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A new hybrid generalization of orthogonal polynomials

Dorota Bród, Mirosław Liana and Anetta Szynal-Liana

Abstract. In this paper, we introduce and study hybrinomials defined by application of orthogonal polynomials. Using selected orthogonal polynomials and hybrid numbers operators, we define Hermite, Laguerre, Legendre and Chebyshev type hybrinomials and present some properties of them.

Keywords. Orthogonal polynomials, complex numbers, hybrid numbers, hybrinomials

1 Introduction and preliminaries

Let $\mathcal{L}_2[a,b]$ denote the family of square-integrable functions on the real interval $[a,b]$. Two functions $f(x)$ and $g(x)$ in $\mathcal{L}_2[a,b]$ are said to be orthogonal on the interval $[a,b]$ with respect to a given continuous and nonnegative weight function $w(x)$ if $\int\limits_a^b w(x)f(x)g(x)dx = 0$ or, equivalently, if $\langle f,g\rangle = 0$, see [13]. Recall that $\langle f,g\rangle$ denote the inner product of the functions f and g. For orthogonal functions basic concepts and results, see for example [4, 17]. In particular, if the functions f and g are polynomials of degree n and m, respectively, we deal with orthogonal polynomials. Among the well-known orthogonal polynomials, there are Hermite polynomials $H_n(x)$, $He_n(x)$, Laguerre polynomials $L_n(x)$ and Jacobi polynomials. The Jacobi polynomials include Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$, $W_n(x)$ and the Legendre polynomials $P_n(x)$ as special cases. We will now recall the definitions and properties of selected orthogonal polynomials, further properties can be found e.g. in [7, 10].

Let $n \geq 0$ be an integer. The physicist's Hermite polynomials $H_n(x)$ are given by $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ and the probabilist's Hermite polynomials $He_n(x)$ are defined by $He_n(x) = (-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}$. The Laguerre polynomials $L_n(x)$ have the form $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$. The Chebyshev polynomials of the first kind $T_n(x)$, second kind $U_n(x)$, third kind $V_n(x)$ and fourth kind $W_n(x)$ are defined by $T_n(x) = \cos(n \arccos x)$, $U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}$, $V_n(X) = \frac{\cos\left((n+\frac{1}{2})\arccos x\right)}{\cos(\frac{1}{2}\arccos x)}$ and $W_n(x) = \frac{\sin\left((n+\frac{1}{2})\arccos x\right)}{\sin(\frac{1}{2}\arccos x)}$, respectively. The Legendre polynomials $P_n(x)$ are given by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.

The above polynomials may be also defined recursively. For $n \geq 2$ we have

$$H_n(x) = 2xH_{n-1}(x) - 2nH_{n-2}(x)$$

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Corresponding author: Anetta Szynal-Liana.

with

$$H_0(x) = 1,\ H_1(x) = 2x,$$

$$He_n(x) = xHe_{n-1}(x) - nHe_{n-2}(x)$$

with

$$He_0(x) = 1,\ He_1(x) = x,$$

$$L_n(x) = \frac{2n-1-x}{n}L_{n-1}(x) - \frac{n-1}{n}L_{n-2}(x)$$

with

$$L_0(x) = 1,\ L_1(x) = 1-x,$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

with

$$T_0(x) = 1,\ T_1(x) = x,$$

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

with

$$U_0(x) = 1,\ U_1(x) = 2x,$$

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x)$$

with

$$V_0(x) = 1,\ V_1(x) = 2x-1,$$

and

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x)$$

with

$$W_0(x) = 1,\ W_1(x) = 2x+1,$$

and

$$P_n(x) = \frac{(2n-1)x}{n}P_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x)$$

with

$$P_0(x) = 1,\ P_1(x) = x.$$

The Table 1 includes initial terms of selected orthogonal polynomials for $n = 0, 1, 2, 3, 4$.

n	0	1	2	3	4
$H_n(x)$	1	$2x$	$4x^2-2$	$8x^3-12x$	$16x^4-48x^2+12$
$He_n(x)$	1	x	x^2-1	x^3-3x	x^4-6x^2+3
$L_n(x)$	1	$1-x$	$\frac{1}{2}\left(x^2-4x+2\right)$	$\frac{1}{6}\left(-x^3+9x^2-18x+6\right)$	$\frac{1}{24}\left(x^4-16x^3+72x^2-96x+24\right)$
$T_n(x)$	1	x	$2x^2-1$	$4x^3-3x$	$8x^4-8x^2+1$
$U_n(x)$	1	$2x$	$4x^2-1$	$8x^3-4x$	$16x^4-12x^2+1$
$V_n(x)$	1	$2x-1$	$4x^2-2x-1$	$8x^3-4x^2-4x+1$	$16x^4-8x^3-12x^2+4x+1$
$W_n(x)$	1	$2x+1$	$4x^2+2x-1$	$8x^3+4x^2-4x-1$	$16x^4+8x^3-12x^2-4x+1$
$P_n(x)$	1	x	$\frac{1}{2}\left(3x^2-1\right)$	$\frac{1}{2}\left(5x^3-3x\right)$	$\frac{1}{8}\left(35x^4-30x^2+3\right)$

Table 1: The orthogonal polynomials.

In the literature, we can find many books and papers concerning polynomials and their properties, see, e.g. [1, 3, 5, 6, 11, 12, 13, 15]. In this paper, we use the concept of orthogonal polynomials in the theory of hybrid numbers. The hybrid numbers were introduced in [14] as a generalization of complex, dual and hyperbolic numbers. A hybrid number $\mathbf{Z}$ has the form $\mathbf{Z} = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$, where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, $\boldsymbol{\varepsilon}$, $\mathbf{h}$ are operators which satisfy the following relations

$$\mathbf{i}^2 = -1,\ \boldsymbol{\varepsilon}^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}. \tag{1.1}$$

The set of hybrid numbers is denoted by $\mathbb{K}$. Let $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}$ and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}$ be any two hybrid numbers. Then

$\mathbf{Z}_1 = \mathbf{Z}_2$ if and only if $a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2,\ d_1 = d_2$,
$\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\boldsymbol{\varepsilon} + (d_1 + d_2)\mathbf{h}$,
$\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\boldsymbol{\varepsilon} + (d_1 - d_2)\mathbf{h}$,
for $\alpha \in \mathbb{R}$ $\quad \alpha\mathbf{Z}_1 = \alpha a_1 + \alpha b_1\mathbf{i} + \alpha c_1\boldsymbol{\varepsilon} + \alpha d_1\mathbf{h}$.

Moreover, the conjugate of the hybrid number $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}$, denoted by $\overline{\mathbf{Z}}_1$, is defined as $\overline{\mathbf{Z}}_1 = a_1 - b_1\mathbf{i} - c_1\boldsymbol{\varepsilon} - d_1\mathbf{h}$.

Using (1.1), we can multiply hybrid numbers. The Table 2 presents products of operators $\mathbf{i}$, $\boldsymbol{\varepsilon}$, and $\mathbf{h}$.

$\cdot$	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$1+\mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$-(\boldsymbol{\varepsilon}+\mathbf{i})$	$\boldsymbol{\varepsilon}$	1

Table 2: The hybrid number multiplication.

It is easy to see that the multiplication of hybrid numbers can be done in the same way as the multiplication of algebraic expressions. Other properties of hybrid numbers are given in [14].

The term 'hybrinomial' was used for the first time in [18]. The authors defined, using Fibonacci and Lucas polynomials, Fibonacci and Lucas hybrinomials. Some generalizations of Fibonacci and Lucas hybrinomials, as well as properties of other hybrinomials, can be found in [2, 8, 9, 16, 19].

Now, we will define selected hybrinomials as follows. For a nonnegative integer n and complex x, the nth physicist's Hermite hybrinomial $\mathcal{H}H_n(x)$, probabilist's Hermite hybrinomial $\mathcal{H}He_n(x)$, Laguerre hybrinomial $\mathcal{H}L_n(x)$, Chebyshev hybrinomial of the first kind $\mathcal{H}T_n(x)$, second kind $\mathcal{H}U_n(x)$, third kind $\mathcal{H}V_n(x)$, fourth kind $\mathcal{H}W_n(x)$ and Legendre hybrinomial $\mathcal{H}P_n(x)$ is defined by

$$\mathcal{H}H_n(x) = H_n(x) + H_{n+1}(x)\mathbf{i} + H_{n+2}(x)\boldsymbol{\varepsilon} + H_{n+3}(x)\mathbf{h},$$
$$\mathcal{H}He_n(x) = He_n(x) + He_{n+1}(x)\mathbf{i} + He_{n+2}(x)\boldsymbol{\varepsilon} + He_{n+3}(x)\mathbf{h},$$
$$\mathcal{H}L_n(x) = L_n(x) + L_{n+1}(x)\mathbf{i} + L_{n+2}(x)\boldsymbol{\varepsilon} + L_{n+3}(x)\mathbf{h}, \tag{1.2}$$
$$\mathcal{H}T_n(x) = T_n(x) + T_{n+1}(x)\mathbf{i} + T_{n+2}(x)\boldsymbol{\varepsilon} + T_{n+3}(x)\mathbf{h}, \tag{1.3}$$
$$\mathcal{H}U_n(x) = U_n(x) + U_{n+1}(x)\mathbf{i} + U_{n+2}(x)\boldsymbol{\varepsilon} + U_{n+3}(x)\mathbf{h}, \tag{1.4}$$
$$\mathcal{H}V_n(x) = V_n(x) + V_{n+1}(x)\mathbf{i} + V_{n+2}(x)\boldsymbol{\varepsilon} + V_{n+3}(x)\mathbf{h},$$
$$\mathcal{H}W_n(x) = W_n(x) + W_{n+1}(x)\mathbf{i} + W_{n+2}(x)\boldsymbol{\varepsilon} + W_{n+3}(x)\mathbf{h}$$

and

$$\mathcal{H}P_n(x) = P_n(x) + P_{n+1}(x)\mathbf{i} + P_{n+2}(x)\boldsymbol{\varepsilon} + P_{n+3}(x)\mathbf{h},$$

respectively.

2 Main results

At the beginning, we will focus on the properties of hybrinomials associated with Chebyshev polynomials. First, let us recall the dependencies between the Chebyshev polynomials (see [13])

$$T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x)),\ n = 2, 3, \dots$$

$$V_n(x) = U_n(x) - U_{n-1}(x),\ n = 1, 2, \dots$$

$$W_n(x) = U_n(x) + U_{n-1}(x),\ n = 1, 2, \dots .$$

Using the above properties, it is easy to show relationships between Chebyshev hybrinomials

$$\mathcal{H}T_n(x) = \frac{1}{2}(\mathcal{H}U_n(x) - \mathcal{H}U_{n-2}(x)),\ n = 2, 3, \dots$$

$$\mathcal{H}V_n(x) = \mathcal{H}U_n(x) - \mathcal{H}U_{n-1}(x),\ n = 1, 2, \dots$$

$$\mathcal{H}W_n(x) = \mathcal{H}U_n(x) + \mathcal{H}U_{n-1}(x),\ n = 1, 2, \dots .$$

In the next part of this section we will study some properties of the Chebyshev hybrinomials of the second kind; the properties of the remaining Chebyshev hybrinomials can be obtained using the dependencies presented.

Theorem 2.1. *For a nonnegative integer n and complex x, $|x| \neq 1$, we have*

$$\mathcal{H}U_n(x) = 2x\mathcal{H}U_{n-1}(x) - \mathcal{H}U_{n-2}(x) \text{ for } n \geq 2 \tag{2.1}$$

with

$$\begin{aligned} \mathcal{H}U_0(x) &= 1 + 2x\mathbf{i} + (4x^2-1)\boldsymbol{\varepsilon} + (8x^3-4x)\mathbf{h} \\ \mathcal{H}U_1(x) &= 2x + (4x^2-1)\mathbf{i} + (8x^3-4x)\boldsymbol{\varepsilon} + (16x^4-12x^2+1)\mathbf{h}. \end{aligned} \tag{2.2}$$

Proof. Let $n = 2$. Then

$$\begin{aligned} \mathcal{H}U_2(x) &= 2x\mathcal{H}U_1(x) - \mathcal{H}U_0(x) \\ &= 2x(2x + (4x^2-1)\mathbf{i} + (8x^3-4x)\boldsymbol{\varepsilon} + (16x^4-12x^2+1)\mathbf{h}) \\ &\quad - 1 - 2x\mathbf{i} - (4x^2-1)\boldsymbol{\varepsilon} - (8x^3-4x)\mathbf{h} \\ &= (4x^2-1) + (8x^3-4x)\mathbf{i} + (16x^4-12x^2+1)\boldsymbol{\varepsilon} + (32x^5-32x^3+6x)\mathbf{h} \\ &= U_2(x) + U_3(x)\mathbf{i} + U_4(x)\boldsymbol{\varepsilon} + U_5(x)\mathbf{h}. \end{aligned}$$

Let $n \geq 3$. Using the definition of Chebyshev polynomials, we get

$$\begin{aligned} \mathcal{H}U_n(x) &= U_n(x) + U_{n+1}(x)\mathbf{i} + U_{n+2}(x)\boldsymbol{\varepsilon} + U_{n+3}(x)\mathbf{h} \\ &= (2xU_{n-1}(x) - U_{n-2}(x)) + (2xU_n(x) - U_{n-1}(x))\mathbf{i} \\ &\quad + (2xU_{n+1}(x) - U_n(x))\boldsymbol{\varepsilon} + (2xU_{n+2}(x) - U_{n+1}(x))\mathbf{h} \\ &= 2x(U_{n-1}(x) + U_n(x)\mathbf{i} + U_{n+1}(x)\boldsymbol{\varepsilon} + U_{n+2}(x)\mathbf{h}) \\ &\quad - (U_{n-2}(x) + U_{n-1}(x)\mathbf{i} + U_n(x)\boldsymbol{\varepsilon} + U_{n+1}(x)\mathbf{h}) \\ &= 2x\mathcal{H}U_{n-1}(x) - \mathcal{H}U_{n-2}(x), \end{aligned}$$

which completes the proof. □

For a nonnegative integer n and complex x, $|x| \neq 1$, we have

$$U_n(x) = \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)}, \tag{2.3}$$

where

$$\alpha(x) = x + \sqrt{x^2 - 1},\ \beta(x) = x - \sqrt{x^2 - 1}, \tag{2.4}$$

see [13].

Theorem 2.2. (*Binet formula for Chebyshev hybrinomials of the second kind*) *For a nonnegative integer n and complex x, $|x| \neq 1$, we have*

$$\begin{aligned} \mathcal{H}U_n(x) &= \frac{\alpha^{n+1}(x)}{\alpha(x) - \beta(x)} \left(1 + \alpha(x)\mathbf{i} + \alpha^2(x)\boldsymbol{\varepsilon} + \alpha^3(x)\mathbf{h}\right) \\ &\quad - \frac{\beta^{n+1}(x)}{\alpha(x) - \beta(x)} \left(1 + \beta(x)\mathbf{i} + \beta^2(x)\boldsymbol{\varepsilon} + \beta^3(x)\mathbf{h}\right), \end{aligned} \tag{2.5}$$

where $\alpha(x)$, $\beta(x)$ are given by (2.4).

Proof. Using (1.4) and (2.3), we have

$$\begin{aligned} \mathcal{H}U_n(x) &= \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)} + \frac{\alpha^{n+2}(x) - \beta^{n+2}(x)}{\alpha(x) - \beta(x)}\mathbf{i} \\ &\quad + \frac{\alpha^{n+3}(x) - \beta^{n+3}(x)}{\alpha(x) - \beta(x)}\boldsymbol{\varepsilon} + \frac{\alpha^{n+4}(x) - \beta^{n+4}(x)}{\alpha(x) - \beta(x)}\mathbf{h} \\ &= \frac{\alpha^{n+1}(x)}{\alpha(x) - \beta(x)} \left(1 + \alpha(x)\mathbf{i} + \alpha^2(x)\boldsymbol{\varepsilon} + \alpha^3(x)\mathbf{h}\right) \\ &\quad - \frac{\beta^{n+1}(x)}{\alpha(x) - \beta(x)} \left(1 + \beta(x)\mathbf{i} + \beta^2(x)\boldsymbol{\varepsilon} + \beta^3(x)\mathbf{h}\right), \end{aligned}$$

which ends the proof. □

For simplicity of notation let

$$\begin{aligned} \hat{\alpha}(x) &= 1 + \alpha(x)\mathbf{i} + \alpha^2(x)\boldsymbol{\varepsilon} + \alpha^3(x)\mathbf{h}, \\ \hat{\beta}(x) &= 1 + \beta(x)\mathbf{i} + \beta^2(x)\boldsymbol{\varepsilon} + \beta^3(x)\mathbf{h}, \end{aligned} \tag{2.6}$$

Then we can write (2.5) as

$$\mathcal{H}U_n(x) = \frac{\alpha^{n+1}(x)\hat{\alpha}(x) - \beta^{n+1}(x)\hat{\beta}(x)}{\alpha(x) - \beta(x)}.$$

Theorem 2.3. (*general bilinear index-reduction formula for Chebyshev hybrinomials of the second kind*) *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then for complex x, $|x| \neq 1$, we have*

$$\begin{aligned} &\mathcal{H}U_a(x) \cdot \mathcal{H}U_b(x) - \mathcal{H}U_c(x) \cdot \mathcal{H}U_d(x) \\ &= \frac{1}{(\alpha(x) - \beta(x))^2} \Big[\left(\alpha^c(x)\beta^d(x) - \alpha^a(x)\beta^b(x)\right) \hat{\alpha}(x)\hat{\beta}(x) \\ &\quad + \left(\beta^c(x)\alpha^d(x) - \beta^a(x)\alpha^b(x)\right) \hat{\beta}(x)\hat{\alpha}(x) \Big], \end{aligned}$$

where $\alpha(x)$, $\beta(x)$ and $\hat{\alpha}(x)$, $\hat{\beta}(x)$ are given by (2.4) and (2.6), respectively.

Proof. By formula (2.5), we get

$$\begin{aligned}&\mathcal{H}U_a(x)\cdot\mathcal{H}U_b(x)-\mathcal{H}U_c(x)\cdot\mathcal{H}U_d(x)\\&=\frac{1}{(\alpha(x)-\beta(x))^2}\left[\left(\alpha^{a+b+2}(x)-\alpha^{c+d+2}(x)\right)\hat{\alpha}(x)\hat{\alpha}(x)\right.\\&+\left(\beta^{a+b+2}(x)-\beta^{c+d+2}(x)\right)\hat{\beta}(x)\hat{\beta}(x)\\&+\left(\alpha^c(x)\beta^d(x)\alpha(x)\beta(x)-\alpha^a(x)\beta^b(x)\alpha(x)\beta(x)\right)\hat{\alpha}(x)\hat{\beta}(x)\\&\left.+\left(\beta^c(x)\alpha^d(x)\beta(x)\alpha(x)-\beta^a(x)\alpha^b(x)\beta(x)\alpha(x)\right)\hat{\beta}(x)\hat{\alpha}(x)\right].\end{aligned}$$

Using $a+b=c+d$ and the fact that $\alpha(x)\cdot\beta(x)=1$, we get

$$\begin{aligned}&\mathcal{H}U_a(x)\cdot\mathcal{H}U_b(x)-\mathcal{H}U_c(x)\cdot\mathcal{H}U_d(x)\\&=\frac{1}{(\alpha(x)-\beta(x))^2}\left[\left(\alpha^c(x)\beta^d(x)-\alpha^a(x)\beta^b(x)\right)\hat{\alpha}(x)\hat{\beta}(x)\right.\\&\left.+\left(\beta^c(x)\alpha^d(x)-\beta^a(x)\alpha^b(x)\right)\hat{\beta}(x)\hat{\alpha}(x)\right],\end{aligned}$$

which ends the proof. □

It is easily seen that for special values of a, b, c and d, by Theorem 2.3, we get some identities for Chebyshev hybrinomials of the second kind.

Corollary 2.4. (*d'Ocagne identity for Chebyshev hybrinomials of the second kind*) *Let $n \geq 0$, $m \geq 0$ be integers. Then for complex x, $|x| \neq 1$, we have*

$$\begin{aligned}&\mathcal{H}U_n(x)\cdot\mathcal{H}U_{m+1}(x)-\mathcal{H}U_{n+1}(x)\cdot\mathcal{H}U_m(x)\\&=\frac{\alpha^n(x)\beta^m(x)\hat{\alpha}(x)\hat{\beta}(x)-\beta^n(x)\alpha^m(x)\hat{\beta}(x)\hat{\alpha}(x)}{\alpha(x)-\beta(x)},\end{aligned}$$

where $\alpha(x)$, $\beta(x)$ and $\hat{\alpha}(x)$, $\hat{\beta}(x)$ are given by (2.4) *and* (2.6), *respectively.*

Corollary 2.5. (*Vajda identity for Chebyshev hybrinomials of the second kind*) *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then for complex x, $|x| \neq 1$, we have*

$$\begin{aligned}&\mathcal{H}U_{m+r}(x)\cdot\mathcal{H}U_{n-r}(x)-\mathcal{H}U_m(x)\cdot\mathcal{H}U_n(x)\\&=\frac{1}{(\alpha(x)-\beta(x))^2}\left[\alpha^m(x)\beta^n(x)\left(1-\left(\frac{\alpha(x)}{\beta(x)}\right)^r\right)\hat{\alpha}(x)\hat{\beta}(x)\right.\\&\left.+\beta^m(x)\alpha^n(x)\left(1-\left(\frac{\beta(x)}{\alpha(x)}\right)^r\right)\hat{\beta}(x)\hat{\alpha}(x)\right],\end{aligned}$$

where $\alpha(x)$, $\beta(x)$ and $\hat{\alpha}(x)$, $\hat{\beta}(x)$ are given by (2.4) *and* (2.6), *respectively.*

Corollary 2.6. (*first Halton identity for Chebyshev hybrinomials of the second kind*) *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers. Then for complex x, $|x| \neq 1$, we have*

$$\begin{aligned}&\mathcal{H}U_{m+r}(x)\cdot\mathcal{H}U_n(x)-\mathcal{H}U_r(x)\cdot\mathcal{H}U_{m+n}(x)\\&=\frac{\beta^m(x)-\alpha^m(x)}{(\alpha(x)-\beta(x))^2}\left[\alpha^r(x)\beta^n(x)\hat{\alpha}(x)\hat{\beta}(x)-\beta^r(x)\alpha^n(x)\hat{\beta}(x)\hat{\alpha}(x)\right],\end{aligned}$$

where $\alpha(x)$, $\beta(x)$ and $\hat{\alpha}(x)$, $\hat{\beta}(x)$ are given by (2.4) *and* (2.6), *respectively.*

Corollary 2.7. (*second Halton identity for Chebyshev hybrinomials of the second kind*) *Let $n \geq 0$, $k \geq 0$, $s \geq 0$ be integers such that $n \geq k$, $n \geq s$. Then for complex x, $|x| \neq 1$, we have*

$$\begin{aligned}&\mathcal{H}U_{n+k}(x) \cdot \mathcal{H}U_{n-k}(x) - \mathcal{H}U_{n+s}(x) \cdot \mathcal{H}U_{n-s}(x)\\ &= \frac{1}{(\alpha(x)-\beta(x))^2}\left[\left(\left(\frac{\alpha(x)}{\beta(x)}\right)^s - \left(\frac{\alpha(x)}{\beta(x)}\right)^k\right)\hat{\alpha}(x)\hat{\beta}(x)\right.\\ &\left.+\left(\left(\frac{\beta(x)}{\alpha(x)}\right)^s - \left(\frac{\beta(x)}{\alpha(x)}\right)^k\right)\hat{\beta}(x)\hat{\alpha}(x)\right],\end{aligned}$$

where $\alpha(x)$, $\beta(x)$ and $\hat{\alpha}(x)$, $\hat{\beta}(x)$ are given by (2.4) and (2.6), respectively.

Corollary 2.8. (*Catalan identity for Chebyshev hybrinomials of the second kind*) *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then for complex x, $|x| \neq 1$, we have*

$$\begin{aligned}&\mathcal{H}U_{n+r}(x) \cdot \mathcal{H}U_{n-r}(x) - \mathcal{H}U_{n}(x) \cdot \mathcal{H}U_{n}(x)\\ &= \frac{1}{(\alpha(x)-\beta(x))^2}\left[\left(1 - \left(\frac{\alpha(x)}{\beta(x)}\right)^r\right)\hat{\alpha}(x)\hat{\beta}(x)\right.\\ &\left.+\left(1 - \left(\frac{\beta(x)}{\alpha(x)}\right)^r\right)\hat{\beta}(x)\hat{\alpha}(x)\right],\end{aligned}$$

where $\alpha(x)$, $\beta(x)$ and $\hat{\alpha}(x)$, $\hat{\beta}(x)$ are given by (2.4) and (2.6), respectively.

Corollary 2.9. (*Cassini identity for Chebyshev hybrinomials of the second kind*) *Let $n \geq 1$ be an integer. Then for complex x, $|x| \neq 1$, we have*

$$\begin{aligned}&\mathcal{H}U_{n+1}(x) \cdot \mathcal{H}U_{n-1}(x) - \mathcal{H}U_{n}(x) \cdot \mathcal{H}U_{n}(x)\\ &= \frac{1}{(\alpha(x)-\beta(x))^2}\left[\left(1 - \frac{\alpha(x)}{\beta(x)}\right)\hat{\alpha}(x)\hat{\beta}(x) + \left(1 - \frac{\beta(x)}{\alpha(x)}\right)\hat{\beta}(x)\hat{\alpha}(x)\right],\end{aligned}$$

where $\alpha(x)$, $\beta(x)$ and $\hat{\alpha}(x)$, $\hat{\beta}(x)$ are given by (2.4) and (2.6), respectively.

The next theorem presents the generating function for Chebyshev hybrinomials of the second kind.

Theorem 2.10. *The generating function for the sequence of the Chebyshev hybrinomials of the second kind $\{\mathcal{H}U_n(x)\}$ is*

$$g(t) = \frac{\mathcal{H}U_0(x) + (\mathcal{H}U_1(x) - 2x\mathcal{H}U_0(x))t}{1-2xt+t^2},$$

where $\mathcal{H}U_0(x)$ and $\mathcal{H}U_1(x)$ are given by (2.2).

Proof. Assume that the generating function of the sequence of the Chebyshev hybrinomials of the second kind $\{\mathcal{H}U_n(x)\}$ has the form $g(t) = \sum\limits_{n=0}^{\infty} \mathcal{H}U_n(x)t^n$. Then

$$g(t) = \mathcal{H}U_0(x) + \mathcal{H}U_1(x)t + \mathcal{H}U_2(x)t^2 + \ldots$$

Hence we get

$$\begin{aligned}-2xt \cdot g(t) &= -2x\mathcal{H}U_0(x)t - 2x\mathcal{H}U_1(x)t^2 - 2x\mathcal{H}U_2(x)t^3 - \ldots\\ t^2 \cdot g(t) &= \mathcal{H}U_0(x)t^2 + \mathcal{H}U_1(x)t^3 + \mathcal{H}U_2(x)t^4 + \ldots\end{aligned}$$

By adding these three equalities above, we get

$$g(t)(1-2xt+t^2) = \mathcal{H}U_0(x) + (\mathcal{H}U_1(x) - 2x\mathcal{H}U_0(x))t$$

since $\mathcal{H}U_n(x) = 2x\mathcal{H}U_{n-1}(x) - \mathcal{H}U_{n-2}(x)$ (see (2.1)) and the coefficients of t^n for $n \geq 2$ are equal to zero. Moreover, by simple calculations we have

$$\mathcal{H}U_1(x) - 2x\mathcal{H}U_0(x) = -\mathbf{i} - 2x\boldsymbol{\varepsilon} + (-4x^2+1)\mathbf{h}.$$

□

Now, we give a matrix representation of Chebyshev hybrinomials of the second kind.

Theorem 2.11. *For a positive integer n and complex x, $|x| \neq 1$, we have*

$$\begin{aligned} &\begin{bmatrix} \mathcal{H}U_{n+1}(x) & -\mathcal{H}U_n(x) \\ H\mathcal{U}_n(x) & -\mathcal{H}U_{n-1}(x) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{H}U_2(x) & -\mathcal{H}U_1(x) \\ \mathcal{H}U_1(x) & -\mathcal{H}U_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix}^{n-1}. \end{aligned} \tag{2.7}$$

Proof. (by induction on n) If $n = 1$, then assuming that the matrix to the power of 0 is the identity matrix, the result is obvious. Assuming the formula (2.7) holds for $n \geq 1$, we shall prove it for $n+1$. Using induction's hypothesis and formula (2.1), we have

$$\begin{aligned} &\begin{bmatrix} \mathcal{H}U_2(x) & -\mathcal{H}U_1(x) \\ \mathcal{H}U_1(x) & -\mathcal{H}U_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix}^{n} \\ &= \begin{bmatrix} \mathcal{H}U_{n+1}(x) & -\mathcal{H}U_n(x) \\ \mathcal{H}U_n(x) & -\mathcal{H}U_{n-1}(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2x\mathcal{H}U_{n+1}(x) - \mathcal{H}U_n(x) & -\mathcal{H}U_{n+1}(x) \\ 2x\mathcal{H}U_n(x) - \mathcal{H}U_{n-1}(x) & -\mathcal{H}U_n(x) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{H}U_{n+2}(x) & -\mathcal{H}U_{n+1}(x) \\ \mathcal{H}U_{n+1}(x) & -\mathcal{H}U_n(x) \end{bmatrix}, \end{aligned}$$

which ends the proof. □

Using some properties of orthogonal polynomials, we can also obtain their hybrid versions. For example, using $T_n(x)+T_{n-1}(x) = (1+x)V_{n-1}(x)$, $n = 1, 2, \ldots$ (see [13]), we obtain $\mathcal{H}T_n(x) + \mathcal{H}T_{n-1}(x) = (1+x)\mathcal{H}V_{n-1}(x)$, $n = 1, 2, \ldots$. Using (1.3), we have

$$\begin{aligned} &\mathcal{H}T_n(x) + \mathcal{H}T_{n-1}(x) = T_n(x) + T_{n+1}(x)\mathbf{i} + T_{n+2}(x)\boldsymbol{\varepsilon} + T_{n+3}(x)\mathbf{h} \\ &\quad + T_{n-1}(x) + T_n(x)\mathbf{i} + T_{n+1}(x)\boldsymbol{\varepsilon} + T_{n+2}(x)\mathbf{h} \\ &\quad = (1+x)V_{n-1}(x) + (1+x)V_n(x)\mathbf{i} + (1+x)V_{n+1}(x)\boldsymbol{\varepsilon} + (1+x)V_{n+2}(x)\mathbf{h} \\ &\quad = (1+x)\mathcal{H}V_{n-1}(x). \end{aligned}$$

In the same way, using Theorem 2.12, we can prove Theorem 2.13.

Theorem 2.12. [13] *For a nonnegative integer n and complex x, $|x| \neq 1$, we have*

(i) $V_n(x) + W_n(x) = 2U_n(x)$,

(ii) $V_n(x) + V_{n-1}(x) = 2T_n(x),\ n = 1, 2, \ldots$

(iii) $W_n(x) - W_{n-1}(x) = 2T_n(x),\ n = 1, 2, \ldots$

(iv) $T_n(x) - T_{n-1}(x) = (x-1)W_{n-1}(x),\ n = 1, 2, \ldots$

(v) $T_n(x) - T_{n-2}(x) = 2(x^2-1)U_{n-2}(x),\ n = 2, 3, \ldots$

Theorem 2.13. *For a nonnegative integer n and complex x, $|x| \neq 1$, we have*

(i) $\mathcal{H}V_n(x) + \mathcal{H}W_n(x) = 2\mathcal{H}U_n(x)$,

(ii) $\mathcal{H}V_n(x) + \mathcal{H}V_{n-1}(x) = 2\mathcal{H}T_n(x),\ n = 1, 2, \ldots$

(iii) $\mathcal{H}W_n(x) - \mathcal{H}W_{n-1}(x) = 2\mathcal{H}T_n(x),\ n = 1, 2, \ldots$

(iv) $\mathcal{H}T_n(x) - \mathcal{H}T_{n-1}(x) = (x-1)\mathcal{H}W_{n-1}(x),\ n = 1, 2, \ldots$

(v) $\mathcal{H}T_n(x) - \mathcal{H}T_{n-2}(x) = 2(x^2-1)\mathcal{H}U_{n-2}(x),\ n = 2, 3, \ldots$

In the next part of this paper, we focus on Hermite, Laguerre and Legendre hybrinomials. Explicit expressions for these orthogonal polynomials have the forms

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}, \tag{2.8}$$

$$He_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!}\frac{x^{n-2m}}{2^m}, \tag{2.9}$$

$$L_n(x) = \sum_{m=0}^{n} \binom{n}{m}\frac{(-1)^m}{m!}x^m, \tag{2.10}$$

and

$$P_n(x) = 2^n \sum_{m=0}^{n} \binom{n}{m}\binom{\frac{n+m-1}{2}}{n}x^m, \tag{2.11}$$

respectively.

For example, using (2.10), we obtain the explicit formula for the nth Laguerre hybrinomial. For a nonnegative integer n and complex x, we have

$$\begin{aligned}\mathcal{H}L_n(x) &= L_n(x) + L_{n+1}(x)\mathbf{i} + L_{n+2}(x)\boldsymbol{\varepsilon} + L_{n+3}(x)\mathbf{h}\\ &= \sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m}{m!}x^m + \sum_{m=0}^{n+1}\binom{n+1}{m}\frac{(-1)^m}{m!}x^m\mathbf{i}\\ &+ \sum_{m=0}^{n+2}\binom{n+2}{m}\frac{(-1)^m}{m!}x^m\boldsymbol{\varepsilon} + \sum_{m=0}^{n+3}\binom{n+3}{m}\frac{(-1)^m}{m!}x^m\mathbf{h}.\end{aligned} \tag{2.12}$$

In a similar way, we can obtain explicit formulas for the remaining hybrinomials. We can also use the fact that the Hermite polynomials can be expressed as a special case of the Laguerre polynomials, i.e.

$$\begin{aligned}H_{2n}(x) &= (-1)^n 2^{2n} n! L_n^{(-\frac{1}{2})}(x^2),\\ H_{2n+1}(x) &= (-1)^n 2^{2n+1} n! x L_n^{(\frac{1}{2})}(x^2),\\ He_n(x) &= 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right).\end{aligned}$$

The Legendre polynomials can also be defined using the generating function

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}}. \tag{2.13}$$

Using (2.13), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{H}P_n(x)t^n &= \sum_{n=0}^{\infty} \left(P_n(x) + P_{n+1}(x)\mathbf{i} + P_{n+2}(x)\boldsymbol{\varepsilon} + P_{n+3}(x)\mathbf{h}\right) \\
&= \sum_{n=0}^{\infty} P_n(x) + \sum_{n=0}^{\infty} P_n(x)\mathbf{i} - P_0(x)\mathbf{i} + \sum_{n=0}^{\infty} P_n(x)\boldsymbol{\varepsilon} - P_0(x)\boldsymbol{\varepsilon} - P_1(x)\boldsymbol{\varepsilon} \\
&+ \sum_{n=0}^{\infty} P_n(x)\mathbf{h} - P_0(x)\mathbf{h} - P_1(x)\mathbf{h} - P_2(x)\mathbf{h} \\
&= \frac{1+\mathbf{i}+\boldsymbol{\varepsilon}+\mathbf{h}}{\sqrt{1-2xt+t^2}} - \mathbf{i} - (1+x)\boldsymbol{\varepsilon} - \left(\frac{3}{2}x^2 + x + \frac{1}{2}\right)\mathbf{h}
\end{aligned}$$

and we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{H}P_n(x)t^n &= \frac{1}{\sqrt{1-2xt+t^2}} + \frac{1-\sqrt{1-2xt+t^2}}{\sqrt{1-2xt+t^2}}\mathbf{i} \\
&+ \frac{1-(1+x)\sqrt{1-2xt+t^2}}{\sqrt{1-2xt+t^2}}\boldsymbol{\varepsilon} + \frac{1-\left(\frac{3}{2}x^2+x+\frac{1}{2}\right)\sqrt{1-2xt+t^2}}{\sqrt{1-2xt+t^2}}\mathbf{h}.
\end{aligned}$$

The use of other properties of polynomials presented will allow us to obtain new properties of the corresponding hybrinomials.

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Dorota Bród Rzeszow University of Technology

E-mail: dorotab@prz.edu.pl

Mirosław Liana Rzeszow University of Technology

E-mail: mliana@prz.edu.pl

Anetta Szynal-Liana Rzeszow University of Technology

E-mail: aszynal@prz.edu.pl