$ be maximal monotone mappings. Let $S_j = J_{r_j}^{M_j}$ be resolvent operators of M_j for $r_j > 0$, and $1 \le j \le M$, and let $T_i = Q_{\mu_i}^{N_i}$ be metric resolvent operators of N_i for $\mu_i > 0$ and $1 \le i \le N$. Then S_j and T_i are 1-demimetric and demiclosed (see [9, 24] for details).
Assume that $\Omega = \Gamma \cap SOl(B, f, \phi)$ is a nonempty set, where

$$\Omega = (\cap_{j=1}^M M_j^{-1}(0)) \cap (\cap_{i=1}^N A_i^{-1}(N_i^{-1}(0)).$$

Then by applying Theorem 3.1 we have the following theorem.

Theorem 4.2. *Let $\{v_n\}$ be a sequence in C such that $v_n \to v$. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} u_n \in C \quad s.t. \;\; f(u_n, y) + \langle Bu_n, y - u_n\rangle + \langle y - u_n, J_E^p u_n - J_E^p x_n\rangle \\ \;+ \phi(u_n, y) - \phi(u_n, u_n) \ge 0, \quad \forall y \in C, \\ y_n = J_{E^*}^q \big[\sum_{j=1}^{M} b_{n,j}\big(J_E^p u_n - \sigma_{n,j} J_E^p(u_n - J_{r_j}^{M_j} u_n)\big)\big], \\ z_n = J_{E^*}^q \big[\sum_{i=1}^{N} a_{n,i}\big(J_E^p y_n - \tau_{n,i} A_i^* J_{E_i}^p (A_i y_n - Q_{\mu_i}^{N_i}) A_i y_n)\big)\big], \\ x_{n+1} = J_{E^*}^q \big[\alpha_n J_E^p u + (1-\alpha_n)(\beta_n J_p^E x_n + (1-\beta_n) J_E^p z_n)\big], \end{cases} \tag{4.3}$$

where the stepsizes are chosen in such a way that

$$\tau_{n,i} = \frac{\rho_{n,i}\|A_i y_n - Q_{\mu_i}^{N_i} A_i y_n\|^{p(p-1)}}{\overline{\tau}_{n,i}^p}, \;\; \textit{for} \;\; i = 1, 2, ..., N,$$

where

$$\overline{\tau}_{n,i} := \max\{\|A_i^* J_{E_i}^p (A_i y_n - Q_{\mu_i}^{N_i} A_i y_n)\|, \gamma\}$$

for $\gamma > 0$, $\rho_{n,i} \subset (0, (\frac{q}{C_q})^{\frac{1}{q-1}})$, and $\sigma_{n,j} \subset (0, (\frac{q}{C_q})^{\frac{1}{q-1}})$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$,$\{a_{n,i}\}$, and $\{b_{n,j}\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subseteq (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^\infty \alpha_n = \infty$.

(ii) $\lim_{n\to\infty} a_{n,j}\rho_{n,j}(1 - \frac{C_q \rho_{n,j}^{q-1}}{q}) > 0$, *and* $\lim_{n\to\infty} b_{n,j}\sigma_{n,j}(1 - \frac{C_q \sigma_{n,j}^{q-1}}{q}) > 0$.

(iii) $\{a_{n,i}\}_{i=1}^N, \{b_{n,j}\}_{j=1}^M \subset (0, 1]$, $\sum_{i=1}^N a_{n,i} = \sum_{j=1}^M b_{n,j} = 1$;

(iv) $0 < \liminf \beta_n \le \limsup \beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, where $x^* = \Pi_\Omega v$.

5 Numerical example

In Theorem 3.1, let $M = N = 1$, $E = \mathbb{R}$ and $E_i = R^2$. We define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x, y) = -3x^2 + xy + 2y^2$, $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $\phi(x, y) = x^2 - xy$, and $B : \mathbb{R} \to \mathbb{R}$ by Bx=x. Through some basic computations, we obtain that $u = K^{f,\phi}x = \frac{x}{6}$. Now we define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = 3x$. Clearly, $x^* = 0$ is the only fixed point of T and we have

$$-2(x - 0)(-2x) = -2\langle x - p, x - Tx\rangle = 4x^2 = \|x - Tx\|^2$$

Thus T is (-2)-generalized demimetric. Note that if we put $x^* = 0$ and $x = 1$, we see that T is not strictly pseudo-contractive and is not quasinonexpansive. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(x_1, x_2) = (3x_1, x_2)$. Clearly $F(S) = \{(0, a) : \forall a \in \mathbb{R}\}$ and S is (-2)-generalized demimetric. Define $A : \mathbb{R} \to \mathbb{R}^2$ by $Ax = (x, 0)$. Clearly A is a bounded linear operator and $A^*(x_1, x_2) = x_1$.

Assume that

$$\alpha_n = \frac{1}{n+10}, \beta_n = \frac{3}{4}, \sigma_{n,j} = \frac{1}{4}, \quad \text{and} \quad \rho_{n,i} = \frac{1}{4}.$$

Figure 1 illustrates the convergence behavior of the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, for the starting points $x_1 = 3.5$ and $x_1 = -4$. Tables 1 shows the numerical values of these sequences for $x_1 = 3.5$. The computations were performed using MATLAB.

Table 1: Numerical results for the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in Algorithm 3.1 with initial point $x_1 = 3.5$

n	x_n	y_n	z_n
1	3.5	0.29167	0.14583
2	2.6627	0.22189	0.11095
3	2.0266	0.16888	0.084442
4	1.5433	0.12861	0.064304
5	1.1760	0.097997	0.067882
6	0.90115	0.075096	0.061544
	-	-	-
	-	-	-
	-	-	-
97	0.002127	0.00017725	0.00017725
98	0.0021065	0.00017554	0.00017554
99	0.0020863	0.00017386	0.00017386
100	0.0020665	0.00017221	0.00017221

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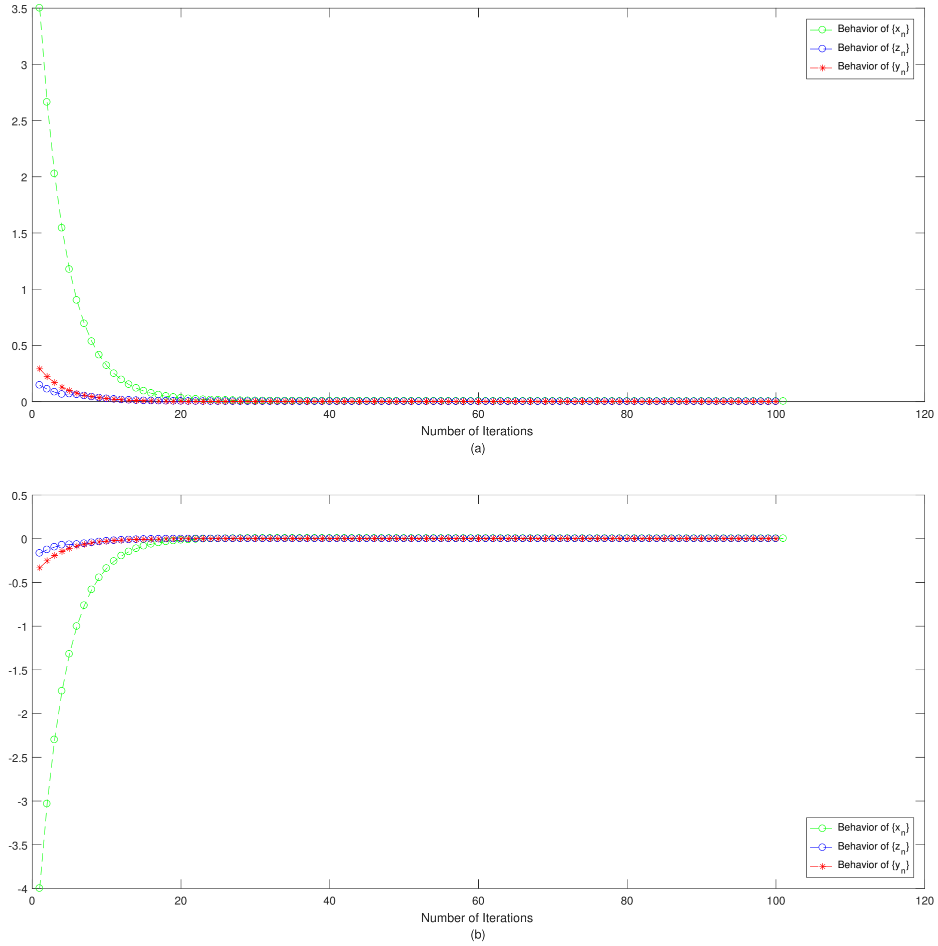

Figure 1: Figure 1 the convergence behavior of Algorithm 3.1 with different choices of starting points: (a) $x_1 = 3.5$, and (b) $x_1 = -4$.

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